Cubic surfaces with 27 rational lines

The simplest way to describe these seems to be as follows. Choose three skew lines L_1, L_2, L_3 on the cubic surface V and let M_1, M_2, M_3 be their three transversals on V; then there is a quadric which contains these six lines. We can take this quadric to be

$$X_0X_1 = X_2X_3$$

and the lines to be

$$\begin{array}{ll} L_1: \ X_0 = X_2 = 0, & M_1: \ X_0 = X_3 = 0, \\ L_2: \ X_1 = X_3 = 0, & M_2: \ X_1 = X_2 = 0, \\ L_3: \ X_0 = X_3, \ X_1 = X_2, & M_3: \ X_0 = X_2, \ X_1 = X_3. \end{array}$$

The general cubic surface containing these six lines has the form

$$\begin{aligned} & c_4(X_0 + X_1 - X_2 - X_3)(X_0X_1 + X_2X_3) \\ & \quad + (c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3)(X_0X_1 - X_2X_3) = 0. \end{aligned}$$

This surface necessarily contains a further nine rational lines, each line being the residual intersection of V with the plane containing L_i and M_j for some i, j; we shall call this line L_{ij}.

The remaining lines on V fall into six pairs, a typical pair being the two lines N, N' on V which meet L_1 and L_2 but not L_3. In general, the individual lines of such a pair are only defined over a quadratic extension; what is crucial for us is that this quadratic extension is the same for each pair, because each line in one pair meets just one of the two lines in any other pair. We have now two ways to proceed: we can choose one line in the ambient space which meets L_1 and L_2 and impose the two linear conditions on the c_i which make this line lie in V, or we can choose two such lines and solve the resulting four linear equations in the c_i.

In either case, the first step is to write down the conditions that the line

$$X_0 = uX_2, \ X_1 = vX_3$$

lies in V. These are

$$\begin{aligned} c_4(u-1)(uv+1) + (uc_0 + c_2)(uv-1) = 0, \\ c_4(v-1)(uv+1) + (vc_1 + c_3)(uv-1) = 0. \end{aligned}$$

The simplicity of these equations leads me to prefer the second alternative above. We therefore ask what is the cubic surface of our type which also contains the two lines

$$N: \ X_0 = u_1 X_2, \ X_1 = v_1 X_3,$$
$$N': \ X_0 = u_2 X_2, \ X_1 = v_2 X_3;$$

and we find that the values of the c_i are

$$\begin{aligned}
c_0 &= u_1 u_2 v_1 v_2 - 1 - (u_1 v_1 - u_2 v_2)(u_1 + u_2 - 2)/(u_1 - u_2),\\
c_1 &= u_1 u_2 v_1 v_2 - 1 - (u_1 v_1 - u_2 v_2)(v_1 + v_2 - 2)/(v_1 - v_2),\\
c_2 &= -u_1 u_2 v_1 v_2 + 1 + (u_1 v_1 - u_2 v_2)(2u_1 u_2 - u_1 - u_2)/(u_1 - u_2),\\
c_3 &= -u_1 u_2 v_1 v_2 + 1 + (u_1 v_1 - u_2 v_2)(2v_1 v_2 - v_1 - v_2)/(v_1 - v_2),\\
c_4 &= -(u_1 v_1 - 1)(u_2 v_2 - 1).
\end{aligned}$$

This is almost certainly the simplest formulation possible.

If you need explicit equations for each of the 27 lines, you should proceed as follows. You already have the equations of the L_i and M_i, together with an easy recipe for the equations of the N_{ij}. You also have the equations of N and N'. Four more lines are the residual intersections of V with the four planes containing either L_1 or L_2 and either N or N'. But N and N' also meet each N_{3i}, and the final six lines are the residual intersections of V with the six planes containing N or N' together with one of the N_{3i}.

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