TAMAGAWA NUMBERS OF DIAGONAL CUBIC SURFACES OF HIGHER RANK

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ABSTRACT. We consider diagonal cubic surfaces defined by an equation of the form

$$ax^3 + by^3 + cz^3 + dt^3 = 0.$$

Numerically, one can find all rational points of height $\leq B$ for B in the range of up to 10^5, thanks to a program due to D. J. Bernstein. On the other hand, there are precise conjectures concerning the constants in the asymptotics of rational points of bounded height due to Manin, Batyrev and the authors. Changing the coefficients we can obtain cubic surfaces with rank of the Picard group varying between 1 and 4 and check that numerical data are compatible with the above conjectures. In a previous paper we considered the case of cubic surfaces with a Picard group of rank one with or without Brauer-Manin obstruction to weak approximation. In this paper, we test the conjectures for diagonal cubic surfaces with a Picard group of higher rank.

CONTENTS

1. INTRODUCTION

As the preceding one, [PT], this paper is devoted to numerical tests of a refined version of a conjecture of Manin about the number of points of bounded height on Fano varieties (see [BM], [FMT], [Pe], or [BT] for a description of the conjectures). The choice of diagonal cubic surfaces to test these conjectures was motivated by the work of Heath-Brown [H-B] in which he treated the cases

$$X^3 + Y^3 + Z^3 + aT^3 = 0$$

for $a = 2$ or 3. The results he obtained were used as a benchmark for the subsequent tentatives of interpretation of the constant (see, in particular, [S-D], [Pe] and [PT]).

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More precisely, we consider a diagonal cubic surface $V \subset \mathbf{P}^3_Q$ given by an equation of the form

$$aX^3 + bY^3 + cZ^3 + dT^3 = 0$$

and use the height zeta function H on V defined by the formula: for any $(x : y : z : t)$ in $V(\mathbf{Q})$, one has

$$H((x : y : z : t)) = \max\{|x|, |y|, |z|, |t|\} \text{ if } \begin{cases} x, y, z, t \in \mathbf{Z}, \\ \gcd(x, y, z, t) = 1. \end{cases}$$

Let U be the complement in V of the 27 lines. Then we are interested in the asymptotic behavior of the cardinal

$$N_{U,H}(B) = \sharp\{\, x \in U(\mathbf{Q}) \mid H(x) \leqslant B \,\}$$

as B goes to infinity.

If $V(\mathbf{Q})$ is Zariski dense, which by a result of Segre (see [Man2, §29,§30]) is equivalent to $V(\mathbf{Q}) \neq \emptyset$, then it is expected that

$$N_{U,H}(B) = B\log(B) + o(B)$$

as B goes to $+\infty$, where P is a polynomial of degree $\operatorname{rk}\operatorname{Pic} V - 1$, with a dominant coefficient $\boldsymbol{\theta}_H(V)$ which have a conjectural description. The aim is to compute explicitely this constant $\boldsymbol{\theta}_H(V)$ and to compare it with numerical data obtained for $B \leqslant 100\,000$. Our previous paper [PT] was devoted to several examples with a Picard group of rank one with or without Brauer-Manin obstruction to the weak approximation. In this paper, we consider examples with Picard group of higher rank. One should note that the error term is expected to decrease more slowly for these examples. Indeed, if the rank of the Picard group of V is one, the error term

$$(N_{U,H}(B) - \boldsymbol{\theta}_H(V)B(\log B)^{\operatorname{rk}\operatorname{Pic}(V)-1})/B(\log B)^{\operatorname{rk}\operatorname{Pic}(V)-1})$$

is expected to decrease as $1/B^\epsilon$ for some $\epsilon > 0$, whereas it should be similar to $1/\log B$ in this higher rank setting. However we observe a quite good accordance, which is even more striking if we take into account that a polynomial P of degree $\operatorname{rk}\operatorname{Pic} V - 1$ should appear and use simple statistical formula to estimate its leading coefficient from the numerical data.

In section 2, we give a short description of the construction of $\boldsymbol{\theta}_H(V)$. Section 3 contains the description of the Galois action on the geometric Picard group $\operatorname{Pic}(\overline{V})$. In section 4, we compute the Euler product corresponding to the good places. In section 5, we explain how to compute the density at the bad places in the cases we are interested in. In section 6 we determine in each case the value of $\alpha(V)$. Section 7 is devoted to the description of the statistical tools used with the numerical data and section 8 to the results of the numerical tests.

2. Description of the conjectural constant

For the sake of self-completeness, we shall now repeat a summary of the description of the conjectural constant for heights defined by an adelic metric on the anticanonical line bundle of a surface (see [Pe] for more details and [BT] for a discussion in a more general setting).

Notations 2.1. For any field E, we denote by $\overline{E}$ an algebraic closure of E. If X is a variety over the field E, then $X(E)$ denotes the set of rational points of X and $\overline{X}$ the product $X \times_{\operatorname{Spec} E} \operatorname{Spec} \overline{E}$. The cohomological Brauer group $\operatorname{Br}(X)$ is defined as the étale cohomology group $H^2_{\text{ét}}(X, \mathbf{G}_m)$. For any A in $\mathrm{Br}(X)$, any extension E' of E and any P in $V(E')$, we denote by $A(P)$ the evaluation of A at P.

If F is a number field, we denote by $\mathrm{Val}(F)$ the set of places of F and by $\mathrm{Val}_f(F)$ the set of finite places. The absolute discriminant of F is denoted by d_F. For any place v of F, let F_v be the v-adic completion of F. If v is finite, then $\mathscr{O}_v$ is the ring of v-adic integers and $\mathbf{F}_v$ the residue field. The global class field theory yields an exact sequence

$$0 \to \mathrm{Br}(F) \to \bigoplus_{v \in \mathrm{Val}(F)} \mathrm{Br}(F_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbf{Q}/\mathbf{Z} \to 0. \tag{2.1}$$

In this paragraph, V denotes a smooth projective geometrically integral variety over a number field F, which satisfies the following conditions:

(i) The group $H^i(V, \mathscr{O}_V) = 0$ is trivial if $i = 1$ or 2,
(ii) $\mathrm{Pic}(\overline{V})$ has no torsion,
(iii) the anticanonical line bundle ω_V^{-1} belongs to the interior of the cone of classes of effective divisors $\Lambda_{\text{eff}} \subset \mathrm{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{R}$.

The adelic space $V(\boldsymbol{A}_F)$ of V coincides with the product $\prod_{v \in \mathrm{Val}(F)} V(F_v)$. By [CT, lemma 1], for any class A in $\mathrm{Br}(V)$, one may consider the map ρ_A defined as the composite

$$\begin{array}{ccccc} V(\boldsymbol{A}_F) & \to & \bigoplus_{v \in \mathrm{Val}(F)} \mathrm{Br}(F_v) & \xrightarrow{\sum \mathrm{inv}_v} & \mathbf{Q}/\mathbf{Z} \\ (P_v)_{v \in \mathrm{Val}(F)} & \mapsto & (A(P_v))_{v \in \mathrm{Val}(F)}. & & \end{array}$$

Then one defines

$$V(\boldsymbol{A}_F)^{\mathrm{Br}} = \bigcap_{A \in \mathrm{Br}\, V} \ker(\rho_A) \subset V(\boldsymbol{A}_F).$$

By the exact sequence (2.1), one has the inclusion

$$\overline{V(F)} \subset V(\boldsymbol{A}_F)^{\mathrm{Br}}$$

where $\overline{V(F)}$ denotes the topological closure of the set of rational points. Conjecturally both sets coincide for cubic surfaces. (See also the text of Swinnerton-Dyer in this volume). The *Brauer-Manin obstruction to weak approximation*, described by Manin in [Man1] and by Colliot-Thélène and Sansuc in [CTS] may be defined as the condition

$$V(\boldsymbol{A}_F)^{\mathrm{Br}} \neq V(\boldsymbol{A}_F).$$

Let us assume that the height H on V is defined by an adelic metric $(\|\cdot\|_v)_{v \in \mathrm{Val}(F)}$ on ω_V^{-1}. By definition, this means that we consider ω_V^{-1} as a line bundle, that the functions $\|\cdot\|_v$ are v-adically continuous metrics on $\omega_V^{-1}(F_v)$ which for almost all place v are given by a smooth model of V, and that the height of a rational point x of V is given by the formula

$$\forall y \in \omega_V^{-1}(x) - \{0\}, \quad H(x) = \prod_{v \in \mathrm{Val}(F)} \|y\|_v^{-1}$$

where $\omega_V^{-1}(x)$ is the fiber of ω_V^{-1} at x.

If v is a place of F, we normalize the Haar measure $\mathrm{d}x_v$ on F_v as follows:

- $\int_{\mathscr{O}_v} \mathrm{d}x_v = 1$ if v is finite,
- $\mathrm{d}x_v$ is the usual Lebesgue measure if $F_v \xrightarrow{\sim} \mathbf{R}$,
- $\mathrm{d}x_v = \mathrm{d}z\, \mathrm{d}\overline{z} = 2\mathrm{d}x\, \mathrm{d}y$ if $F_v \xrightarrow{\sim} \mathbf{C}$.

The metric $\|\cdot\|_v$ then defines a measure $\boldsymbol{\omega}_{H,v}$ on the locally compact space $V(F_v)$ given locally by the formula

$$\boldsymbol{\omega}_{H,v} = \left\| \frac{\partial}{\partial x_{1,v}} \wedge \cdots \wedge \frac{\partial}{\partial x_{n,v}} \right\|_v \mathrm{d}x_{1,v} \ldots \mathrm{d}x_{n,v}$$

where $x_{1,v} \ldots x_{n,v}$ are local v-adic analytic coordinates on $V(F_v)$.

If M is a discrete representation of $\mathrm{Gal}(\overline{F}/F)$ over $\mathbf{Q}$, then for any finite place $\mathfrak{p}$ of F, the local term of the corresponding L-function is defined as follows: we choose an algebraic closure $\overline{F}_\mathfrak{p}$ of $F_\mathfrak{p}$ containing $\overline{F}$. We get an exact sequence

$$1 \to I_\mathfrak{p} \to D_\mathfrak{p} \to \mathrm{Gal}(\overline{\mathbf{F}_\mathfrak{p}}/\mathbf{F}_\mathfrak{p}) \to 1$$

where $D_\mathfrak{p}$ is the decomposition group and $I_\mathfrak{p}$ the inertia. We denote by $\widetilde{\mathrm{Fr}}_\mathfrak{p}$ a lifting of the Frobenius map in $D_\mathfrak{p} \subset \mathrm{Gal}(\overline{F}/F)$ which up to conjugation depends only on $\mathfrak{p}$, and

$$L_\mathfrak{p}(s, M) = \frac{1}{\mathrm{Det}(1 - (\sharp \mathbf{F}_\mathfrak{p})^{-s} \widetilde{\mathrm{Fr}}_\mathfrak{p} \mid M^{I_\mathfrak{p}})}.$$

We fix a finite set S of bad places containing the archimedean ones so that V admits a smooth projective model $\mathscr{V}$ over the ring of S-integers $\mathscr{O}_S$. For any $\mathfrak{p}$ in $\mathrm{Val}(F) - S$ we consider

$$L_\mathfrak{p}(s, \mathrm{Pic}(\overline{V})) = L_\mathfrak{p}(s, \mathrm{Pic}(\overline{V}) \otimes_\mathbf{Z} \mathbf{Q})$$

The corresponding global L-function is given by the Euler product

$$L_S(s, \mathrm{Pic}(\overline{V})) = \prod_{\mathfrak{p} \in \mathrm{Val}(F) - S} L_\mathfrak{p}(s, \mathrm{Pic}\, \overline{V})$$

which converges for $\mathrm{Re}\, s > 1$ and has a meromorphic continuation to $\mathbf{C}$ with a pole of order $t = \mathrm{rk}\, \mathrm{Pic}(V)$ at 1. One then introduces the local converging factors λ_v given by

$$\lambda_v = \begin{cases} L_v(1, \mathrm{Pic}(\overline{V})) \text{ if } v \in \mathrm{Val}(F) - S, \\ 1 \text{ otherwise.} \end{cases}$$

It then follows from the Weil conjectures proven by Deligne that the Tamagawa measure

$$\prod_{v \in \mathrm{Val}(F)} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}$$

converges on $V(\boldsymbol{A}_F)$ (see [Pe, proposition 2.2.2]).

Definition 2.2. The *Tamagawa measure* on $V(\boldsymbol{A}_F)$ corresponding to the adelic metric $(\|\cdot\|_v)_{v \in \mathrm{Val}(F)}$ is defined by

$$\boldsymbol{\omega}_H = \frac{1}{\sqrt{d_F}^{\dim V}} \lim_{s \to 1} (s-1)^t L_S(s, \mathrm{Pic}\, \overline{V}) \prod_{v \in \mathrm{Val}(F)} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}.$$

Following a suggestion of Salberger, we use here $V(\boldsymbol{A}_F)^{\mathrm{Br}}$ instead of $\overline{V(F)}$ to define the corresponding Tamagawa number.

$$\tau_H(V) = \boldsymbol{\omega}_H(V(\boldsymbol{A})^{\mathrm{Br}}) \tag{2.2}$$

Let $\mathrm{Pic}(V)^\vee$ be the dual lattice to $\mathrm{Pic}(V)$. We denote by $\mathrm{d}\boldsymbol{y}$ the corresponding Lebesgue measure on $\mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R}$ and by

$$\Lambda_{\mathrm{eff}}(V)^\vee = \{\, x \in \mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R} \mid \forall y \in \Lambda_{\mathrm{eff}}(V),\ \langle x, y\rangle \geqslant 0 \,\}$$

the dual cone of $\Lambda_{\mathrm{eff}}(V)$.

Definition 2.3. We define

$$\alpha(V) = \frac{1}{(t-1)!} \int_{\Lambda_{\mathrm{eff}}(V)^\vee} e^{-\langle \omega_V^{-1}, \boldsymbol{y}\rangle} \mathrm{d}\boldsymbol{y}$$

and

$$\beta(V) = \sharp H^1(k, \mathrm{Pic}(\overline{V})).$$

The theoritical constant attached to V and H is defined as

$$\boldsymbol{\theta}_H(V) = \alpha(V)\beta(V)\tau_H(V). \tag{2.3}$$

The aim of the following sections is to give an explicit expression of this constant $\boldsymbol{\theta}_H(V)$ for various diagonal cubic surfaces.

3. The Galois module $\mathrm{Pic}(\overline{V})$

As usual the description of this Galois module is based upon the study of the 27 lines of the cubic. We therefore fix notations for these lines extending the ones given by Colliot-Thélène, Kanevsky and Sansuc in [CTKS, p. 9].

Notations 3.1. From now on we restrict ourselves to the case of a diagonal cubic surface V given by an equation of the form

$$aX^3 + bY^3 + cZ^3 + dT^3 = 0 \tag{3.1}$$

where a,b,c and d are coprime strictly positive integers. In the following we also take

$$S = \{\infty, 3\} \cup \{\, p \mid p | abc\}$$

We fix a cubic root α (resp. α', α'') of b/a (resp. c/a, d/a) which is assumed to be in $\mathbf{Q}$ if b/a (resp. c/a, d/a) is a cube in $\mathbf{Q}$ and we put

$$\beta = \frac{\alpha''}{\alpha'} = \sqrt[3]{\frac{d}{c}}, \qquad \beta' = \frac{\alpha''}{\alpha} = \sqrt[3]{\frac{d}{b}} \qquad \text{and} \qquad \beta'' = \frac{\alpha'}{\alpha} = \sqrt[3]{\frac{c}{b}}.$$

We also consider

$$\gamma = \frac{\alpha''}{\alpha\alpha'} = \sqrt[3]{\frac{ad}{bc}}, \qquad \gamma' = \frac{\alpha}{\alpha'\alpha''} = \sqrt[3]{\frac{ab}{cd}} \qquad \text{and} \qquad \gamma'' = \frac{\alpha'}{\alpha\alpha''} = \sqrt[3]{\frac{ac}{bd}}.$$

We denote by θ a primitive third root of one. The 27 lines of the cubic surface (3.1) are then given by the following equations, where i belongs to $\mathbf{Z}/3\mathbf{Z}$:

$$L(i){:}\begin{cases} X+\theta^i\alpha Y = 0,\\ Z+\theta^i\beta T = 0.\end{cases} \qquad L'(i){:}\begin{cases} X+\theta^i\alpha Y = 0,\\ Z+\theta^{i+1}\beta T = 0.\end{cases} \qquad L''(i){:}\begin{cases} X+\theta^i\alpha Y = 0,\\ Z+\theta^{i+2}\beta T = 0.\end{cases}$$

$$M(i){:}\begin{cases} X+\theta^i\alpha' Z = 0,\\ Y+\theta^{i+1}\beta' T = 0.\end{cases} \qquad M'(i){:}\begin{cases} X+\theta^i\alpha' Z = 0,\\ Y+\theta^{i+2}\beta' T = 0.\end{cases} \qquad M''(i){:}\begin{cases} X+\theta^i\alpha' Z = 0,\\ Y+\theta^{i}\beta' T = 0.\end{cases}$$

$$N(i){:}\begin{cases} X+\theta^i\alpha'' T = 0,\\ Y+\theta^{i+2}\beta'' Z = 0.\end{cases} \qquad N'(i){:}\begin{cases} X+\theta^i\alpha'' T = 0,\\ Y+\theta^{i}\beta'' Z = 0.\end{cases} \qquad N''(i){:}\begin{cases} X+\theta^i\alpha'' T = 0,\\ Y+\theta^{i+1}\beta'' Z = 0.\end{cases}$$

Let K be the field $\mathbf{Q}(j, \alpha, \alpha', \alpha'')$. It is a Galois extension of $\mathbf{Q}$. In the generic case, K is an extension of degree 54 with a Galois group isomorphic to

$$(\mathbf{Z}/3\mathbf{Z})^3 \rtimes \mathbf{Z}/2\mathbf{Z}.$$

It is generated by the elements c, τ, τ' and τ'' characterized as follows by their action on θ, α, α' and α''.

	θ	α	α'	α''
c	θ^2	α	α'	α''
τ	θ	$\theta\alpha$	α'	α''
τ'	θ	α	$\theta\alpha'$	α''
τ''	θ	α	α'	$\theta\alpha''$

Their action on the 27 lines is given as follows: for τ we have

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L''(i+1) \arrow[dl] \\ & L'(i+2) \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M''(i) \arrow[dl] \\ & M'(i) \arrow[ul] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N''(i) \arrow[dl] \\ & N'(i) \arrow[ul] & \end{tikzcd} \tag{3.2}$$

for τ':

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L''(i) \arrow[dl] \\ & L'(i) \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M''(i+1) \arrow[dl] \\ & M'(i+2) \arrow[ul] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N'(i) \arrow[dl] \\ & N''(i) \arrow[ul] & \end{tikzcd} \tag{3.3}$$

for τ'':

$$\begin{tikzcd}[column sep=small] L(i) \arrow[rr] & & L'(i) \arrow[dl] \\ & L''(i) \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] M(i) \arrow[rr] & & M'(i) \arrow[dl] \\ & M''(i) \arrow[ul] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}[column sep=small] N(i) \arrow[rr] & & N''(i+1) \arrow[dl] \\ & N'(i+2) \arrow[ul] & \end{tikzcd} \tag{3.4}$$

for c:

$$\begin{tikzcd} L(0) & L'(0) \arrow[r, leftrightarrow] & L''(0) & M(0) \arrow[r, leftrightarrow] & M'(0) & M''(0) \\ L(1) \arrow[d, leftrightarrow] & L'(1) \arrow[dr, leftrightarrow] & L''(1) \arrow[dl, leftrightarrow] & M(1) \arrow[dr, leftrightarrow] & M'(1) \arrow[dl, leftrightarrow] & M''(1) \arrow[d, leftrightarrow] \\ L(2) & L'(2) & L''(2) & M(2) & M'(2) & M''(2) \end{tikzcd}$$

$$\begin{tikzcd} N(0) \arrow[rr, leftrightarrow, bend left] & N'(0) & N''(0) \\ N(1) \arrow[ddrr, leftrightarrow, phantom] \arrow[rrd, leftrightarrow] & N'(1) \arrow[d, leftrightarrow] & N''(1) \arrow[lld, leftrightarrow] \\ N(2) & N'(2) & N''(2). \end{tikzcd} \tag{3.5}$$

To describe the relations between the classes of these divisors in $\mathrm{Pic}(\overline{V})$, one way is to see $\overline{V}$ as the plane $\mathbf{P}^2_{\mathbf{Q}}$ blown up in six points P_1, P_2, P_3, P_4, P_5 and P_6. In this setting, the

27 lines may be described as the 6 exceptional divisors $E_1, \dots, E_6$, the 15 strict liftings $L_{i,j}$ of the projective lines (P_iP_j) for $1 \leqslant i < j \leqslant 6$ and the 6 strict liftings of the conics Q_i going through all points except P_i. Let Λ be the lifting of a line of $\mathbf{P}^2_{\mathbf{Q}}$ which does not contain any of the points $P_1, \dots, P_6$. Then

$$([\Lambda], [E_1], [E_2], [E_3], [E_4], [E_5], [E_6])$$

is a basis of $\mathrm{Pic}(\overline{V})$ and we have the following relations in $\mathrm{Pic}(\overline{V})$:

$$\begin{aligned} [L_{i,j}] &= [\Lambda] - [E_i] - [E_j] \qquad \text{for } 1 \leqslant i < j \leqslant 6, \\ [Q_i] &= 2[\Lambda] - \sum_{j \neq i} [E_j]. \end{aligned} \tag{3.6}$$

In the following, we choose the projection of $\overline{V}$ to $\mathbf{P}^2_{\mathbf{Q}}$ so that we have the equalities:

$$\begin{array}{lll} E_1 = L(0), & E_2 = L(1), & E_3 = L(2), \\ E_4 = M(0), & E_5 = M(1), & E_6 = M(2), \\ Q_1 = L'(1), & Q_2 = L'(2), & Q_3 = L'(0), \\ Q_4 = M''(2), & Q_5 = M''(0), & Q_6 = M''(1), \\ L_{1,2} = L''(1), & L_{2,3} = L''(2), & L_{3,1} = L''(0), \\ L_{4,5} = M'(0), & L_{5,6} = M'(1), & L_{6,4} = M'(2), \\ L_{1,4} = N'(2), & L_{1,5} = N'(0), & L_{1,6} = N'(1), \\ L_{2,4} = N''(0), & L_{2,5} = N''(1), & L_{2,6} = N''(2), \\ L_{3,4} = N(1), & L_{3,5} = N(2), & L_{3,6} = N(0). \end{array} \tag{3.7}$$

Notations 3.2. We consider the étale algebra E_1 over $\mathbf{Q}$ defined as $\mathbf{Q}(\gamma)$ if ad/bc is not a cube in $\mathbf{Q}$ and as $\mathbf{Q}(\theta) \times \mathbf{Q}$ otherwise. Similarly, we define the algebra E_2 (resp. E_3) corresponding to γ' (resp. γ'') and we put

$$E = E_1 \times E_2 \times E_3.$$

We also consider the following elements of $\mathrm{Pic}(\overline{V})$

$$\begin{array}{ll} e^1_0 = [L(0)] + [L(1)] + [L(2)], & e^1_1 = [L'(0)] + [L'(1)] + [L'(2)], \\ e^1_2 = [L''(0)] + [L''(1)] + [L''2], & e^2_0 = [M''(0)] + [M(1)] + [M'(2)], \\ e^2_1 = [M'(0)] + [M''(1)] + [M(2)], & e^2_2 = [M(0)] + [M'(1)] + [M''(2)], \\ e^3_0 = [N'(0)] + [N'(1)] + [N'(2)], & e^3_1 = [N''(0)] + [N''(1)] + [N''(2)], \\ e^3_2 = [N(0)] + [N(1)] + [N(2)] & \end{array}$$

and the sets

$$\mathscr{E}_1 = \{e^1_0, e^1_1, e^1_2\}, \quad \mathscr{E}_2 = \{e^2_0, e^2_1, e^2_2\}, \quad \mathscr{E}_3 = \{e^3_0, e^3_1, e^3_2\},$$

and $\mathscr{E} = \mathscr{E}_1 \sqcup \mathscr{E}_2 \sqcup \mathscr{E}_3$.

Lemma 3.3. *The sets $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ are globally invariant under the action of* $\mathrm{Gal}(K/\mathbf{Q})$ *and the étale algebra corresponding to the set $\mathscr{E}_i$ is isomorphic to E_i.*

Proof. The fact that the sets $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ are globally invariant follows immediately from the descriptions (3.2)–(3.5). The étale algebra F corresponding to a finite $\mathrm{Gal}(K/\mathbf{Q})$-set $\mathscr{F}$ may be defined as the algebra

$$(K[\mathscr{F}])^{\mathrm{Gal}(K/\mathbf{Q})}$$

where $K[\mathscr{F}]$ is the algebra $K^{\mathscr{F}}$ and where $\mathrm{Gal}(K/\mathbf{Q})$ acts simultaneously on K and $\mathscr{F}$. In the generic case, let us consider

$$\sigma = \tau\tau', \quad \sigma' = \tau'\tau'' \quad \text{and} \quad \sigma'' = \tau''\tau.$$

Then σ sends γ on $\theta\gamma$ and acts trivially on γ', γ'' and θ. We may describe similarly the actions of σ' and σ''. The action of $\mathrm{Gal}(K/\mathbf{Q})$ on $\mathscr{E}_1$ in the generic case is then given by the table

	e_0^1	e_1^1	e_2^1
c	e_0^1	e_2^1	e_1^1
σ	e_1^1	e_2^1	e_0^1
σ'	e_0^1	e_1^1	e_2^1
σ''	e_0^1	e_1^1	e_2^1

This implies that if ad/bc is not a cube in $\mathbf{Q}$, then $\mathscr{E}_1$ is isomorphic to

$$\mathrm{Gal}(K/\mathbf{Q})/\,\mathrm{Gal}(K/\mathbf{Q}(\gamma))$$

as a $\mathrm{Gal}(K/\mathbf{Q})$-set. Then the corresponding étale algebra is

$$(K[\mathrm{Gal}(K/\mathbf{Q})/\,\mathrm{Gal}(K/\mathbf{Q}(\gamma))])^{\mathrm{Gal}(K/\mathbf{Q})} \xrightarrow{\sim} K^{\mathrm{Gal}(K/\mathbf{Q}(\gamma))} = \mathbf{Q}(\gamma) = E_1.$$

Similarly if ad/bc is a cube in $\mathbf{Q}$, then we may decompose $\mathscr{E}_1$ into two orbits and we get that the corresponding étale algebra is $\mathbf{Q}(\theta) \times \mathbf{Q} = E_1$. The proofs for $\mathscr{E}_2$ and $\mathscr{E}_3$ are similar. $\square$

Lemma 3.4. *There exists an exact sequence of* $\mathrm{Gal}(K/\mathbf{Q})$ *modules*

$$0 \to \mathbf{Q}^2 \to \mathbf{Q}[\mathscr{E}] \to \mathrm{Pic}(\overline{V}) \otimes_{\mathbf{Z}} \mathbf{Q} \to 0.$$

Proof. By (3.6) and (3.7), we have in $\mathrm{Pic}(\overline{V})$ the relations

$$\begin{aligned}
e_0^1 &= [E_1] + [E_2] + [E_3],\\
e_1^1 &= 6[\Lambda] - 2[E_1] - 2[E_2] - 2[E_3] - 3[E_4] - 3[E_5] - 3[E_6],\\
e_2^1 &= 3[\Lambda] - 2[E_1] - 2[E_2] - 2[E_3],\\
e_0^2 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] - 2[E_4] + [E_5] - 2[E_6],\\
e_1^2 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] - 2[E_4] - 2[E_5] + [E_6],\\
e_2^2 &= 3[\Lambda] - [E_1] - [E_2] - [E_3] + [E_4] - 2[E_5] - 2[E_6],\\
e_0^3 &= 3[\Lambda] - 3[E_1] - [E_4] - [E_5] - [E_6],\\
e_1^3 &= 3[\Lambda] - 3[E_2] - [E_4] - [E_5] - [E_6],\\
e_2^3 &= 3[\Lambda] - 3[E_3] - [E_4] - [E_5] - [E_6]
\end{aligned}$$

which proves that the natural projection from $\mathbf{Q}[\mathscr{E}]$ to $\mathrm{Pic}(\overline{V})\otimes_{\mathbf{Z}}\mathbf{Q}$ is surjective. Moreover one has the relations

$$3\omega_V^{-1} = \sum_{x\in\mathscr{E}_1} x = \sum_{x\in\mathscr{E}_2} x = \sum_{x\in\mathscr{E}_3} x,$$

which gives a morphism of $\mathrm{Gal}(K/\mathbf{Q})$-modules

$$\mathbf{Q}^2 \to \mathbf{Q}[\mathscr{E}]$$

and the exact sequence of the lemma. □

Notations 3.5. For any prime p and any finite field extension F of $\mathbf{Q}$, we consider the local factor $\zeta_{F,p}$ of the function ζ_F at p which is defined by

$$\zeta_{F,p}(s) = \prod_{\{\, v \in \mathrm{Val}(F) \,|\, v|p \,\}} (1 - \sharp \mathbf{F}_v^{-s})^{-1}$$

If F is an étale algebra over $\mathbf{Q}$ and $F = \prod_{i \in I} F_i$ its decomposition in fields. Then we put

$$\zeta_F(s) = \prod_{i \in I} \zeta_{F_i}(s) \quad \text{and} \quad \zeta_{F,p}(s) = \prod_{i \in I} \zeta_{F_i,p}(s)$$

For any prime p, we denote by $\nu_F(p)$ the number of components of $F \otimes_{\mathbf{Q}} \mathbf{Q}_p$ of degree one over $\mathbf{Q}_p$.

Proposition 3.6. *With notation as above, for any prime p not in S, one has*

(i) $L_p(s, \mathrm{Pic}(\overline{V})) = \dfrac{\zeta_{E,p}(s)}{\zeta_{\mathbf{Q},p}(s)^2}$,

(ii) $\mathrm{Tr}(\widetilde{\mathrm{Fr}}_p \mid \mathrm{Pic}(\overline{V})) = \nu_E(p) - 2$.

Proof. By lemma 3.4, we have

$$L_p(s, \mathrm{Pic}(\overline{V})) = \frac{L_p(s, \mathbf{Q}[\mathscr{E}])}{L_p(s, \mathbf{Q})^2}$$

Thus it is enough to prove that if E is an arbitrary étale algebra over $\mathbf{Q}$ corresponding to a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-set $\mathscr{E}$ and if p is a prime such that $E/\mathbf{Q}$ is not ramified at p, then

$$\zeta_{E,p}(s) = L_p(s, \mathbf{Q}[\mathscr{E}]).$$

This well-known assertion follows from the fact the the components of $E \otimes \mathbf{Q}_p$ are in one to one correspondance with the orbits of Fr_p in $\mathscr{E}$, and the degree of each component is the length of the corresponding orbit. This proves (i).

But this also shows that

$$\mathrm{Tr}(\widetilde{\mathrm{Fr}}_p \mid \mathbf{Q}[\mathscr{E}]) = \nu_E(p)$$

which implies (ii). □

Remark 3.7. This implies that the factor λ'_p defined in proposition 5.1 in [PT] coincide with $L_p(1, \mathrm{Pic}(\overline{V}))$ at the good places as was suggested by the referee of that paper.

4. Euler product for the good places

We first need to compute the number of solutions of (3.1) modulo p for any prime not in S.

Proposition 4.1. *For any prime p not in S, one has*

$$\frac{\sharp V(\mathbf{F}_p)}{p^2} = 1 + \frac{\nu_E(p) - 2}{p} + \frac{1}{p^2}$$

where E is the étale algebra defined in §3.

Proof. By a result of Weil (see [Man2, theorem 23.1]), one has

$$\sharp V(\mathbf{F}_p) = 1 + \operatorname{Tr}(\operatorname{Fr}_p \mid \operatorname{Pic}(\overline{V}))p + p^2$$

and it follows from proposition 3.6 that

$$\operatorname{Tr}(\operatorname{Fr}_p \mid \operatorname{Pic}(\overline{V})) = \nu_E(p) - 2. \quad \square$$

Remark 4.2. In fact, we could also prove this result directly as in [PT]. Let $N(p)$ be the number of solutions of (3.1) in $\mathbf{F}_p^4$. We have

$$\sharp V(\mathbf{F}_p) = \frac{N(p)-1}{p-1}.$$

By [IR, §8.7 theorem 5], one has

$$N(p) = p^3 + \sum \overline{\chi}_1(a)\overline{\chi}_2(b)\overline{\chi}_3(c)\overline{\chi}_4(d)J_0(\chi_1,\chi_2,\chi_3,\chi_4)$$

where the sum is taken over all quadruples $(\chi_1,\dots,\chi_4)$ of nontrivial cubic characters from $\mathbf{F}_p^*$ to $\mathbf{C}^*$ such that $\chi_1\chi_2\chi_3\chi_4 = 1$ and where

$$J_0(\chi_1,\chi_2,\chi_3,\chi_4) = \sum_{t_1+\dots+t_4=0}\prod_{i=1}^{4}\chi_i(t_i)$$

the characters being extended by $\chi_i(0) = 0$. if $p \equiv 2 \bmod 3$ there are no nontrivial character and the formula is obvious. Otherwise there are exactly two non nontrivial χ and $\overline{\chi}$ which are conjugate. By [PT, proof of prop. 4.1], we have

$$J_0(\chi_1,\chi_2,\chi_3,\chi_4) = p(p-1)$$

and

$$\sharp V(\mathbf{F}_p) = 1 + p(1 + \sum \chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d)) + p^2$$

where the sum is taken over the same quadruples as above and is given by the formula

$$\sum \chi_1(a)\chi_2(b)\chi_3(c)\chi_4(d) = \chi\left(\frac{ab}{cd}\right) + \overline{\chi}\left(\frac{ab}{cd}\right) + \chi\left(\frac{ac}{bd}\right) + \overline{\chi}\left(\frac{ac}{bd}\right) + \chi\left(\frac{ad}{bc}\right) + \overline{\chi}\left(\frac{ad}{bc}\right)$$

which implies the result.

Notations 4.3. For any place v of $\mathbf{Q}$, we put

$$\lambda_v = \begin{cases} \frac{\zeta_{E,v}(s)}{\zeta_{\mathbf{Q},v}(s)^2} & \text{if } v \text{ is finite,} \\ 1 & \text{otherwise.} \end{cases}$$

Remark 4.4. By proposition 3.6, $\lambda_p = L_p(1, \operatorname{Pic}(\overline{V}))$ if $p \in \operatorname{Val}(\mathbf{Q}) - S$. Thus the Tamagawa measure $\boldsymbol{\omega}_H$ is given by the formula

$$\boldsymbol{\omega}_H = \lim_{s\to 1}(s-1)^{\operatorname{rk}\operatorname{Pic}(V)}\left(\frac{\zeta_E(s)}{\zeta_{\mathbf{Q}}(s)^2}\right) \times \prod_{v\in\operatorname{Val}(\mathbf{Q})} \lambda_v^{-1}\boldsymbol{\omega}_{H,v}.$$

By lemmata 3.2 and 3.4 in [PT] and lemma 5.4.6 in [Pe], for any p in $\operatorname{Val}(\mathbf{Q}) - S$ one has

$$\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) = \frac{\sharp V(\mathbf{F}_p)}{p^2}$$

(See also [Pe, lemma 2.2.1] and [PT, remark 5.2]). Therefore the local factor at a good place p is given by

$$\begin{array}{ll}
\left(1-\frac{1}{p}\right)^7\left(1+\frac{7}{p}+\frac{1}{p^2}\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } \nu_E(p)=9\\
\left(1-\frac{1}{p}\right)^4\left(1-\frac{1}{p^3}\right)\left(1+\frac{4}{p}+\frac{1}{p^2}\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } \nu_E(p)=6\\
\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)^2\left(1+\frac{1}{p}+\frac{1}{p^2}\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } \nu_E(p)=3\\
\left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{1}{p^3}\right)^3\left(1-\frac{2}{p}+\frac{1}{p^2}\right) & \text{if } p\equiv 1 \bmod 3 \text{ and } \nu_E(p)=0\\
\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^3\left(1+\frac{1}{p}+\frac{1}{p^2}\right) & \text{if } p\equiv 2 \bmod 3.
\end{array}$$

and we get for the good places the factors C_0, C_1, C_2 and C_3 where

$$C_0 = \prod_{\substack{p\nmid 3abcd,\\ p\equiv 2 \bmod 3.}} \left(1-\frac{1}{p^3}\right)\left(1-\frac{1}{p^2}\right)^3,$$

$$C_1 = \prod_{\substack{p\nmid 3abcd,\\ p\equiv 1 \bmod 3,\\ \nu_E(p)=9.}} \left(1-\frac{1}{p}\right)^7\left(1+\frac{7}{p}+\frac{1}{p^2}\right),$$

$$C_2 = \prod_{\substack{p\nmid 3abcd,\\ p\equiv 1 \bmod 3,\\ \nu_E(p)=6.}} \left(1-\frac{1}{p^3}\right)\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right),$$

$$C_3 = \prod_{\substack{p\nmid 3abcd,\\ p\equiv 1 \bmod 3,\\ \nu_E(p)=0 \text{ or } 3.}} \left(1-\frac{1}{p^3}\right)^3.$$

These products converge quickly and are easy to approximate.

5. Density at the bad places

In this section we only consider the cases of cubic surfaces with equations of the form

$$X^3+Y^3+qZ^3+q^2T^3=0 \tag{5.1}$$

with q prime and

$$aX^3+aY^3+qZ^3+qT^3=0 \tag{5.2}$$

with q prime and a a prime to q integer.

Notations 5.1. If V is defined by the equation (3.1), and p is a prime, then we consider

$$N^*(p^r)=\sharp\{(x,y,z,t)\in(\mathbf{Z}/p^r\mathbf{Z})^4-(p\mathbf{Z}/p^r\mathbf{Z})^4 \mid ax^3+by^3+cz^3+dt^3=0 \text{ in } \mathbf{Z}/p^r\mathbf{Z}\}$$

Remark 5.2. By [PT, lemmata 3.2 and 3.4], there is an explicit integer r_0 such that

$$\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p))=\frac{1}{1-p^{-1}}\times\frac{N^*(p^{r_0})}{p^{3r_0}}.$$

If $p=3$ and $3\nmid abcd$, then a direct computation in $(\mathbf{Z}/9\mathbf{Z})^4$ gives the value of $N^*(9)$ and thus of $\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p))$. therefore in the following lemma, we restrict ourselves to the case where V is given by (5.1) or (5.2) and $p=q$.

Lemma 5.3. *If V is given by the equation*

$$X^3+Y^3+pZ^3+p^2T^3=0,$$

then for $r \geqslant 2$,

$$\frac{N^*(p^r)}{p^{3r}}=\begin{cases}1-\frac{1}{p} & \text{if } p\equiv 2 \text{ mod } 3,\\ 3\left(1-\frac{1}{p}\right) & \text{if } p\equiv 1 \text{ mod } 3,\\ \frac{2}{3} & \text{if } p=3.\end{cases}$$

If v is given by the equation

$$aX^3+aY^3+pZ^3+pT^3=0,$$

with $p \not| a$ then for $r \geqslant 3$,

$$\frac{N^*(p^r)}{p^{3r}}=\begin{cases}1-\frac{1}{p^2} & \text{if } p\equiv 2 \text{ mod } 3,\\ 3\left(1-\frac{1}{p^2}\right) & \text{if } p\equiv 1 \text{ mod } 3,\\ \frac{4}{3} & \text{if } p=3.\end{cases}$$

Remark 5.4. This lemma implies that if V is given by the first equation then the local factor at p is given by

$$\lambda_p\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p))=\begin{cases}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p}\right) & \text{if } p\equiv 2 \text{ mod } 3,\\ 3\left(1-\frac{1}{p}\right)^3 & \text{if } p\equiv 1 \text{ mod } 3,\\ \frac{4}{9} & \text{if } p=3,\end{cases}$$

and if V is given by the second one then this factor is

$$\lambda_p\boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p))=\begin{cases}\left(1-\frac{1}{p^2}\right)^3 & \text{if } p\equiv 2 \text{ mod } 3,\\ 3\left(1-\frac{1}{p}\right)^4\left(1-\frac{1}{p^2}\right) & \text{if } p\equiv 1 \text{ mod } 3,\\ \frac{16}{27} & \text{if } p=3.\end{cases}$$

Proof. Let us consider the set of quadruples (x,y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z})^4-(p\mathbf{Z}/p^r\mathbf{Z})^4$ such that

$$x^3+y^3+pz^3+p^2t^3=0 \qquad \text{in } \mathbf{Z}/p^r\mathbf{Z}. \tag{5.3}$$

If $p\,|\,x$ then $p\,|\,y$, $p\,|\,z$ and $p\,|\,t$. Therefore, for any (x,y,z,t) as above, $p \not| x$ and $p \not| y$. But for any triples (y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z}-p\mathbf{Z}/p^r\mathbf{Z})\times(\mathbf{Z}/p^r\mathbf{Z})^2$, there exists exactly one x verifying (5.3) if $p\equiv 2$ mod 3 and exactly three of them if $p\equiv 1$ mod 3. If $p=3$ and y belongs to $\mathbf{Z}/3^r\mathbf{Z}-3\mathbf{Z}/3^r\mathbf{Z}$ then (5.3) implies that $3\,|\,z$ For any triple (y,z,t) with y in $(\mathbf{Z}/3^r\mathbf{Z})-(3\mathbf{Z}/3^r\mathbf{Z})$, z in $(3\mathbf{Z}/3^r\mathbf{Z})$ and t in $(\mathbf{Z}/3^r\mathbf{Z})$ there exists exactly three x in $\mathbf{Z}/3^r\mathbf{Z}$ which satisfy (5.3). We get that

$$\frac{N^*(p^r)}{p^{3r}}=\begin{cases}\frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}}=1-\frac{1}{p} & \text{if } p\equiv 2 \text{ mod } 3,\\ 3\frac{(p-1)p^{r-1}\times p^r\times p^r}{p^{3r}}=3\left(1-\frac{1}{p}\right) & \text{if } p\equiv 1 \text{ mod } 3,\\ 3\frac{2\times 3^{r-1}\times 3^{r-1}\times 3^r}{3^{3r}}=\frac{2}{3} & \text{if } p=3,\end{cases}$$

Let us now turn to the set of (x,y,z,t) in $(\mathbf{Z}/p^r\mathbf{Z})^4-(p\mathbf{Z}/p^r\mathbf{Z})^4$ such that

$$ax^3+ay^3+pz^3+pt^3=0$$

We decompose this set in two as follows

$$N_1^*(p^r) = \sharp\left\{ (x,y,z,t) \in (\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4 \,\middle|\, \begin{cases} p \nmid x, \\ ax^3 + ay^3 + pz^3 + pt^3 = 0. \end{cases} \right\}$$

$$N_2^*(p^r) = \sharp\left\{ (x,y,z,t) \in (\mathbf{Z}/p^r\mathbf{Z})^4 - (p\mathbf{Z}/p^r\mathbf{Z})^4 \,\middle|\, \begin{cases} p|x, \quad p \nmid z, \\ ax^3 + ay^3 + pz^3 + pt^3 = 0. \end{cases} \right\}$$

As above we have the formula

$$\frac{N_1^*(p^r)}{p^{3r}} = \begin{cases} \frac{(p-1)p^{r-1}\times p^r \times p^r}{p^{3r}} = 1 - \frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3 \times \frac{(p-1)p^{r-1}\times p^r \times p^r}{p^{3r}} = 3\Big(1 - \frac{1}{p}\Big) & \text{if } p \equiv 1 \bmod 3, \\ 3\frac{2\times 3^{r-1}\times 3^{r-1}\times 3^r}{3^{3r}} = \frac{2}{3} & \text{if } p = 3, \end{cases}$$

where for $p = 3$ we use the equality

$$3^{r-1} \times 3^r = \sharp\{\, (z,t) \in (\mathbf{Z}/3^r\mathbf{Z})^2 \mid z^3 \equiv t^3 \bmod 3 \,\}.$$

On the other hand

$$N_2^*(p^r) = p^2 \left\{ (x,y,z,t) \in (\mathbf{Z}/p^{r-1}\mathbf{Z})^4 \,\middle|\, \begin{cases} p \nmid z \\ ap^2x^3 + ap^2y^3 + z^3 + t^3 = 0. \end{cases} \right\}$$

and

$$\frac{N_2^*(p^r)}{p^{3r}} = \frac{p^2}{p^3} \times \begin{cases} \frac{(p-1)p^{r-2}\times p^{r-1}\times p^{r-1}}{p^{3(r-1)}} = 1 - \frac{1}{p} & \text{if } p \equiv 2 \bmod 3, \\ 3\frac{(p-1)p^{r-2}\times p^{r-1}\times p^{r-1}}{p^{3(r-1)}} = 3\Big(1 - \frac{1}{p}\Big) & \text{if } p \equiv 2 \bmod 3, \\ 3\frac{2\times 3^{r-2}\times 3^{r-1}\times 3^{r-1}}{3^{3(r-1)}} = 2 & \text{if } p = 3. \end{cases}$$

And we get the result:

$$\frac{N^*(p^r)}{p^{3r}} = \begin{cases} 1 - \frac{1}{p} + \frac{1}{p} - \frac{1}{p^2} = 1 - \frac{1}{p^2} & \text{if } p \equiv 2 \bmod 3, \\ 3\Big(1 - \frac{1}{p^2}\Big) & \text{if } p \equiv 1 \bmod 3, \\ \frac{2}{3} + \frac{2}{3} = \frac{4}{3} & \text{if } p = 3. \quad \square \end{cases}$$

6. THE CONSTANT $\alpha(V)$

Since the cubic surfaces we consider in this paper are $\mathbf{Q}$-rational, it only remains to compute the rank t of the Picard group and the value of $\alpha(V)$.

Proposition 6.1. *If V is given by the equation*

$$X^3 + Y^3 + aZ^3 + aT^3 = 0 \tag{6.1}$$

where a is not a cube in $\mathbf{Q}$ then $\operatorname{rk}\operatorname{Pic}(V) = 2$ *and* $\alpha(V) = 2$.

If V is given by the equation

$$aX^3 + aY^3 + bZ^3 + bT^3 = 0 \tag{6.2}$$

where a and b are strictly positive integers and b/a is not a cube in $\mathbf{Q}$, then $\operatorname{rk}\operatorname{Pic}(V) = 3$ *and* $\alpha(V) = 1$.

if V is given by the equation

$$X^3 + Y^3 + Z^3 + T^3 = 0 \tag{6.3}$$

then $\operatorname{rk}\operatorname{Pic}(V) = 4$ *and* $\alpha(V) = 7/18$.

Proof. To compute $\alpha(V)$ we shall use its original definition [Pe, §2], namely

$$\alpha(V) = \mathrm{Vol}\{\, x \in \Lambda_{\mathrm{eff}}(V) \mid \langle \omega_V^{-1}, x\rangle = 1 \,\}$$

where the Lebesgue measure on the affine hyperplane

$$\mathscr{H}(\lambda) = \{\, x \in \mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R} \mid \langle \omega_V^{-1}, x\rangle = \lambda \,\}$$

is defined by the $(t-1)$-form $\mathrm{d}\boldsymbol{x}$ which is characterized by the relation

$$\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\omega_V^{-1} = \mathrm{d}\boldsymbol{y}$$

where $\mathrm{d}\omega_V^{-1}$ is the linear form defined by ω_V^{-1} on $\mathrm{Pic}(V)^\vee$ and $\mathrm{d}\boldsymbol{y}$ is the form corresponding to the natural Lebesgue measure on $\mathrm{Pic}(V)^\vee \otimes_{\mathbf{Z}} \mathbf{R}$. More explicitely, if $(e_1, \dots, e_t)$ is a basis of $\mathrm{Pic}(V)$ and $(e_1^\vee, \dots, e_t^\vee)$ the dual basis, if

$$\omega_V^{-1} = \sum_{i=1}^{t} \lambda_i e_i$$

with $\lambda_t \neq 0$, then let $f_1, \dots, f_{t-1}$ be the projection of $e_1^\vee, \dots, e_{t-1}^\vee$ on $\mathscr{H}(0)$ along $e_t^\vee$, then

$$\mathrm{d}\boldsymbol{x} = \frac{1}{\lambda_t} \mathrm{d}f_1^\vee \wedge \dots \wedge \mathrm{d}f_{t-1}^\vee .$$

If V is given by the equation (6.1), then $\mathrm{Gal}(K/\mathbf{Q})$ may be described as

$$\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$$

and the orbits of the Galois action on the 27 lines are

$$\begin{aligned}
O_1 &= \{L(0), L'(0), L''(0)\},\\
O_2 &= \{L(1), L(2), L'(1), L'(2), L''(1), L''(2)\},\\
O_3 &= \{M(0), M(2), M'(0), M'(1), M''(1), M''(2)\},\\
O_4 &= \{M(1), M'(2), M''(0)\},\\
O_5 &= \{N(0), N(1), N'(1), N'(2), N''(0), N''(2)\},\\
O_6 &= \{N(2), N'(0), N''(1)\}.
\end{aligned}$$

In the basis $([\Lambda], [E_1], \dots, [E_6])$, a basis of $\mathrm{Pic}(V) = (\mathrm{Pic}\, \overline{V})^{\mathrm{Gal}(K:\mathbf{Q})}$ is given by

$$e_1 = \omega_V^{-1}, \qquad e_2 = [E_4] - 2[E_5] + [E_6].$$

The effective cone $\Lambda_{\mathrm{eff}}(V)$ is generated by the classes $[O_i] = \sum_{x \in O_i} x$, which in the basis (e_0, e_1) are given by

$$\begin{aligned}
&[O_1] = e_1, && [O_2] = 2e_1, && [O_3] = 2e_1 + e_2,\\
&[O_4] = e_1 - e_2, && [O_5] = 2e_1 - e_2, && [O_6] = e_1 + e_2.
\end{aligned}$$

Therefore this cone is generated by the elements $e_1 - e_2$ and $e_1 + e_2$ and $\alpha(V)$ is given as the volume of the domain

$$x = 1, \quad x + y > 0 \quad \text{and} \quad x - y > 0$$

that is as the volume of the segment $[-1, 1]$ and $\alpha(V) = 2$.

If V is given by the equation (6.2) then $\mathrm{Gal}(K/\mathbf{Q})$ is also isomorphic to

$$\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$$

and the orbits of the Galois action on the 27 lines are

$$\begin{aligned}
O_1 &= \{L(0)\},\\
O_2 &= \{L(1), L(2)\},\\
O_3 &= \{L'(0), L''(0)\},\\
O_4 &= \{L'(1), L''(2)\},\\
O_5 &= \{L'(2), L''(1)\},\\
O_6 &= \{M(0), M(1), M(2), M'(0), M'(1), M'(2)\},\\
O_7 &= \{M''(0), M''(1), M''(2)\},\\
O_8 &= \{N(0), N(1), N(2), N''(0), N''(1), N''(2)\},\\
O_9 &= \{N'(0), N'(1), N'(2)\}.
\end{aligned}$$

A basis of $\mathrm{Pic}(V)$ is given by

$$e_1 = \omega_V^{-1}, \quad e_2 = [E_1], \quad e_3 = [E_2] + [E_3],$$

and the cone $\Lambda_{\text{eff}}(V)$ is generated by

$$\begin{array}{lll}
[O_1] = e_2, & [O_2] = e_3, & [O_3] = e_1 - e_2,\\
[O_4] = e_1 + e_2 - e_3, & [O_5] = e_1 - e_2, & [O_6] = e_1 + e_2 + e_3,\\
[O_7] = 2e_1 - e_2 - e_3, & [O_8] = 2e_1 + 2e_2 - e_3, & [O_9] = e_1 - 2e_2 + e_3.
\end{array}$$

that is by

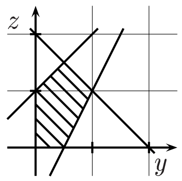
$$e_2, \quad e_3, \quad e_1 + e_2 - e_3, \quad 2e_1 - e_2 - e_3, \quad e_1 - 2e_2 + e_3$$

since $3[O_3] = [O_7] + [O_9]$ and $\alpha(V)$ is the volume of the domain given by

$$\begin{cases} x = 1,\, y > 0,\, z > 0,\\ x + y - z > 0,\\ 2x - y - z > 0,\\ x - 2y + z > 0. \end{cases}$$

that is of the domain

$$\begin{cases} 0 < y,\, 0 < z,\\ z - y < 1,\\ y + z < 2,\\ 2y - z < 1. \end{cases}$$



Therefore $\alpha(V) = 1$.

If V is given by the equation (6.3), then $\mathrm{Gal}(K/\mathbf{Q})$ is $\mathbf{Z}/2\mathbf{Z}$ and the orbits of the Galois action on the 27 lines are given by

$$\begin{array}{lll}
O_1 = \{L(0)\}, & O_2 = \{L(1), L(2)\}, & O_3 = \{L'(0), L''(0)\}, \\
O_4 = \{L'(1), L''(2)\}, & O_5 = \{L'(2), L''(1)\}, & \\
O_6 = \{M(0), M'(0)\}, & O_7 = \{M(1), M'(2)\}, & O_8 = \{M(2), M'(1)\}, \\
O_9 = \{M''(0)\}, & O_{10} = \{M''(1), M''(2)\}, & \\
O_{11} = \{N(0), N''(0)\}, & O_{12} = \{N(1), N''(2)\}, & O_{13} = \{N(2), N''(1)\}, \\
O_{14} = \{N'(0)\}, & O_{15} = \{N'(1), N'(2)\}. &
\end{array}$$

A basis of the Picard group is given by

$$e_1 = [\Lambda] - [E_5], \quad e_2 = [E_1], \quad e_3 = [E_2] + [E_3], \quad e_4 = [E_4] - 2[E_5] + [E_6].$$

The effective cone $\Lambda_{\mathrm{eff}}(V)$ is generated by

$$\begin{array}{ll}
[O_1] = e_2, & [O_2] = e_3, \\
[O_3] = 3e_1 - 2e_2 - e_3 - e_4, & [O_4] = 3e_1 - 2e_3 - e_4, \\
[O_5] = 3e_1 - 2e_2 - e_3 - e_4, & [O_6] = e_1, \\
[O_7] = e_1 - e_4, & [O_8] = e_1, \\
[O_9] = 2e_1 - e_2 - e_3 - e_4, & [O_{10}] = 4e_1 - 2e_2 - 2e_3 - e_4, \\
[O_{11}] = 2e_1 - e_3 - e_4, & [O_{12}] = 2e_1 - e_3 - e_4, \\
[O_{13}] = 2e_1 - e_3, & [O_{14}] = e_1 - e_2, \\
[O_{15}] = 2e_1 - 2e_2 - e_4. &
\end{array}$$

Since $[O_3] = [O_5] = [O_9] + [O_{14}]$ and $[O_{11}] = [O_{12}] = [O_9] + [O_2]$, we get that $\Lambda_{\mathrm{eff}}(V)$ is generated by

$$\begin{gathered}
e_2, \quad e_3, \quad 3e_1 - 2e_3 - e_4, \quad e_1 - e_4, \quad 2e_1 - e_2 - e_3 - e_4, \\
4e_1 - 2e_2 - 2e_3 - e_4, \quad 2e_1 - e_3, \quad e_1 - e_2, \quad 2e_1 - 2e_2 - e_4.
\end{gathered}$$

Moreover the anticanonical sheaf is given by

$$\omega_V^{-1} = 3e_1 - e_2 - e_3 - e_4.$$

Thus $\alpha(V)$ is the volume of the domain

$$\begin{cases}
3x - y - z - t = 1, \\
y > 0, \quad z > 0, \\
x - y > 0, \\
2x - z > 0, \\
x - t > 0, \\
3x - 2z - t > 0, \\
2x - y - z - t > 0, \\
4x - 2y - 2z - t > 0, \\
2x - 2y - t > 0.
\end{cases}$$

That is of the domain P of $\mathbf{R}^3$ given by

$$\begin{cases} y > 0, \quad z > 0, \\ x - y > 0, \\ 2x - z > 0, \\ 1 - 2x + y + z > 0, \\ 1 + y - z > 0, \\ 1 - x > 0, \\ 1 + x - y - z > 0, \\ 1 - x - y + z > 0. \end{cases}$$

To compute this we use the following general method: we decompose P into cones with appex $(0,0,0)$ and supported by the faces not containing this point. So we have to consider the following faces of P:

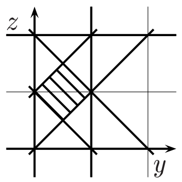
$$\begin{array}{llll} F_1: & 1 - x = 0, & F_2: & 1 - 2x + y + z = 0, \\ F_3: & 1 + y - z = 0, & F_4: & 1 + x - y - z = 0, \\ F_5: & 1 - x - y + z = 0. & & \end{array}$$

More precisely, one has

$$\alpha(V) = \mathrm{Vol}(P) = \frac{1}{3} \sum_{i=1}^{5} \mathrm{Area}(F_i).$$

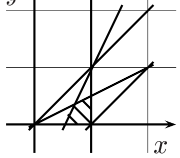
But the area of F_1 is the volume of the domain

$$\begin{cases} y > 0, \quad z > 0, \\ 1 - y > 0, \\ 2 - z > 0, \\ -1 + y + z > 0, \\ 1 + y - z > 0, \\ 2 - y - z > 0, \\ z - y > 0. \end{cases}$$



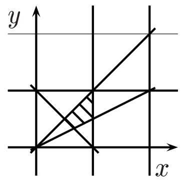
we get that $\mathrm{Area}(F_1) = \frac{1}{2}$. For F_2 we get the equations

$$\begin{cases} y > 0, \\ -1 + 2x - y > 0, \\ x - y > 0, \\ 1 + y > 0, \\ 2 - 2x + 2y > 0, \\ 1 - x > 0, \\ 2 - x > 0, \\ x - 2y > 0. \end{cases}$$



We get $\mathrm{Area}(F_2) = \frac{1}{6}$. For F_3 we get the same equations and the same area. For F_4 we get the equations

$$\begin{cases} y > 0, \\ 1 + x - y > 0, \\ x - y > 0, \\ -1 + x + y > 0, \\ 2 - x > 0, \\ -x + 2y > 0, \\ 1 - x > 0, \\ 2 - 2y > 0. \end{cases}$$



We find that $\mathrm{Area}(F_4) = 1/8 + 1/24 = 1/6$. The face F_5 is given by the same equations and $\mathrm{Area}(F_5) = 1/6$. Finally

$$\alpha(V) = \frac{1}{3}\Big(\frac{1}{2} + \frac{4}{6}\Big) = \frac{7}{18}. \quad \square$$

7. Trivial statistical formulae

With the program of D. J. Bernstein [Be] it is easy to get in addition to the value of $N_{U,H}(B_0)$ for some fixed B_0 the value of $N_{U,H}(2^r)$ for $2^r \leqslant B_0$. Thus we get a family of pairs $(B_i, N_{U,H}(B_i))_{1 \leqslant i \leqslant N}$. But conjecturally the asymptotic behavior of $N_{U,H}(B)$ is expected to be of the form

$$N_{U,H}(B) = BP(\log B) + o(B)$$

where P is a polynomial of degree $t - 1$ with a dominant coefficient equal to $\boldsymbol{\theta}_H(V)$. For any i between 1 and N, let

$$x_i = \log(B_i) \quad \text{and} \quad y_i = N_{U,H}(B_i)/B_i$$

then using statistics one may look for a polynomial Q of degree $t - 1$ such that

$$\sum_{i=1}^{N} (Q(x_i) - y_i)^2$$

is minimal. We recall below the formulae for the leading coefficient of the polynomial Q which will be compared with $\boldsymbol{\theta}_H(V)$

Notations 7.1. If $R(X, Y)$ is a polynomial in $\mathbf{Q}[X, Y]$, we denote by $\langle R(X, Y)\rangle$ the mean value of $(R(x_i, y_i))_{1 \leqslant i \leqslant N}$, that is

$$\langle R(X,Y)\rangle = \frac{1}{N} \sum_{i=1}^{N} R(x_i, y_i).$$

If $t = 2$ the leading coefficient of Q if it is uniquely defined is given by

$$A_1 = \frac{\langle XY\rangle - \langle Y\rangle\langle X\rangle}{\langle X^2\rangle - \langle X\rangle^2}$$

if $t = 3$ the leading coefficient is

$$A_2 = \frac{\langle YX^2\rangle - \langle Y\rangle\langle X^2\rangle - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)(\langle YX\rangle - \langle Y\rangle\langle X\rangle)}{\langle X^2\rangle - \langle X\rangle^2}}{\langle X^4\rangle - \langle X^2\rangle^2 - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2}}$$

If $t = 4$, the leading coefficient is

$$A_3 = \frac{\langle YX^3\rangle - \langle Y\rangle\langle X^3\rangle - \frac{(\langle X^4\rangle - \langle X\rangle\langle X^3\rangle)(\langle YX\rangle - \langle Y\rangle\langle X\rangle)}{\langle X^2\rangle - \langle X\rangle^2} - \frac{\beta\delta}{\gamma}}{\langle X^6\rangle - \langle X^3\rangle^2 - \frac{(\langle X^4\rangle - \langle X\rangle\langle X^3\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2} - \frac{\beta^2}{\gamma}}$$

with

$$\beta = \langle X^5\rangle - \langle X^3\rangle\langle X^2\rangle - \frac{\langle X^3\rangle - \langle X\rangle\langle X^2\rangle}{\langle X^2\rangle - \langle X\rangle^2}(\langle X^4\rangle - \langle X^3\rangle\langle X\rangle)$$

$$\gamma = \langle X^4\rangle - \langle X^2\rangle^2 - \frac{(\langle X^3\rangle - \langle X\rangle\langle X^2\rangle)^2}{\langle X^2\rangle - \langle X\rangle^2}$$

$$\delta = \langle YX^2\rangle - \langle Y\rangle\langle X^2\rangle - \frac{\langle X^3\rangle - \langle X\rangle\langle X^2\rangle}{\langle X^2\rangle - \langle X\rangle^2}(\langle YX\rangle - \langle Y\rangle\langle X\rangle)$$

The coefficient A_{t-1} will be denoted by $\boldsymbol{\theta}_H^{\mathrm{stat}}(V)$ in the next section.

8. Presentation of the numerical tests

All the examples we consider are of the form (6.1), (6.2), or (6.3). By [CTKS, Lemme 1], the corresponding surface V is $\mathbf{Q}$-rational and, in particular, $\operatorname{Br} V = 0$. Thus the Brauer-Manin obstruction to weak approximation is void and

$$V(\boldsymbol{A}_{\mathbf{Q}})^{\mathrm{Br}} = V(\boldsymbol{A}_{\mathbf{Q}}) = \prod_{v \in \mathrm{Val}(\mathbf{Q})} V(\mathbf{Q}_v).$$

Moreover

$$\beta(V) = \sharp H^1(\mathbf{Q}, \mathrm{Pic}(\overline{V})) = 1.$$

By (2.2) and (2.3), the constant $\boldsymbol{\theta}_H(V)$ may be written as

$$\boldsymbol{\theta}_H(V) = \alpha(V)\boldsymbol{\omega}_H(V(\boldsymbol{A}_{\mathbf{Q}}))$$

Using remark 4.4 we get that

$$\boldsymbol{\theta}_H(V) = \alpha(V)\lim_{s\to 1}(s-1)^{t+2}\zeta_E(s) \times \boldsymbol{\omega}_{H,\infty}(V(\mathbf{R})) \times \prod_{p | 3abcd} \lambda_p \boldsymbol{\omega}_{H,p}(V(\mathbf{Q}_p)) \times \prod_{i=0}^{3} C_i,$$

where E is the étale algebra given in notation 3.2. The residue of the zeta function could have been computed directly (see, for example, [Co, chapter 4], but we chose instead to use the PARI system for this computation. The volume at the real place is given by the formula

$$\frac{1}{2}\int_{\left\{(x,y,z,t) \,\middle|\, \left\{\substack{ax^3+by^3+cz^3+dt^3=0 \\ \sup(|x|,|y|,|z|,|t|)\leqslant 1}\right.\right\}} \boldsymbol{\omega}_L(x,y,z,t)$$

where $\boldsymbol{\omega}_L$ is the Leray form which may be described by the formula

$$\boldsymbol{\omega}_L(x,y,z,t) = \frac{\sqrt[3]{d}}{3\sqrt[3]{ax^3+by^3+cz^3}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

By decomposing the domain of integration and using the various expressions of the Leray form, it is possible to remove the singularities of this integral which is then easy to estimate on a computer. The factors corresponding to the bad places have been described in section

5 and the constants C_0, C_1, C_2, and C_3 may be computed directly as described in section 4.

The examples we consider are the following ones: for the cubic surfaces with a Picard group of rank 2 we used

$$X^3 + Y^3 + 2Z^3 + 4T^3 = 0, \tag{S_1}$$

$$X^3 + Y^3 + 5Z^3 + 25T^3 = 0, \tag{S_2}$$

$$X^3 + Y^3 + 3Z^3 + 9T^3 = 0. \tag{S_3}$$

For the rank 3 case:

$$X^3 + Y^3 + 2Z^3 + 2T^3 = 0, \tag{S_4}$$

$$X^3 + Y^3 + 5Z^3 + 5T^3 = 0, \tag{S_5}$$

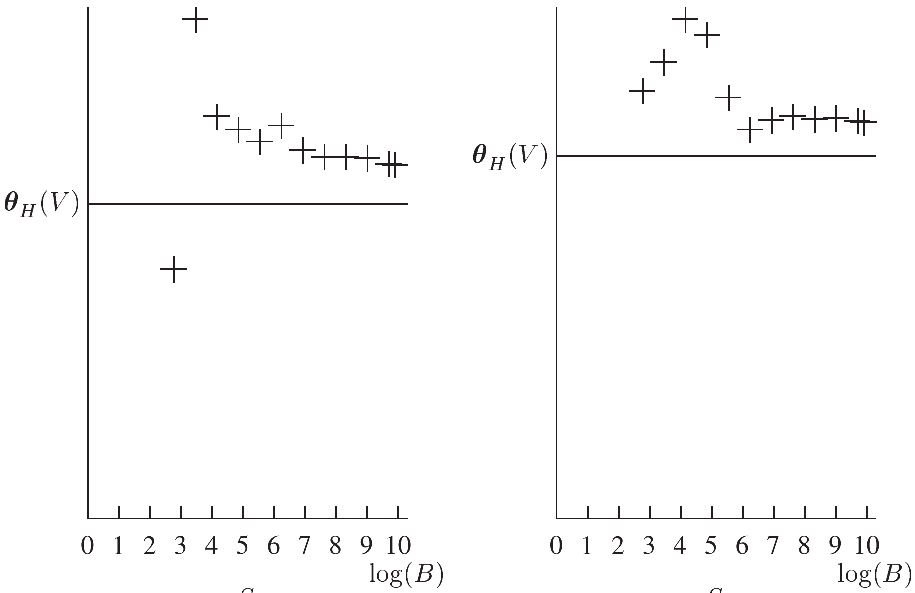
$$X^3 + Y^3 + 7Z^3 + 7T^3 = 0, \tag{S_6}$$

$$2X^3 + 2Y^3 + 3Z^3 + 3T^3 = 0, \tag{S_7}$$

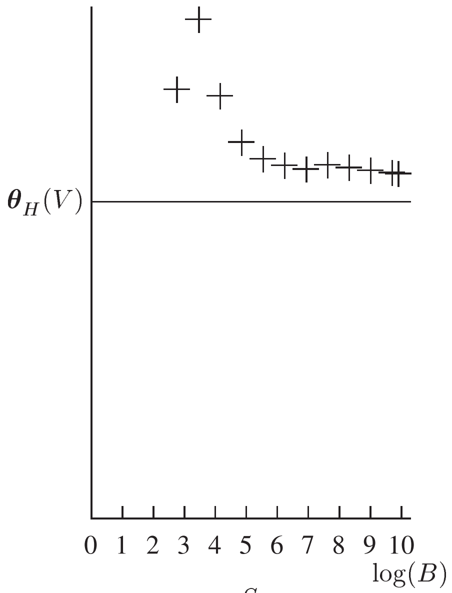
and for the rank 4:

$$X^3 + Y^3 + Z^3 + T^3 = 0. \tag{S_8}$$

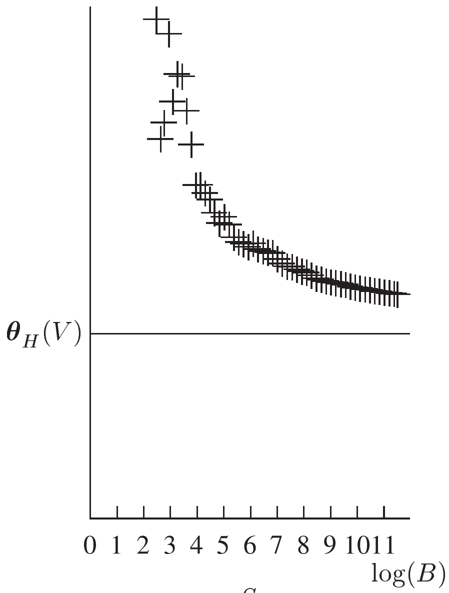
We draw below the corresponding experimental curves in which we compare the value of $N_{U,H}(B)/(B(\log B)^{t-1})$ with $\boldsymbol{\theta}_H(V)$.



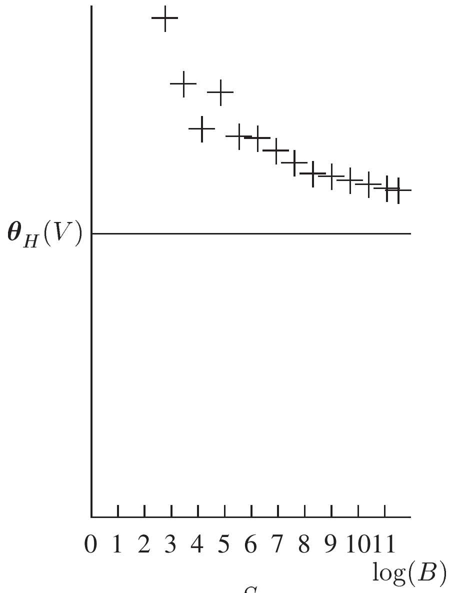
S_1 S_2


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 10
$\log(B)$

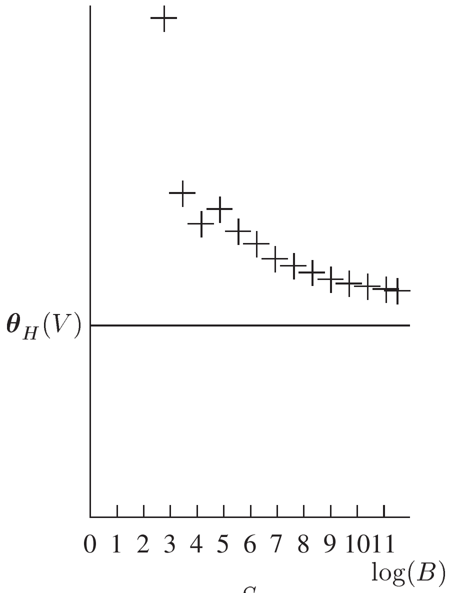
S_3


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 10 11
$\log(B)$

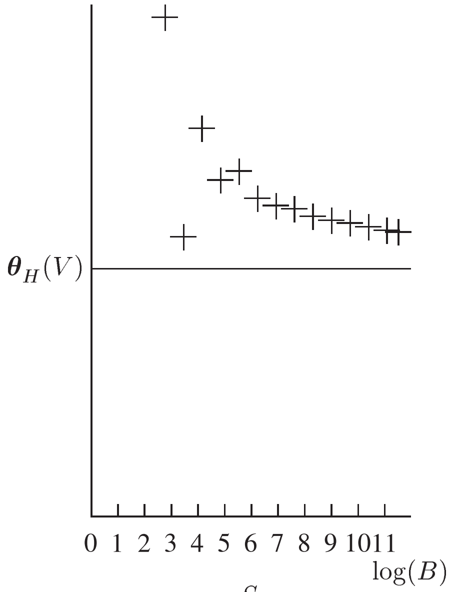
S_4


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 10 11
$\log(B)$

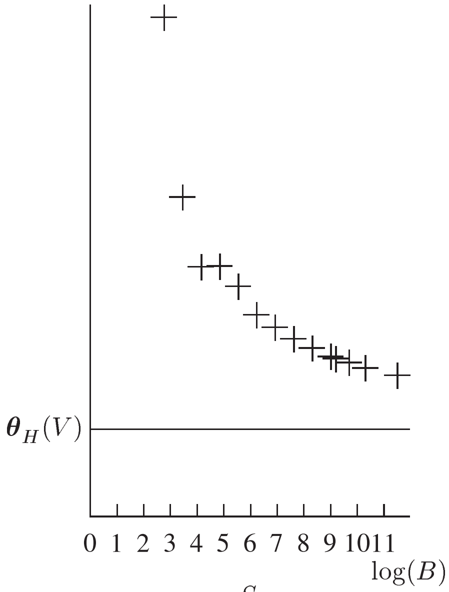
S_5


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 10 11
$\log(B)$

S_6


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

S_7


$\boldsymbol{\theta}_H(V)$
0 1 2 3 4 5 6 7 8 9 1011
$\log(B)$

S_8

We finish with tables of numerical results. The value of $\boldsymbol{\theta}_H^{\text{stat}}(V)$ is obtained from the pairs $(B_i, N_{U,H}(B_i))$ as described in section 7. We also denote by $\zeta^*_{E_i}(1)$ the limit

$$\zeta^*_{E_i}(1) = \lim_{s\to 1}(s-1)^{t_i}\zeta^*_{E_i}(s)$$

where t_i is the number of components of E_i. Note that for the examples with a Picard group of rank 2, C_2 is equal to 1.

Surface	S_1	S_2	S_3
B	20000	20000	20000
$N_{U,H}(B)$	75984	49608	78980
$\alpha(V)$	2	2	2
ad/cb	2	5	3
$\zeta^*_{E_1}(1)$	8.146241×10^{-1}	1.163730	1.017615
ab/cd	$1/8$	$1/125$	$1/27$
$\zeta^*_{E_2}(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
ac/bd	$1/2$	$1/5$	$1/3$
$\zeta^*_{E_3}(1)$	8.146241×10^{-1}	1.163730	1.017615
$\lambda_3\boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$4/9$	$4/9$	$4/9$
p_0	2	5	
$\lambda_{p_0}\boldsymbol{\omega}_H(V(\mathbf{Q}_{p_0}))$	$3/8$	$96/125$	
C_0	8.306815×10^{-1}	3.493824×10^{-1}	3.066383×10^{-1}
C_1	9.540383×10^{-1}	8.704106×10^{-1}	9.762028×10^{-1}
C_3	9.893865×10^{-1}	9.906098×10^{-1}	9.892790×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	3.255161	1.360417	2.221359
$\boldsymbol{\theta}_H(V)$	3.413500×10^{-1}	2.290769×10^{-1}	3.660885×10^{-1}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V)B\log(B)$	1.123839	1.093332	1.089213
$\boldsymbol{\theta}_H^{\text{stat}}(V)/\boldsymbol{\theta}_H(V)$	1.008178	0.958517	1.021637

For the examples with a Picard group of rank 3, C_3 is equal to 1.

Surface	S_4	S_5	S_6	S_7
B	100000	100000	100000	100000
$N_{U,H}(B)$	3051198	1976482	3420784	1966160
$\alpha(V)$	1	1	1	1
ad/bc	1	1	1	1
$\zeta^*_{E_1}(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
ab/cd	$1/4$	$1/25$	$1/49$	$4/9$
$\zeta^*_{E_2}(1)$	8.146241×10^{-1}	1.163730	1.265025	1.028996
ac/bd	1	1	1	1
$\zeta^*_{E_3}(1)$	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}	6.045998×10^{-1}
$\lambda_3 \boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$16/27$	$16/27$	$16/27$	$16/27$
p_0	2	5	7	2
$\lambda_{p_0} \boldsymbol{\omega}_H(V(\mathbf{Q}_{p_0}))$	$27/64$	$13824/15625$	$186624/117649$	$27/64$
C_0	8.306815×10^{-1}	3.493824×10^{-1}	3.066383×10^{-1}	8.306815×10^{-1}
C_1	9.540383×10^{-1}	8.704106×10^{-1}	9.297617×10^{-1}	8.196347×10^{-1}
C_2	7.827314×10^{-1}	8.112747×10^{-1}	9.228033×10^{-1}	8.294515×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	4.105301	2.347970	1.910125	2.430506
$\boldsymbol{\theta}_H(V)$	1.895795×10^{-1}	1.291945×10^{-1}	2.184437×10^{-1}	1.290720×10^{-1}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V) B \log(B)^2$	1.214249	1.154191	1.181448	1.149252
$\boldsymbol{\theta}^{\mathrm{stat}}_H(V)/\boldsymbol{\theta}_H(V)$	0.980969	1.056178	1.044971	0.981831

For the last example we have $C_2 = C_3 = 1$ and $E_1 = E_2 = E_3$ and we get

Surface	S_8
B	100000
$N_{U,H}(B)$	12137664
$\alpha(V)$	$7/18$
$\zeta^*_{E_i}(1)$	6.045998×10^{-1}
$\lambda_3 \boldsymbol{\omega}_H(V(\mathbf{Q}_3))$	$16/27$
C_0	3.066383×10^{-1}
C_1	5.129319×10^{-1}
$\boldsymbol{\omega}_H(V(\mathbf{R}))$	6.121864
$\boldsymbol{\theta}_H(V)$	4.904057×10^{-2}
$N_{U,H}(B)/\boldsymbol{\theta}_H(V) B \log(B)^3$	1.621894
$\boldsymbol{\theta}_H^{\mathrm{stat}}(V)/\boldsymbol{\theta}_H(V)$	1.024630

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