FUJITA'S PROGRAM AND RATIONAL POINTS

by

Yuri Tschinkel

1. Introduction

A classical theme in mathematics is the study of integral solutions of diophantine equations, that is, equations with integral coefficients. The main problems are

– decide the existence (or nonexistence) of solutions;
– find (some or all) solutions;
– describe (qualitatively or quantitatively) the set of solutions.

Even the first of these questions is difficult, in full generality. As we know from logic, it is, in a sense, equivalent to *all* (formal) mathematics: for a given formal language, there exists *one* (nonhomogeneous) polynomial $f(t, x_1, ..., x_n)$ (with $\mathbb{Z}$-coefficients) such that the Statement $\#t$ is provable in this language iff the equation $f(t, x_1, ..., x_n) = 0$ has a solution with $(x_1, ..., x_n) \in \mathbb{Z}^n$. Thus, at least theoretically, one can convert a problem in any field of mathematics, e.g. topology, to a problem in number theory. This is convincing evidence that general diophantine equations are extraordinarily complex.

Nevertheless, there exist classes of equations, for which the above questions may have a satisfactory answer. Below I will show how some of these classes arise in practice. But first notice that the questions are "invariant" under coordinate changes. This central observation signals the entry of algebraic geometry into the subject. Algebraic geometry does not only simplify and streamline manipulations of bulky equations (providing "models"). More importantly, it opens the field to geometric intuition and geometric constructions.

From now on, let $X/\mathbb{Q}$ be a projective algebraic variety. Thus X is defined by a system of homogeneous equations with $\mathbb{Q}$-coefficients. Rational points on X are (equivalence classes of) rational solutions to these equations. It is a natural idea that in the *stable* range, that is, after passing to a finite extension of $\mathbb{Q}$ and restricting to some Zariski open subset X^0 of X there should be a relation between the set of rational points of X^0 and geometric invariants of X. This idea not only explains and unifies "statistical" data (numerical experiments, theoretical results) but also arms us with predictive power. It is a source of inspiration to both fields: arithmetic and geometry.

The focus of this survey is the relationship between the asymptotic distribution of rational points of bounded height on varieties with many rational points and their global geometric invariants: the cone of (classes of) effective divisors and the position of the anticanonical class with respect to this cone. We will be mostly concerned with *rational* varieties so that the points are already Zariski dense. In fact, most of the varieties are compactifications of algebraic groups or homogeneous varieties and the proofs of asymptotics of rational points rely on harmonic analysis on the corresponding adelic spaces. My main goal in these notes is to show where the geometric invariants enter the analysis and how the analysis forces us to introduce new geometric tools.

Here is a brief outline of the paper. In Section 2 we introduce the main geometric invariants of interest to us. In Section 3 we explain the problem and state known results. In Section 4 we define Tamagawa numbers which appear in asymptotics of rational points of bounded height. Finally, in Section 5 we illustrate the interaction between geometry and arithmetic in several examples.

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2. Geometry

In this section we introduce some geometric background material and define certain geometric invariants which will be used in arithmetic investigations later on. Here we work over an algebraically closed field of characteristic zero.

2.1. Main invariants. — Let X be a smooth projective algebraic variety. The main global geometric invariants of interest to us are

- the Picard group $\mathrm{Pic}(X)$ and the Néron-Severi group $\mathrm{NS}(X)$;
- the (closed) cone of effective divisors

$$\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}} = \mathrm{Pic}(X) \otimes \mathbb{R}$$

  and (the closure of) the ample cone $\Lambda_{\mathrm{ample}}(X) \subset \Lambda_{\mathrm{eff}}(X)$;
- the canonical class K_X.

One can think of these as being combinatorial data encoding the geometry of X (fibration structures etc.). In all of our applications, $\mathrm{Pic}(X) = \mathrm{NS}(X)$, $\Lambda_{\mathrm{eff}}(X)$ is finitely generated and $-K_X$ is contained in the interior of $\Lambda_{\mathrm{eff}}(X)$.

Example 2.1.1. — Let $X = X_d \subset \mathbb{P}^n$ be the hypersurface defined by a (smooth) homogeneous form f of degree d in $n+1$ variables (with $n \geq 4$). Then

- $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ (where H is the hyperplane class);
- $\Lambda_{\mathrm{eff}}(X) = \Lambda_{\mathrm{ample}}(X)$ is generated by H;
- $-K_X = (n+1-d)H$.

Example 2.1.2. — Let $X = \bar{M}_{0,5}$ be the moduli space of stable curves of genus 0 with 5 punctures, classically known as the Del Pezzo surface of degree 5. It can be realized as a blow-up of $\mathbb{P}^2$ in four points. We write H for the (proper transform) of the hyperplane class on $\mathbb{P}^2$ and $E_1, \ldots, E_4$ for the classes of the exceptional curves. Then

- $\mathrm{Pic}(X) = \mathbb{Z}^5 = \langle H, E_1, \ldots, E_4 \rangle$;
- $\Lambda_{\mathrm{eff}}(X)$ is generated by the classes $E_i, H - E_i - E_j$, and $\Lambda_{\mathrm{ample}}(X)$ is dual to it (with respect to the intersection pairing);
- $-K_X = 3H - (E_1 + \ldots + E_4)$.

For more information, see [33] or [24].

Example 2.1.3. — Let $X = \bar{M}_{0,6}$ be the moduli space of stable curves of genus 0 with 6 punctures. It can be realized as a blow-up of $\mathbb{P}^3$ in 5 points and 10 lines joining pairs of these points. We write H for the hyperplane class, $E_1, ..., E_5$ for the proper transforms of the 5 points and E_{ij} for the proper transforms of lines through the points i and j. By [24], we have:

- $\mathrm{Pic}(X) = \mathbb{Z}^{16} = \langle H, E_i, E_{ij}\rangle$ with $i \neq j \in [1, ..., 5]$, $E_{i,j} = E_{j,i}$;
- $\Lambda_{\mathrm{eff}}(X)$ is generated by
$$E_i, E_{ij}, H - (E_i + E_j + E_k + E_{ij} + E_{ik} + E_{jk}),$$
$$2H - (E_1 + ... + E_5) - (E_{ik} + E_{i\ell} + E_{jk} + E_{j\ell})$$
(altogether 40 generators);
- $-K_X = 4H - 2(E_1 + ... + E_5) - (\sum_{i<j} E_{ij})$.

The $\mathbb{S}_6$-invariant ample cone has two generators, denoted by L_3 and L_4. The associated line bundles give maps
$$f_3 \,:\, \bar{M}_{0,6} \to S_3 \subset \mathbb{P}^4,$$
$$f_4 \,:\, \bar{M}_{0,6} \to I_4 \subset \mathbb{P}^4,$$
with images the *Segre* cubic:
$$S_3 \;:\; \sum_{i=0}^{5} x_i^3 = \sum_{i=0}^{5} x_i = 0,$$
respectively, the *Igusa* quartic:
$$I_4 \;:\; (x_0x_1 + x_0x_2 + x_1x_2 - x_3x_4)^2 - 4x_0x_1x_2(x_0 + x_1 + x_2 + x_3 + x_4) = 0,$$
(see [26] for more information about these varieties).

Example 2.1.4. — Let X be an equivariant compactification of a unipotent algebraic group. Then

- $\mathrm{Pic}(X) = \oplus_{j\in\mathcal{J}} D_j$, where D_j are classes of the irreducible components of the boundary;
- $\Lambda_{\mathrm{eff}}(X)$ is generated by the D_j;
- $-K_X = \sum_{j\in\mathcal{J}} \kappa_j D_j$, with $\kappa_j \geq 2$ for all $j \in \mathcal{J}$ (see [23]).

Example 2.1.5. — Let $X = \mathrm{B}\backslash\mathrm{G}$ be a generalized flag variety: G is a semi-simple algebraic group and $\mathrm{B} \subset \mathrm{G}$ a Borel subgroup. Then

- $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathfrak{X}^*(\mathrm{T})_{\mathbb{Q}}$ (the characters of the maximal torus $\mathrm{T} \subset \mathrm{B}$);

– $\Lambda_{\rm eff}(X) = \Lambda_{\rm ample}(X)$ is generated by the fundamental weights ω_j, $j = 1, ..., \mathrm{rk}\, \mathrm{G}$;
– $-K_X = 2\rho$ (sum of roots occurring in the unipotent radical of B).

Example 2.1.6. — Let X be the *wonderful* compactification of a semi-simple group G of adjoint type (see [15]). Then

– $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathfrak{X}^*(\mathrm{T})_{\mathbb{Q}}$ (where T is the maximal torus in G);
– $\Lambda_{\rm eff}(X)$ is generated by the positive simple roots, α_j, $j = 1, ..., \mathrm{rk}\, \mathrm{G}$;
– $\Lambda_{\rm ample}(X)$ is generated by the fundamental weights ω_j, $j = 1, ..., \mathrm{rkG}$;
– $-K_X = 2\rho + \sum_j \alpha_j$.

2.2. Rough classification. — Let me recall the very rough classification of smooth algebraic varieties (over an algebraically closed field) according to the position of the anticanonical class with respect to the ample cone. There are three main cases:

– **Fano:** $-K_X$ ample;
– **general type:** K_X ample;
– **intermediate type:** none of the above.

In finer classification schemes and in the minimal model program it is important to include varieties with certain "mild" singularities (see [29]). Similarly, in many arithmetic questions, the passage to singular varieties is inevitable (see Section 4). Some examples of varieties in each group are contained in the following table:

dim	**Fano**	**Intermediate type**	**General type**
1	$\mathbb{P}^1$	elliptic curves	$C,\ g(C) \geq 2$
2	$\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, $\bar{M}_{0,5}, X_3 \subset \mathbb{P}^3, ...$	$K3$ surfaces : $X_4 \subset \mathbb{P}^3, ...$ abelian surfaces, ...	...
3	smooth : ~ 120 families singular : $\bar{M}_{0,6}, ...$	Calabi – Yau varieties ...	...

Smooth Fano surfaces, also known as Del Pezzo surfaces, have been classified by the Italian school. The classification of smooth Fano threefolds, initiated by Fano, advanced by Iskovskikh [27] and completed by Mori-Mukai [35], is a major achievement of modern algebraic geometry. After Kollár and Mori [31], one knows that there are only finitely many families of smooth Fano varieties in each dimension. It seems plausible, that certain classes of varieties of intermediate type will also admit a good description (for example, holomorphic symplectic varieties). For a recent approach to the classification of algebraic varieties see [8].

In our arithmetic applications we will mostly encounter varieties on the left side of the table.

2.3. Cones. — Let (A, Λ) be a pair consisting of a lattice and a strictly convex (closed) cone in $A_{\mathbb{R}}$: $\Lambda \cap -\Lambda = 0$. Let $(\check{A}, \check{\Lambda})$ the pair consisting of the dual lattice and the dual cone defined by

$$\check{\Lambda} := \{\check{\lambda} \in \check{A}_{\mathbb{R}} \mid \langle \lambda', \check{\lambda} \rangle \geq 0, \ \ \forall \lambda' \in \Lambda\}.$$

The lattice $\check{A}$ determines the normalization of the Lebesgue measure $d\check{a}$ on $\check{A}_{\mathbb{R}}$ (covolume =1). For $a \in A_{\mathbb{C}}$ define

$$\mathcal{X}_{\Lambda}(a) := \int_{\check{\Lambda}} e^{-\langle a, \check{a} \rangle} d\check{a}. \tag{2.1}$$

The integral converges absolutely and uniformly for $\Re(a)$ in compacts contained in the interior Λ° of Λ.

Definition 2.3.1. — *Assume that X is smooth,* $\mathrm{NS}(X) = \mathrm{Pic}(X)$ *and that $-K_X$ is in the interior of $\Lambda_{\mathrm{eff}}(X)$. We define*

$$\gamma(X) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(X)}(-K_X).$$

Remark 2.3.2. — This constant measures the volume of the polytope obtained by intersecting the affine hyperplane $\langle -K_X, \cdot \rangle = 1$ with the *dual* to the cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$ in the dual to the Néron-Severi group. The explicit determination of $\gamma(X)$ can be a serious problem in linear programming. For example, let X be the moduli space $\bar{M}_{0,6}$ (see Example 2.1.3). The dual to the cone $\Lambda_{\mathrm{eff}}(X)$ has 3905 generators (in a 16-dimensional vector space), forming 25 orbits under the action of the symmetric group $\mathbb{S}_6$.

Let $(A, \Lambda, -K)$ be a triple consisting of a (torsion free) lattice $A = \mathbb{Z}^n$, a (closed) strictly convex polyhedral cone in $A_{\mathbb{R}}$ generated by finitely many vectors in A and a vector $-K \subset \Lambda^\circ$ (the interior of Λ). For $L \in \Lambda^\circ$ we define

$$a(\Lambda, L) = \inf\{a \,|\, aL + K \in \Lambda\}$$

and $b(\Lambda, L)$ as the codimension of the minimal face of Λ containing the class $a(\Lambda, L)L + K$. Obviously, for $L = -K$ we get that $a(\Lambda, -K) = 1$ and that $b(\Lambda, -K)$ equals the rank of A.

2.4. Cones in geometry. — In this section we assume that X is a smooth projective Fano variety ($-K_X \in \Lambda_{\mathrm{ample}}(X)$). We have a sequence of inclusions:

$$\Lambda_{\mathrm{ample}}(X) \subset \Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}.$$

Projectivity implies that $\Lambda_{\mathrm{ample}}(X)$ contains an open subset. The finer classification theories of Mori (resp. Fujita) are based on the study of $\Lambda_{\mathrm{ample}}(X)$ (resp. $\Lambda_{\mathrm{eff}}(X)$). We now give a simplified picture of these theories by stating some basic theorems and conjectures. We refer to [29] and [20] for more details. The conjectures are proved for all Fano varieties of dimension ≤ 3 and certain classes of varieties, like toric varieties, in arbitrary dimension [31], [1].

Theorem 2.4.1. — *The cone $\Lambda_{\mathrm{ample}}(X)$ is finitely generated.*

Conjecture 2.4.2. — *The cone $\Lambda_{\mathrm{eff}}(X)$ is finitely generated.*

Remark 2.4.3. — The ample cone $\Lambda_{\mathrm{ample}}(X)$ can be finitely generated even when $-K_X$ is not ample (or even effective). For example, $\Lambda_{\mathrm{ample}}(X)$ is finitely generated for *every* toric variety or every variety with $\mathrm{rk}\,\mathrm{NS}(X) = 1$.

Let $\Lambda \subset \mathrm{Pic}(X)_{\mathbb{R}}$ be a finitely generated rational cone. For $L \in \Lambda^\circ$, with $a(\Lambda, L) \in \mathbb{Q}$, we define

$$\mathrm{R}(\Lambda, L) \quad := \oplus_{\nu \geq 0} H^0(X, (L^{\otimes a(\Lambda,L)} \otimes K_X)^{\otimes kv})$$

(where k is the denominator of $a(\Lambda, L)$).

Conjecture 2.4.4. — *Let X be a smooth Fano variety and L (the class of) a very ample line bundle on X. Then the rings* $\mathrm{R}(\Lambda_{\mathrm{ample}}(X), L)$ *and* $\mathrm{R}(\Lambda_{\mathrm{eff}}(X), L)$ *are finitely generated.*

Moreover, there should be diagrams:

$$\begin{array}{cclccl} & \textbf{Mori} & & & \textbf{Fujita} & \\ X & & L|_{X_y} \sim -K_{X_y} & X & & L|_{X_y} \sim -K_{X_y} + E \\ \vdots & & & \vdots & & \\ \downarrow & & & \downarrow & & \\ Y & = & \mathrm{Proj}(\mathrm{R}(\Lambda_{\mathrm{ample}}(X), L)) & Y & = & \mathrm{Proj}(\mathrm{R}(\Lambda_{\mathrm{eff}}(X), L)) \end{array}$$

where L is ample and E a rigid effective divisor (and $\sim$ means proportional). Recall that a divisor E is called *rigid* if $h^0(\mathcal{O}(\nu E)) = 1$ for $\nu \gg 0$. Intuitively, the following is happening: at $a = a(\Lambda_{\mathrm{ample}}(X), L)$ (resp. $a = (\Lambda_{\mathrm{ample}}(X), L)$) the divisor $aL + K_X$ just fails to be ample (resp. *big*). But a divisor in this class could still move and define a fibration, rather than an embedding.

Remark 2.4.5. — Fujita's program follows from Mori's Minimal Model Program [1]. We formulated it independently since in arithmetic applications we need the *effective* cone. Moreover, in our examples the anticanonical class is in the interior of the effective cone but not always ample (for example, toric varieties) and the cone $\Lambda_{\mathrm{ample}}(X)$ is not always finitely generated (for example, equivariant compactifications of unipotent groups).

3. Arithmetic

3.1. Asymptotics. — Let F be a finite extension of $\mathbb{Q}$. The conjectures of Bombieri, Lang and Vojta predict that on a variety X (over F) of general type the set $X(F)$ of F-rational points is not Zariski dense (see [49]). Faltings proved this for subvarieties of abelian varieties [18]. On the other hand, one expects that Fano varieties should have a Zariski dense set of rational points, at least after a finite extension of F. See [24], [7] for some partial results and [22] for a recent survey. Most of the time, we will be interested in *rational* varieties, so that rational points are *a priori* Zariski dense. We will study the asymptotic distribution of rational points of bounded height.

For $X = \mathbb{P}^n$ one can define a *height*

$$\begin{array}{rccc} H: & \mathbb{P}^n(F) & \rightarrow & \mathbb{R}_{>0}, \\ & (x_0 : \ldots : x_n) & \mapsto & \prod_v \max_j(|x_j|_v), \end{array}$$

where the product is over the set $\mathrm{Val}(F)$ of all valuations of F. The choice of the norm $\max_j(|x_j|_v)$ in the vector space F^{n+1} is to some extent arbitrary. Namely, replacing the norms by comparable (homogeneous) norms at *finitely* many places one obtains a new height. For example, at the infinite places one could replace

$$\max(|x_j|_v) \text{ by } (\sum_j |x_j|_v^2)^{1/2}.$$

One refers to such choices of norms as fixing the *metrization* (see Section 4.1 for more details).

Let X be projective, L a very ample line bundle on X and

$$f_L : X \rightarrow \mathbb{P}^n = \mathbb{P}(H^0(X, L))$$

the corresponding embedding into a projective space. We have an induced (exponential) height:

$$\begin{array}{rccc} H_{\mathcal{L}}: & X(F) & \rightarrow & \mathbb{R}_{>0}, \\ & x & \mapsto & H(f_L(x)). \end{array}$$

We wrote $\mathcal{L}$ to stress the dependence of the height on the metrization. The set of points of $\mathcal{L}$-height bounded by $B > 0$ is finite. We define the *counting function*

$$N(U, \mathcal{L}, B) := \#\{x \in U(F) \,|\, H_{\mathcal{L}}(x) \leq B\},$$

where $U \subset X$ is a Zariski open subset.

The following theorem summarizes most of the known results concerning asymptotics of rational points of bounded height on algebraic varieties.

Theorem 3.1.1. — *Let X/F be one of the following varieties:*

- *smooth complete intersection of small degree (for example,* [**6**]*);*
- *flag variety* [**19**]*;*
- *smooth toric variety* [**5**]*;*
- *smooth equivariant compactification of* G/U *- horospherical variety, where* G *is a semi-simple group and* $\mathrm{U} \subset \mathrm{G}$ *a maximal unipotent subgroup* [**47**]*;*

– *smooth equivariant compactification of* $\mathbb{G}_a^n$ [12]*;*
– *smooth bi-equivariant compactification of a unipotent group* [44], [45]*;*
– *a wonderful compactification of an anisotropic form of a simple algebraic group of adjoint type* [42], [43].

Let $H_{\mathcal{L}}$ *be an appropriate height such that the corresponding class* $L \in \mathrm{Pic}(X)$ *is contained in the interior of the cone of effective divisors and*

$$a(L) = a(\Lambda_{\mathrm{eff}}(X), L), \;\; b(L) = b(\Lambda_{\mathrm{eff}}(X), L).$$

There exist a dense Zariski open subset $U \subset X$ *and a constant* $c(U, \mathcal{L}) > 0$ *such that*

$$N(U, \mathcal{L}, B) = \frac{c(U, \mathcal{L})}{a(L)(b(L)-1)!} B^{a(L)}(\log(B))^{b(L)-1}(1 + o(1)),$$

as $B \to \infty$*.*

Remark 3.1.2. — The constant $c(U, \mathcal{L})$ depends, of course, not only on the geometric data (U, L) but also on the metrization. The definitions are postponed until Section 4.

Remark 3.1.3. — The theorem holds for *singular* varieties as well. One has to apply the methods to some (equivariant) desingularization and use the functoriality of heights: if $\pi : X' \to X$ is a birational map, inducing an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} U \subset X$ then

$$H_{\pi^*\mathcal{L}}(\pi^{-1}(x)) = H_{\mathcal{L}}(x).$$

Consequently,

$$N(U, \mathcal{L}, B) = N(\pi^{-1}(U), \pi^*\mathcal{L}, B).$$

Remark 3.1.4. — The only other classes of varieties (that I am aware of), for which similar asymptotic formulas have been proved, is the class of certain twisted products of flag varieties considered in [46] and $\bar{M}_{0,5}$ in its anticanonical embedding [16].

Remark 3.1.5. — Notice that with the exception of complete intersections the varieties from Theorem 3.1.1 have a rather simple "cellular" structure. In particular, we can parametrize all rational points in some dense Zariski open subset. The theorem is to be understood as a statement about *heights*: even the torus $\mathbb{G}_m^2$ has very nontrivial embeddings into projective spaces and in each of these embeddings we have a different counting problem.

Remark 3.1.6. — The restriction to Zariski open subvarieties is necessary, as X may contain proper subvarieties Z such that

$$\lim_{B\to\infty} \frac{N(X \setminus Z, \mathcal{L}, B)}{N(Z, \mathcal{L}, B)} = 0.$$

It is futile to try to relate the asymptotic of rational points on Z to geometric invariants of X. Such varieties are called $\mathcal{L}$-*accumulating*. The simplest example is the blow-up of $\mathbb{P}^2$ in one point: the exceptional curve is accumulating with respect to $-K_X$.

Stratifications by accumulating subvarieties have been considered in [**34**].

4. Tamagawa numbers

In this section we show how to associate to a metrized ample line bundle $\mathcal{L}$ a constant $c(\mathcal{L})$ which should appear in the leading term of asymptotic expansions for the number of rational points of bounded $\mathcal{L}$-height. If L is proportional to $-K_X$ then the main ingredient of $c(\mathcal{L})$ is an adelic integral. Otherwise, we use Fujita's program to reduce to this situation.

The fact that a metrized canonical line bundle determines a measure is well known in differential geometry. The same reasoning applies over the adeles. Integrals of such (suitably regularized) measures give Tamagawa numbers which appear as factors in leading coefficients in Laurent expansions of zeta functions.

4.1. Metrizations of line bundles. —

Notations 4.1.1. — Let F be a number field and $\mathrm{disc}(F)$ the discriminant of F (over $\mathbb{Q}$). The set of places of F will be denoted by $\mathrm{Val}(F)$. We shall write $v|\infty$ if v is archimedean and $v \nmid \infty$ if v is nonarchimedean. For any place v of F we denote by F_v the completion of F at v and by $\mathfrak{o}_v$ the ring of v-adic integers (for $v \nmid \infty$). Let q_v be the cardinality of the residue field $\mathbb{F}_v$ of F_v for nonarchimedean valuations. The local absolute value $|\cdot|_v$ on F_v is the multiplier of the Haar measure, i.e., $d(ax_v) = |a|_v dx_v$ for some Haar measure dx_v on F_v. We denote by $\mathbb{A} = \mathbb{A}_F = \prod'_v F_v$ the adele ring of F.

Definition 4.1.2. — *Let X be an algebraic variety over F and L a line bundle on X. A v-adic metric on L is a family $(\|\cdot\|_x)_{x\in X(F_v)}$ of v-adic Banach norms on L_x such that for every Zariski open $U \subset X$ and every section $g \in H^0(U, L)$ the map*

$$U(F_v) \to \mathbb{R}, \ \ x \mapsto \|g\|_x,$$

is continuous in the v-adic topology on $U(F_v)$.

Example 4.1.3. — Assume that L is generated by global sections. Choose a basis $(g_j)_{j\in[0,\dots,n]}$ of $H^0(X, L)$ (over F). If g is a section such that $g(x) \neq 0$ then define

$$\|g\|_x := \max_{0\le j\le n} (|\frac{g_j}{g}(x)|_v)^{-1},$$

otherwise $\|0\|_x := 0$. This defines a v-adic metric on L. Of course, this metric depends on the choice of $(g_j)_{j\in[0,\dots,n]}$.

Definition 4.1.4. — *Assume that L is generated by global sections. An adelic metric on L is a collection of v-adic metrics, for every $v \in \mathrm{Val}(F)$, such that for all but finitely many $v \in \mathrm{Val}(F)$ the v-adic metric on L is defined by means of some* fixed *basis $(g_j)_{j\in[0,\dots,n]}$ of $H^0(X, L)$.*

We shall write $\|\cdot\| := (\|\cdot\|_v)$ for an adelic metric on L and call a pair $\mathcal{L} = (L, \|\cdot\|)$ an adelically metrized line bundle. Metrizations extend naturally to tensor products and duals of metrized line bundles, which shows a way to define adelic metrizations on arbitrary line bundles L (on projective X): represent L as $L = L_1 \otimes L_2^{-1}$ with very ample L_1 and L_2. Assume that L_1, L_2 are adelically metrized. An adelic metrization of L is any metrization which for all but finitely many v is induced from the metrizations on L_1, L_2.

Definition 4.1.5. — *Let $\mathcal{L} = (L, \|\cdot\|)$ be an adelically metrized line bundle on X and g an F-rational section of L. Let $U \subset X$ be the maximal Zariski open subset of X where g is defined and is $\neq 0$. For all $x = (x_v)_v \in U(\mathbb{A})$ we define the local*

$$H_{\mathcal{L},g,v}(x_v) := \|g\|_{x_v}^{-1}$$

and the global height function

$$H_{\mathcal{L}}(x) := \prod_{v \in \mathrm{Val}(F)} H_{\mathcal{L},g,v}(x_v).$$

By the product formula, the restriction of the global height to $U(F)$ does not depend on the choice of g.

EXAMPLE 4.1.6. — For $X = \mathbb{P}^1 = (x_0 : x_1)$ one has $\mathrm{Pic}(X) = \mathbb{Z}$, spanned by the class $L = [(1:0)]$. For all $g = x_0/x_1 \in \mathbb{G}_a(\mathbb{A})$ we define

$$H_{\mathcal{L},g,v}(g_v) = \max(1, |g|_v).$$

The restriction of $H_{\mathcal{L}} = \prod_v H_{\mathcal{L},g,v}$ to $\mathbb{G}_a(F) \subset \mathbb{P}^1$ is the usual height on $\mathbb{P}^1$ (with respect to the usual metrization of $\mathcal{L} = \mathcal{O}(1)$).

4.2. Geometry. — I follow closely the exposition in [5]. Let E/F be some finite Galois extension such that all of the following constructions are defined over E. Let $(V, \mathcal{L})$ be a smooth *quasi-projective* d-dimensional variety together with a metrized very ample line bundle $\mathcal{L}$ which embeds V in some projective space $\mathbb{P}^n$. We denote by $\overline{V}^{\mathcal{L}}$ the normalization of the projective closure of $V \subset \mathbb{P}^n$. In general, $\overline{V}^{\mathcal{L}}$ is singular. We will introduce several notions relying on a resolution of singularities

$$\rho : X \to \overline{V}^{\mathcal{L}}.$$

Naturally, the defined objects will be independent of the choice of the resolution. For $\Lambda \subset \mathrm{NS}(X)_{\mathbb{R}}$ we define

$$a(\Lambda, \mathcal{L}) := a(\Lambda, \rho^*\mathcal{L}).$$

We will always assume that $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) > 0$.

DEFINITION 4.2.1. — *A pair $(V, \mathcal{L})$ is called* primitive *if $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) \in \mathbb{Q}_{>0}$ and if there exists a resolution of singularities*

$$\rho : X \to \overline{V}^{\mathcal{L}}$$

such that for some $k \in \mathbb{N}$

$$((\rho^*\mathcal{L})^{\otimes a(\Lambda_{\mathrm{eff}}(X),\mathcal{L})} \otimes K_X)^{\otimes k} = \mathcal{O}(D),$$

where D is a rigid *effective divisor ($h^0(X, \mathcal{O}(\nu D)) = 1$ for all $\nu \gg 0$).*

EXAMPLE 4.2.2. — of a primitive pair: $(V, -\mathcal{K}_V)$, where V is a smooth projective Fano variety and $-\mathcal{K}_V$ is a metrized anticanonical line bundle.

Let $k \in \mathbb{N}$ be such that $a(\Lambda, \mathcal{L})k \in \mathbb{N}$ and consider

$$\mathrm{R}(\Lambda, \mathcal{L}) := \oplus_{\nu \geq 0} H^0(X, (((\rho^* \mathcal{L})^{a(\Lambda,\mathcal{L})} \otimes K_X)^{\otimes k})^{\otimes \nu}).$$

As explained in Section 2.4, in both cases ($\Lambda = \Lambda_{\mathrm{ample}}$ or $\Lambda = \Lambda_{\mathrm{eff}}$) it is expected that $\mathrm{R}(\Lambda, \mathcal{L})$ is finitely generated and that we have a fibration

$$\pi = \pi_{\mathcal{L}} : X \to Y^{\mathcal{L}},$$

where $Y^{\mathcal{L}} = \mathrm{Proj}(\mathrm{R}(\mathcal{L}, \Lambda))$. For $\Lambda = \Lambda_{\mathrm{eff}}(X)$ the generic fiber of π is (expected to be) a primitive variety in the sense of Definition 4.2.1. More precisely, there should be a diagram:

$$\begin{array}{cccc} \rho: & X & \to & \overline{V}^{\mathcal{L}} \supset V \\ & \downarrow & & \\ & Y^{\mathcal{L}} & & \end{array}$$

such that:
- $\dim(Y^{\mathcal{L}}) < \dim(X)$;
- there exists a Zariski open $U \subset Y^{\mathcal{L}}$ such that for all $y \in U(\mathbb{C})$ the pair $(V_y, \mathcal{L}_y)$ is primitive (here $V_y = \pi^{-1}(y) \cap V$ and $\mathcal{L}_y$ is the restriction of $\mathcal{L}$ to V_y);
- for all $y \in U(\mathbb{C})$ we have $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L}) = a(\Lambda_{\mathrm{eff}}(V_y), \mathcal{L}_y)$;
- For all $k \in \mathbb{N}$ such that $a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})k \in \mathbb{N}$ the vector bundle

$$\mathcal{L}_k := R^0 \pi_*(((\rho^* \mathcal{L})^{\otimes a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})} \otimes K_X)^{\otimes k})$$

is in fact an ample invertible sheaf on $Y^{\mathcal{L}}$.

Such a fibration will be called an *$\mathcal{L}$-primitive fibration.* A variety may admit several primitive fibrations.

Example 4.2.3. — Let $X \subset \mathbb{P}_1^n \times \mathbb{P}_2^n$ ($n \geq 2$) be a hypersurface given by a bi-homogeneous form of bi-degree (d_1, d_2). Both projections $X \to \mathbb{P}_1^n$ and $X \to \mathbb{P}_2^n$ are $\mathcal{L}$-primitive, for appropriate $\mathcal{L}$. In particular, for $n = 3$ and $(d_1, d_2) = (1, 3)$ there are *two* distinct $-\mathcal{K}_X$-primitive fibrations: one onto a point and another onto $\mathbb{P}_1^3$.

4.3. Tamagawa numbers. — For smooth projective Fano varieties X with an adelically metrized anticanonical line bundle Peyre defined in [39] a Tamagawa number, generalizing the classical construction for linear algebraic groups. We need to further generalize this to primitive pairs.

Abbreviate $a(\mathcal{L}) = a(\Lambda_{\mathrm{eff}}(X), \mathcal{L})$ and let $(V, \mathcal{L})$ be a primitive pair such that

$$\mathcal{O}(D) := ((\rho^*\mathcal{L})^{\otimes a(\mathcal{L})} \otimes K_X)^{\otimes k},$$

where k is such that $a(\mathcal{L})k \in \mathbb{N}$ and D is a rigid effective divisor as in Definition 4.2.1. Choose an F-rational section $g \in H^0(X, \mathcal{O}(D))$; it is unique up to multiplication by F^*. Choose local analytic coordinates $x_{1,v}, ..., x_{d,v}$ in a neighborhood U_x of $x \in X(F_v)$. In U_x the section g has a representation

$$g = f^{ka(\mathcal{L})}(dx_{1,v} \wedge ... \wedge dx_{d,v})^k,$$

where f is a local section of L. This defines a local v-adic measure in U_x by

$$\omega_{\mathcal{L},g,v} := \|f\|_{x_v}^{a(\mathcal{L})} dx_{1,v} \cdots dx_{d,v},$$

where $dx_{1,v} \cdots dx_{d,v}$ is the Haar measure on F_v^d normalized by $\mathrm{vol}(\mathfrak{o}_v^d) = 1$. A standard argument shows that $\omega_{\mathcal{L},g,v}$ glues to a v-adic measure on $X(F_v)$. The restriction of this measure to $V(F_v)$ does not depend on the choice of the resolution $\rho : X \to \overline{V}^{\mathcal{L}}$. Thus we have a measure on $V(F_v)$.

Denote by $(D_j)_{j\in\mathcal{J}}$ the irreducible components of the support of D and by

$$\mathrm{Pic}(V, \mathcal{L}) := \mathrm{Pic}(X \setminus \cup_{j\in\mathcal{J}} D_j).$$

The Galois group Γ acts on $\mathrm{Pic}(V, \mathcal{L})$. Let S be a finite set of valuations of bad reduction for the data (ρ, D_j, etc.), including the archimedean valuations. Put $\lambda_v = 1$ for $v \in S$, $\lambda_v = L_v(1, \mathrm{Pic}(V, \mathcal{L}))$ (for $v \notin S$) and

$$\omega_{\mathcal{L}} := L_S^*(1, \mathrm{Pic}(V, \mathcal{L}))|\mathrm{disc}(F)|^{-d/2} \prod_v \lambda_v^{-1} \omega_{\mathcal{L},g,v}.$$

(Here L_v is the local factor of the Artin L-function associated to the Γ-module $\mathrm{Pic}(V, \mathcal{L})$ and $L_S^*(1, \mathrm{Pic}(V, \mathcal{L}))$ is the residue at $s = 1$ of the partial Artin L-function.) By the product formula, the measure does not depend on the choice of the F-rational section g. Define

$$\tau_{\mathcal{L}}(V) := \int_{\overline{X(F)}} \omega_{\mathcal{L}},$$

where $\overline{X(F)} \subset X(\mathbb{A})$ is the closure of $X(F)$ in the direct product topology. The convergence of the Euler product follows from

$$h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0.$$

We have a map

$$\tilde{\rho} : \mathrm{Pic}(X)_{\mathbb{R}} \to \mathrm{Pic}(V, \mathcal{L})_{\mathbb{R}}$$

and we denote by

$$\Lambda_{\mathrm{eff}}(V, \mathcal{L}) := \tilde{\rho}(\Lambda_{\mathrm{eff}}(X)) \subset \mathrm{Pic}(V, \mathcal{L})_{\mathbb{R}}.$$

Definition 4.3.1. — *Let $(V, \mathcal{L})$ be a primitive pair as above. Define*

$$c(V, \mathcal{L}) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(V,\mathcal{L})}(\tilde{\rho}([-K_X])) \cdot |H^1(\Gamma, \mathrm{Pic}(V, \mathcal{L}))| \cdot \tau_{\mathcal{L}}(V).$$

Example 4.3.2. — Let us return to the Example 2.1.3. For the image of $\bar{M}_{0,6}$ under f_3 (the Segre cubic) one knows an upper and lower bound:

$$c' B^2 \log(B)^5 \le N(U, L_3, B) \le c'' B^2 \log(B)^5,$$

for an appropriate Zariski open U and some constants $c', c'' \ge 0$ [**50**]. An *asymptotic* formula for $N(U, L_3, B)$ of this shape would be compatible with the description in Theorem 3.1.1. Moreover, we are now in the position to specify a constant c which should appear in this asymptotic (answering a question in [**50**]). Indeed, the Segre cubic threefold is singular (it has 10 isolated double points). The blow-up $X = \tilde{S}_3$ of 5 points on $\mathbb{P}^3$ is a desingularization of S_3. Its Picard group $\mathrm{Pic}(X)$ is freely generated by the classes H, E_j (for $j = 1, ..., 5$). The effective cone is generated by the classes $E_j, H - (E_i + E_j + E_k)$ and the anticanonical class is given by

$$-K_X = 4H - 2(E_1 + ... + E_5).$$

The line bundle on S_3 giving the Segre embedding pulls back to $L = -1/2 \cdot K_X$. Clearly, $a(L) = 2$ and $b(L) = \mathrm{rk}\,\mathrm{Pic}(X) = 6$ (see also [**5**], Section 5.2). The predicted leading constant $c = \gamma \cdot \tau$, where

$$\gamma = \frac{2^5}{5!} \mathcal{X}_{\Lambda}(-K_X)$$

and

$$\tau = \tau_\infty \cdot \prod_p (1 - 1/p)^6 (1 + 6/p + 6/p^2 + 1/p^3),$$

(τ_∞ is the "singular integral" - the archimedean density of X).

If $(V, \mathcal{L})$ is not primitive then, by Section 4.2, some Zariski open subset $U \subset V$ admits a primitive fibration: there is a diagram

$$\begin{array}{ccc} X & \rightarrow & \overline{V}^{\mathcal{L}} \\ \downarrow & & \\ Y^{\mathcal{L}} & & \end{array}$$

such that for all $y \in Y^{\mathcal{L}}(F)$ the pair $(U_y, \mathcal{L}_y)$ is primitive. Then

$$c(U, \mathcal{L}) := \sum_{y \in Y^0} c(U_y, \mathcal{L}_y),$$

where the right side is a (possibly infinite converging (!)) sum over the subset $Y^0 \subset Y^{\mathcal{L}}(F)$ of all those fibers U_y where

$$a(\mathcal{L}) = a(\mathcal{L}_y) \text{ and } \operatorname{rk} \operatorname{Pic}(V, \mathcal{L})^{\Gamma} = \operatorname{rk} \operatorname{Pic}(V_y, \mathcal{L}_y)^{\Gamma}.$$

In Section 5 we will see that even if we start with pairs $(V, \mathcal{L})$ where V is a smooth projective variety and $\mathcal{L}$ is a very ample adelically metrized line bundle on V we still need to consider singular varieties.

5. Analysis

5.1. Height zeta functions. — We illustrate our approach to asymptotics of rational points in the case of equivariant compactifications X of $\mathrm{G} = \mathbb{G}_a^n$. The geometry of these varieties is quite complicated already in dimension three (see [**23**] for more details). Nevertheless, we are able to prove asymptotic formulas for *every* such compactification and *every* (big) line bundle on it.

The idea is to use harmonic analysis on the adelic points $\mathrm{G}(\mathbb{A})$. We may assume that X is projective and that the boundary $X \setminus \mathrm{G}$ is a strict normal crossing divisor (using equivariant resolution of singularities, if necessary). Recall that $\operatorname{Pic}(X) = \mathrm{NS}(X)$ is generated by the irreducible boundary components (see Example 2.1.4). Next we define a height pairing

$$H = \prod_v H_v \, : \, \operatorname{Pic}(X)_{\mathbb{C}} \times \mathrm{G}(\mathbb{A}) \to \mathbb{C}.$$

such that its restriction to $L \in \operatorname{Pic}(X) \times \mathrm{G}(F)$ is the usual height (with respect to some metrized line bundle $\mathcal{L}$ in this class). Indeed, $\operatorname{Pic}(X)$ has a basis of (classes of) ample line bundles with sections which do not vanish in G and we can apply the constructions in Section 4.1. The equivariance of X implies that H is invariant under some compact subgroup $\mathbf{K} \subset \mathrm{G}(\mathbb{A}_f)$ (this may fail for general compactifications of the affine space).

Now we consider the *height zeta function*

$$\mathcal{Z}(\mathrm{G}, \mathbf{s}) = \sum_{x \in \mathrm{G}(F)} H(\mathbf{s}; x)^{-1}.$$

The projectivity of X implies that $\mathcal{Z}(\mathrm{G}, \mathbf{s})$ converges *a priori* for $\Re(\mathbf{s})$ in some (shifted) open cone in $\mathrm{Pic}(X)_{\mathbb{R}}$ (see Lemma 3.2 in [**9**]). Tauberian theorems relate analytic properties of $\mathcal{Z}$ to asymptotics of rational points of bounded height. Thus we need to determine the domain of holomorphy of $\mathcal{Z}$, find its poles and obtain its meromorphic continuation beyond those poles.

To achieve this we pass to a "spectral" expansion of $\mathcal{Z}$ (Poisson formula) to obtain a representation

$$\mathcal{Z}(\mathrm{G}, \mathbf{s}) = \sum_{(\mathrm{G}(\mathbb{A})/\mathrm{G}(F)\mathbf{K})^*} \hat{H}(\mathbf{s}; \psi), \tag{5.1}$$

where the sum is over the group of unitary characters of $\mathrm{G}(\mathbb{A})$ which are trivial on $\mathrm{G}(F)\mathbf{K}$. The triviality of ψ on $\mathbf{K}$ follows from the invariance of H under $\mathbf{K}$. This is crucial for the subsequent analysis - the right side in (5.1) is a summation over a *lattice* (and the trivial representation is isolated). The next step is the determination of the contribution of the trivial representation to $\mathcal{Z}$, achieved in the following section.

5.2. Height integrals. — Let X be a smooth d-dimensional equivariant compactification of a linear algebraic group G over F. Assume that over some finite Galois extension E/F with Galois group $\Gamma = \mathrm{Gal}(E/F)$ the boundary is a *strict normal crossing* divisor consisting of geometrically irreducible components

$$X \setminus \mathrm{G} = \cup_{j \in \mathcal{J}} D_j.$$

Then Γ acts by permutations on the set of boundary components. Denote by $\mathcal{J}/\Gamma$ (resp. $\mathcal{J}/\Gamma_v$, where Γ_v is the decomposition group at v) the set of orbits of Γ (resp. Γ_v) on $\mathcal{J}$. We put $D_\emptyset = \mathrm{G}$ and define for every subset $J \subset \mathcal{J}/\Gamma_v$

$$\begin{array}{rcl} D_J & = & \cap_{j\in J} D_j \\ D_J^0 & = & D_J \setminus \cup_{J'\supsetneq J} D_{J'}. \end{array}$$

For every $j \in \mathcal{J}/\Gamma$ (resp. $j \in \mathcal{J}/\Gamma_v$) we denote by F_j/F (resp. $F_{j,v}/F_v$) the minimal extension over which the orbit j splits completely and by $f_{j,v}$ the degree $[F_{j,v} : F_v]$. Choose for each v a Haar measure dg_v on $\mathrm{G}(F_v)$ such that for almost all v

$$\int_{\mathrm{G}(\mathfrak{o}_v)} dg_v = 1,$$

(to define $\mathrm{G}(\mathfrak{o}_v)$ one fixes a model for G over the integers). As in Section 4.1, one can define a *height* pairing between

$$\mathrm{Div}_{\mathbb{C}} := \oplus_j \mathbb{C} D_j$$

and $\mathrm{G}(\mathbb{A})$ (for *unipotent* groups, $\mathrm{Div}_{\mathbb{C}} = \mathrm{Pic}(X)_{\mathbb{C}}$, by Example 2.1.4). In the above basis, we have coordinates $\mathbf{s} = (s_j)_{j\in\mathcal{J}/\Gamma}$ on $\mathrm{Div}_{\mathbb{C}}$. Choose an F-rational (bi-)invariant differential form d-form on G. Then it has poles along each boundary component, and we denote by κ_j the corresponding multiplicities. For all but finitely many nonarchimedean valuations v, one has (see [12])

(5.2)
$$\int_{\mathrm{G}(F_v)} H_v(\mathbf{s}; g_v)^{-1} dg_v = \tau_v(\mathrm{G})^{-1} \left(\sum_{J\subset\mathcal{J}/\Gamma_v} \frac{\#D_J^0(\mathbb{F}_v)}{q_v^d} \prod_{j\in J} \frac{q_v^{f_{j,v}} - 1}{q_v^{f_{j,v}(s_j-\kappa_j+1)} - 1} \right),$$

where $\tau_v(\mathrm{G})$ is the local Tamagawa number of G (see [51],[36]).

Corollary 5.2.1. — *Let X be an equivariant compactification of a unipotent group. We have a height pairing*

$$\mathrm{Pic}(X)_{\mathbb{C}} = \oplus_{j\in\mathcal{J}/\Gamma} \mathbb{C} D_j \times \mathbb{G}_a^n(\mathbb{A}) \to \mathbb{C},$$

generalizing the usual heights. There exists a function $\varphi(\mathbf{s})$ (with $\mathbf{s} = (s_j)_{j\in\mathcal{J}/\Gamma}$) which is holomorphic for $s_j \geq \kappa_j - \delta$ (for some $\delta > 0$) such that

$$(5.3) \qquad \prod_v \int_{\mathrm{G}(F_v)} H_v(\mathbf{s}; g_v)^{-1} dg_v = \prod_{j\in\mathcal{J}/\Gamma} \zeta_{F_j}(s_j - \kappa_j + 1) \cdot \varphi(\mathbf{s})$$

(here we used that for unipotent groups $\tau_v(\mathrm{G}) = 1$).

Example 5.2.2. — Let $X = \mathbb{P}^1$, $H_v(g_v)$ the local height as in Example 4.1.6. Then

$$\int_{\mathbb{G}_a(F_v)} H_v(g_v)^{-s} dg_v = \frac{1 - q_v^{-s}}{1 - q_v^{-(s-2)}}. \tag{5.4}$$

Remark 5.2.3. — As often in harmonic analysis on *compact* spaces, the trivial representation contributes the main term in "asymptotics". Indeed, the Euler product above will provide the poles of "highest" order in the expansion of the zeta function $\mathcal{Z}(\mathbf{s})$ around the point $-K_X$. Recall that (in the basis D_j) the effective cone is the positive octant (see Example 2.1.4). We observe that the polar hyperplanes of $\mathcal{Z}$ cut out the shifted, by the anticanonical class, effective cone. Even if we didn't know from geometric considerations (see 2.1.4) what the effective cone or the anticanonical class are - the adelic integral (5.3) is "telling" us the answer.

Example 5.2.4. — Let us consider a wonderful compactification X of a (split) semi-simple group G of adjoint type (see 2.1.6). By construction, X is a bi-equivariant compactification of G (the group G acts on both sides). For almost all v, the local heights H_v are invariant under $\mathbf{K}_v = \mathrm{G}(\mathfrak{o}_v)$. This implies that $H_v(\mathbf{s}; g_v)$ depends only on the a_v^+-component in the Cartan decomposition

$$g_v = k_v a_v^+ k_v'$$

of g_v (here $\mathbf{s} \in \mathfrak{X}^*(\mathrm{T})_{\mathbb{C}} = \mathrm{Pic}(X)_{\mathbb{C}}$ and $a_v^+ \in \mathrm{T}(F_v)$). One can check that the height pairing

$$H_v \,:\, \mathfrak{X}(\mathrm{T})_{\mathbb{C}} \times \mathrm{G}(F_v) \to \mathbb{C}$$

is given by

$$(\mathbf{s}, g_v) \mapsto q_v^{\langle \mathbf{s}, \overline{a}_v^+ \rangle},$$

where $\overline{a}_v^+$ is the image of $a_v^+ \in \mathrm{T}(F_v)$ in the Lie algebra of T (under the logarithmic map). The local height integral can be computed geometrically as in (5.2) or directly

$$\int_{\mathrm{G}(F_v)} H_v(\mathbf{s}; g_v)^{-1} dg_v = \sum_{\overline{a}_v^+} q_v^{-\langle \mathbf{s}, \overline{a}_v^+ \rangle} \mathrm{vol}(\mathbf{K}_v a_v^+ \mathbf{K}_v).$$

Comparing these two expressions one obtains closed formulas for the volumes

$$\mathrm{vol}(\mathbf{K}_v a_v^+ \mathbf{K}_v).$$

Notice that the boundary strata $D_{\mathcal{J}}$ have a nice inductive description: they are fibrations over $(\mathrm{P}_{\mathcal{J}}\backslash\mathrm{G}) \times (\mathrm{P}_{\mathcal{J}}\backslash\mathrm{G})$ with fibers wonderful compactifications of the Levi components of the parabolic $\mathrm{P}_{\mathcal{J}}$ (where $\mathcal{J} \subset [1, ..., \mathrm{rk}\,\mathrm{G}]$). This allows a transparent inductive computation of the number of $\mathbb{F}_v$-points in each stratum. A combinatorial approach to these volumes for *split* G can be found in [**32**]. I am not aware of a treatment of the general case in the literature.

Finally, as in Remark 5.2.3, the local computation is telling us that the effective cone $\Lambda_{\mathrm{eff}}(X) \subset \mathfrak{X}^*(\mathrm{T})$ "has to be" the cone dual to the cone A_v^+ (parametrizing the $\mathbf{K}_v a_v^+ \mathbf{K}_v$-cosets in the Cartan decomposition), that is exactly the cone spanned by simple roots. Similarly, we can read off the anticanonical class.

5.3. Height zeta functions continued. — We return to the study of analytic properties of the height zeta function $\mathcal{Z}(\mathbf{s}; g)$ for equivariant compactifications of $\mathbb{G}_a^n$ and consider now nontrivial characters ψ, occurring in the expansion (5.1). For each ψ we compute the Fourier transforms $\hat{H}_v$ at almost all nonarchimedean places and find estimates at the remaining places. At good reduction places, this generalizes the computation in Section 5.2. A character $\psi = \psi_{\mathbf{a}}$ of $\mathbb{G}_a^n(\mathbb{A})/\mathbb{G}_a^n(F)$ is defined by a linear form $f_a := \langle \mathbf{a}, \cdot \rangle$, with $\mathbf{a} \in \mathbb{G}_a^n(F)$. The divisor of the corresponding rational function on X can be written as

$$\mathrm{div}(f_{\mathbf{a}}) = E(\mathbf{a}) - \sum_{j \in \mathcal{J}/\Gamma} d_j(\mathbf{a}) D_j,$$

where all $d_j \geq 0$ (by Example 2.1.4). Denote by $\mathcal{J}_0(\mathbf{a})$ the set of all $j \in \mathcal{J}/\Gamma$ such that the corresponding $d_j(\mathbf{a}) = 0$. For nontrivial $\mathbf{a}$ the set $\mathcal{J}_0(\mathbf{a})$ is a *proper* subset of $\mathcal{J}/\Gamma$. "Motivic" integration (combined with estimates at bad reduction places) yields

$$\hat{H}(\mathbf{s}; \psi_{\mathbf{a}}) = \prod_{j \in \mathcal{J}_0(\mathbf{a})} \zeta_{F_j}(s_j - \kappa_j + 1) \varphi(\mathbf{s}; \mathbf{a})$$

where φ is holomorphic in the neighborhood of $-K_X$. At this stage we observe that *each* term in (5.1) admits a meromorphic continuation and that the poles of each term are contained in the faces of the (shifted) cone $\Lambda_{\mathrm{eff}}(X) - K_X$. To obtain a meromorphic continuation for the *sum* we need a bound

$$|\varphi(\mathbf{s}; \mathbf{a})| \ll (1 + \|\mathbf{a}\|)^{-\mathsf{n}}$$

(for sufficiently large n). For this we use integration by parts at the archimedean places. Once again, the equivariance of X is essential.

In the neighborhood of $-K_X$ the pole of $\mathcal{Z}$ of highest order is contributed by the trivial character and the leading constant at this pole is an adelic integral (a Tamagawa number) times a rational number (the γ-constant). Let us consider a line bundle L which is *not* proportional to $-K_X$ and restrict $\mathcal{Z}(\mathbf{s})$ to sL. Denote by $\mathcal{J}(L)$ the set of boundary divisors which do not lie in the face of the cone $\Lambda_{\mathrm{eff}}(X)$ containing $a(L)L+K_X$. The one-parameter zeta function $\mathcal{Z}(sL)$ is holomorphic for $\Re(s)>a(L)$, admits a meromorphic continuation to $\Re(s)>a(L)-\delta$ (for some $\delta>0$) and has a pole of order *at most* $b(L)$ at $s=a(L)$. Notice that *every* term in the sum

$$\sum_{\mathbf{a}:\mathcal{J}_0(\mathbf{a})\supseteq\mathcal{J}(L)} \hat{H}(\mathbf{s};\psi_{\mathbf{a}}) \tag{5.5}$$

contributes to this pole (the other terms in $\mathcal{Z}$ have poles of smaller order). We need to show that the (in general, infinite) sum of leading coefficients of these terms (complex numbers) converges to a *non-zero* real number. Further, we need to identify this real number.

One possible approach, suggested by harmonic analysis, is to use the Poisson formula one more time to convert the sum over Fourier transforms of the height to an integral over some subgroup of $\mathrm{G}(\mathbb{A})$. The common kernel of the characters appearing in (5.5) is a subgroup

$$\mathrm{G}^L(F)\cdot\mathrm{G}_L(\mathbb{A})\subset\mathrm{G}(\mathbb{A}),$$

where G_L is the intersection of kernels of $f_{\mathbf{a}}$, for all $\mathbf{a}$ occurring in (5.5). Applying the Poisson formula we find that the leading term equals

$$\lim_{s\to a(L)}(s-a(L))^{b(L)}\cdot\left(\sum_{g\in\mathrm{G}(F)/\mathrm{G}_L(F)}\int_{\mathrm{G}_L(\mathbb{A})} H(s;g+g')^{-1}dg'\right) \tag{5.6}$$

Now we want to find a geometric explanation of this formula. First of all, the outer sum indicates that we are dealing with a fibration. It turns out that this fibration is exactly of the type appearing in Fujita's program (the map onto the base of the fibration is given by the *sections* $f_{\mathbf{a}}$ as in (5.5)). Secondly, the adelic integrals are very similar to those already treated in Section 5.2. In fact, they are height integrals for the induced equivariant compactifications of translates of G_L. There is one important difference: previously, we started with a smooth equivariant compactification X, with boundary a strict normal crossing divisor. We cannot guarantee, however, that the compactification of G_L (and its shifts) inside X has the same property! This explains the technical setup of Section 4: analysis tells us that we need to deal with fibrations and singular varieties.

EXAMPLE 5.3.1. — Consider the simplest case $X = \mathbb{P}^n \times \mathbb{P}^n$, $\mathcal{L}$ the product of the line bundles $\mathcal{O}(a)$ and $\mathcal{O}(b)$. The Picard group $\mathrm{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$, so that the class of $\mathcal{L} = (a, b)$ and $-K_X = (n+1, n+1)$. The height zeta function factors as well

$$\mathcal{Z}_X(s_1, s_2) = \mathcal{Z}_{\mathbb{P}^n}(s_1)\mathcal{Z}_{\mathbb{P}^n}(s_2).$$

Its restriction to the line $s(n+1, n+1)$ has a pole at $s = 1$ of order 2, with leading coefficient an adelic integral (the Tamagawa number τ^2) times a rational number (the γ-constant). The restriction to the line $s(a, b)$, for $a > b$, has a pole at $s = (n+1)/b$ with residue $\tau \cdot \gamma \cdot \mathcal{Z}_{\mathbb{P}^n}(a(n+1)/b)$. The (converging) sum

$$\sum_{x \in \mathbb{P}^n(F)} \tau \cdot \gamma \cdot H_{\mathcal{O}(1)}(x)^{a(n+1)/b}$$

is exactly the constant $c(X, \mathcal{L})$ defined in Section 4.

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