

## Noether's Problem for Some $p$ -groups

Shou-Jen Hu

Ming-chang Kang

Department of Mathematics    and    Department of Mathematics

Tamkang University

National Taiwan University

Tamsui, Taiwan

Taipei, Taiwan

E-mail:kang@math.ntu.edu.tw

**Abstract** Let  $K$  be any field and  $G$  be a finite group. Let  $G$  act on the rational function field  $K(x_g : g \in G)$  by  $K$ -automorphisms defined by  $g \cdot x_h = x_{gh}$  for any  $g, h \in G$ . Noether's problem asks whether the fixed field  $K(G) = K(x_g : g \in G)^G$  is rational (=purely transcendental) over  $K$ . We will prove that if  $G$  is a non-abelian  $p$ -group of order  $p^n$  containing a cyclic subgroup of index  $p$  and  $K$  is any field containing a primitive  $p^{n-2}$ -th root of unity, then  $K(G)$  is rational over  $K$ . As a corollary, if  $G$  is a non-abelian  $p$ -group of order  $p^3$  and  $K$  is a field containing a primitive  $p$ -th root of unity, then  $K(G)$  is rational.

---

Mathematics Subject Classification (2000): Primary 12F12, 13A50, 11R32, 14E08.

Keywords and phrases: Noether's problem, the rationality problem, the inverse Galois problem,  $p$ -group actions.

## §1. Introduction

Let  $K$  be any field and  $G$  be a finite group. Let  $G$  act on the rational function field  $K(x_g : g \in G)$  by  $K$ -automorphisms such that  $g \cdot x_h = x_{gh}$  for any  $g, h \in G$ . Denote by  $K(G)$  the fixed field  $K(x_g : g \in G)^G$ . Noether's problem asks whether  $K(G)$  is rational (=purely transcendental) over  $K$ . Noether's problem for abelian groups was studied by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. See the survey article [Sw] for more details. Consequently we will restrict our attention to the non-abelian case in this article.

First we will recall several results of Noether's problem for non-abelian  $p$ -groups.

**Theorem 1.1.** (Chu and Kang [CK, Theorem 1.6]) *Let  $G$  be a non-abelian  $p$ -group of order  $\leq p^4$  and exponent  $p^e$ . Assume that  $K$  is any field such that either (i)  $\text{char } K = p > 0$ , or (ii)  $\text{char } K \neq p$  and  $K$  contains a primitive  $p^e$ -th root of unity. Then  $K(G)$  is rational over  $K$ .*

**Theorem 1.2.** ([Ka2, Theorem 1.5]) *Let  $G$  be a non-abelian metacyclic  $p$ -group of exponent  $p^e$ . Assume that  $K$  is any field such that either (i)  $\text{char } K = p > 0$ , or (ii)  $\text{char } K \neq p$  and  $K$  contains a primitive  $p^e$ -th root of unity. The  $K(G)$  is rational over  $K$ .*

**Theorem 1.3.** (Saltman [Sa1]) *Let  $K$  be any field with  $\text{char } K \neq p$  ( in particular,  $K$  may be any algebraically closed field with  $\text{char } K \neq p$  ). There exists a non-abelian  $p$ -group  $G$  of order  $p^9$  such that  $K(G)$  is not rational over  $K$ .*

**Theorem 1.4.** (Bogomolov [Bo]) *There exists a non-abelian  $p$ -group  $G$  of order  $p^6$  such that  $\mathbb{C}(G)$  is not rational over  $\mathbb{C}$ .*

All the above theorems deal with fields  $K$  containing enough roots of unity.

For a field  $K$  which doesn't have enough roots of unity, so far as we know, the only two known cases are the following Theorem 1.5 and Theorem 1.6.

**Theorem 1.5.** (Saltman [Sa2, Theorem 1]) *Let  $G$  be a non-abelian  $p$ -group of order  $p^3$ . Assume that  $K$  is any field such that either (i)  $\text{char } K = p > 0$  or (ii)  $\text{char } K \neq p$  and  $K$  contains a primitive  $p$ -th root of unity. Then  $K(G)$  is stably rational over  $K$ .*

**Theorem 1.6.** (Chu, Hu and Kang [CHK; Ka1]) *Let  $K$  be any field. Suppose that  $G$  is a non-abelian group of order 8 or 16. Then  $K(G)$  is rational over  $K$  except when  $G = Q$ , the generalized quaternion group of order 16 (see Theorem 1.9 for its definition). When  $G = Q$  and  $K(\zeta)$  is cyclic over  $K$  where  $\zeta$  is an primitive 8-th root of unity, then  $K(G)$  is also rational over  $K$ .*

We will remark that, if  $G = Q$  is the generalized quaternion group of order 16, then  $\mathbb{Q}(G)$  is not rational over  $\mathbb{Q}$  by a theorem of Serre [GMS, Theorem 34.7, p.92]. The main result of this article is the following.

**Theorem 1.7.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^n$  such that  $G$  contains a cyclic subgroup of index  $p$ . Assume that  $K$  is any field such that either (i)  $\text{char } K = p > 0$  or (ii)  $\text{char } K \neq p$  and  $[K(\zeta) : K] = 1$  or  $p$  where  $\zeta$  is a primitive  $p^{n-1}$ -th root of unity. Then  $K(G)$  is rational over  $K$ .*

As a corollary of Theorem 1.1 and Theorem 1.7, we have

**Theorem 1.8.** *Let  $G$  be a non-abelian  $p$ -group of order  $p^3$ . Assume that  $K$  is any field such that either (i)  $\text{char } K = p > 0$  or (ii)  $\text{char } K \neq p$  and  $K$  contains a primitive  $p$ -th root of unity. Then  $K(G)$  is rational over  $K$ .*

Noether's problem is studied for the inverse Galois problem and the construction of a generic Galois  $G$ -extension over  $K$ . See [DM] for details.

We will describe the main ideas of the proof of Theorem 1.7 and Theorem 1.8. All the  $p$ -groups containing cyclic subgroups of index  $p$  are classified by the following theorem.

**Theorem 1.9.** ([Su, p.107]) *Let  $G$  be a non-abelian  $p$ -group of order  $p^n$  containing a cyclic subgroup of index  $p$ .*

- (i) *If  $p$  is an odd prime number, then  $G$  is isomorphic to  $M(p^n)$ ; and*
- (ii) *If  $p = 2$ , then  $G$  is isomorphic to  $M(2^n)$ ,  $D(2^{n-1})$ ,  $SD(2^{n-1})$  where  $n \geq 4$ , and  $Q(2^n)$  where  $n \geq 3$  such that*

$$M(p^n) = \langle \sigma, \tau : \sigma^{p^{n-1}} = \tau^p = 1, \tau^{-1}\sigma\tau = \sigma^{1+p^{n-2}} \rangle,$$

$$D(2^{n-1}) = \langle \sigma, \tau : \sigma^{2^{n-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle,$$

$$SD(2^{n-1}) = \langle \sigma, \tau : \sigma^{2^{n-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-2}} \rangle,$$

$$Q(2^n) = \langle \sigma, \tau : \sigma^{2^{n-1}} = \tau^4 = 1, \sigma^{2^{n-2}} = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle.$$

The groups  $M(p^n)$ ,  $D(2^{n-1})$ ,  $SD(2^{n-1})$ ,  $Q(2^n)$  are called the modular group, the dihedral group, the quasi-dihedral group and the generalized quaternion group respectively.

Thus we will concentrate on the rationality of  $K(G)$  for  $G = M(p^n)$ ,  $D(2^{n-1})$ ,  $SD(2^{n-1})$ ,  $Q(2^n)$  with the assumption that  $[K(\zeta) : K] = 1$  or  $p$  where  $G$  is a group of exponent  $p^e$  and  $\zeta$  is a primitive  $p^e$ -th root of unity. If  $\zeta \in K$ , then Theorem 1.7 follows from Theorem 1.2. Hence we may assume that  $[K(\zeta) : K] = p$ . If  $p$  is an odd prime number, the condition on  $[K(\zeta) : K]$  implies that  $K$  contains a primitive  $p^{e-1}$ -th root of unity. If  $p = 2$ , the condition  $[K(\zeta) : K] = 2$  implies that  $\lambda(\zeta) = -\zeta$ ,  $\pm\zeta^{-1}$  where  $\lambda$  is a generator of the Galois group of  $K(\zeta)$  over  $K$ . (The case  $\lambda(\zeta) = -\zeta$  is equivalent to that the primitive  $2^{e-1}$ -th root of unity belongs to  $K$ .) In case  $K$  contains a primitive  $p^{e-1}$ -th root of unity, we

construct a faithful representation  $G \longrightarrow GL(V)$  such that  $\dim V = p^2$  and  $K(V)$  is rational over  $K$ . For the remaining cases i.e.  $p = 2$ , we will add the root  $\zeta$  to the ground field  $K$  and show that  $K(G) = K(\zeta)(G)^{<\lambda>}$  is rational over  $K$ . In the case  $p = 2$  we will construct various faithful representations according to the group  $G = M(2^n), D(2^{n-1}), SD(2^{n-1}), Q(2^n)$  and the possible image  $\lambda(\zeta)$  because it seems that a straightforward imitation of the case for  $K$  containing a primitive  $p^{e-1}$ -th root of unity doesn't work.

We organize this article as follows. Section 2 contains some preliminaries which will be used subsequently. In Section 3, we first prove Theorem 1.7 for the case when  $K$  contains a primitive  $p^{e-1}$ -th root of unity. This result will be applied to prove Theorem 1.8. In Section 4 we continue to complete the proof of Theorem 1.7. The case when  $\text{char } K = p > 0$  will be taken care by the following theorem due to Kuniyoshi.

**Theorem 1.10.** (Kuniyoshi [CK, Theorem 1.7]) *If  $\text{char } K = p > 0$  and  $G$  is a finite  $p$ -group, then  $K(G)$  is rational over  $K$ .*

**Standing Notations.** The exponent of a finite group, denoted by  $\exp(G)$ , is defined as  $\exp(G) = \max\{\text{ord}(g) : g \in G\}$  where  $\text{ord}(g)$  is the order of the element  $g$ . Recall the definitions of modular groups, dihedral groups, quasi-dihedral groups and generalized quaternion groups which are defined in Theorem 1.9. If  $K$  is a field with  $\text{char } K = 0$  or  $\text{char } K \nmid m$ , then  $\zeta_m$  denotes a primitive  $m$ -th root of unity in some extension field of  $K$ . If  $L$  is any field and we write  $L(x, y), L(x, y, z)$  without any explanation, we mean that these fields  $L(x, y), L(x, y, z)$  are rational function fields over  $K$ .

## §2. Generalities

We list several results which will be used in the sequel.

**Theorem 2.1.** ([CK, Theorem 4.1]) *Let  $G$  be a finite group acting on  $L(x_1, \dots, x_m)$ , the rational function field of  $m$  variables over a field  $L$  such that*

- (i) *for any  $\sigma \in G$ ,  $\sigma(L) \subset L$ ;*
- (ii) *the restriction of the action of  $G$  to  $L$  is faithful;*
- (iii) *for any  $\sigma \in G$ ,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} + B(\sigma)$$

*where  $A(\sigma) \in GL_m(L)$  and  $B(\sigma)$  is an  $m \times 1$  matrix over  $L$ . Then there exist  $z_1, \dots, z_m \in L(x_1, \dots, x_m)$  so that  $L(x_1, \dots, x_m) = L(z_1, \dots, z_m)$  with  $\sigma(z_i) = z_i$  for any  $\sigma \in G$ , any  $1 \leq i \leq m$ .*

**Theorem 2.2.** ([AHK, Theorem 3.1]) *Let  $G$  be a finite group acting on  $L(x)$ , the rational function field of one variable over a field  $L$ . Assume that, for any  $\sigma \in G$ ,  $\sigma(L) \subset L$  and  $\sigma(x) = a_\sigma x + b_\sigma$  for any  $a_\sigma, b_\sigma \in L$  with  $a_\sigma \neq 0$ . Then  $L(x)^G = L^G(z)$  for some  $z \in L[x]$ .*

**Theorem 2.3.** ([CHK, Theorem 2.3]) *Let  $K$  be any field,  $K(x, y)$  the rational function field of two variables over  $K$ , and  $a, b \in K \setminus \{0\}$ . If  $\sigma$  is a  $K$ -automorphism on  $K(x, y)$  defined by  $\sigma(x) = a/x$ ,  $\sigma(y) = b/y$ , then  $K(x, y)^{\langle \sigma \rangle} = K(u, v)$  where*

$$u = \frac{x - \frac{a}{x}}{xy - \frac{ab}{xy}}, \quad v = \frac{y - \frac{b}{y}}{xy - \frac{ab}{xy}}.$$

*Moreover,  $x + (a/x) = (-bu^2 + av^2 + 1)/v$ ,  $y + (b/y) = (bu^2 - av^2 + 1)/u$ ,  $xy + (ab/(xy)) = (-bu^2 - av^2 + 1)/(uv)$ .*

**Lemma 2.4.** *Let  $K$  be any field whose prime field is denoted by  $\mathbb{F}$ . Let  $m \geq 3$  be an integer. Assume that  $\text{char } \mathbb{F} \neq 2$ ,  $[K(\zeta_{2^m}) : K] = 2$  and  $\lambda(\zeta_{2^m}) = \zeta_{2^m}^{-1}$  (resp.  $\lambda(\zeta_{2^m}) = -\zeta_{2^m}^{-1}$ ) where  $\lambda$  is the non-trivial  $K$ -automorphism on  $K(\zeta_{2^m})$ . Then  $K(\zeta_{2^m}) = K(\zeta_4)$  and  $K \cap \mathbb{F}(\zeta_4) = \mathbb{F}$ .*

*Proof.* Since  $m \geq 3$ , it follows that  $\lambda(\zeta_4) = \zeta_4^{-1}$  no matter whether  $\lambda(\zeta_{2^m}) = \zeta_{2^m}^{-1}$  or  $-\zeta_{2^m}^{-1}$ . Hence  $\lambda(\zeta_4) \neq \zeta_4$ . It follows that  $\zeta_4 \in K(\zeta_{2^m}) \setminus K$ . Thus  $K(\zeta_{2^m}) = K(\zeta_4)$ . In particular,  $\zeta_4 \notin \mathbb{F}$ . Since  $[K(\zeta_4) : K] = 2$  and  $[\mathbb{F}(\zeta_4) : \mathbb{F}] = 2$ , it follows that  $K \cap \mathbb{F}(\zeta_4) = \mathbb{F}$ .  $\square$

### §3. Proof of Theorem 1.8

Because of Theorem 1.10 we will assume that  $\text{char } K \neq p$  for any field  $K$  considered in this section.

**Theorem 3.1.** *Let  $p$  be any prime number,  $G = M(p^n)$  the modular group of order  $p^n$  where  $n \geq 3$  and  $K$  be any field containing a primitive  $p^{n-2}$ -th root of unity. Then  $K(G)$  is rational over  $K$ .*

*Proof.* Let  $\xi$  be a primitive  $p^{n-2}$ -th root of unity in  $K$ .

Step 1.

Let  $\bigoplus_{g \in G} K \cdot x(g)$  be the representation space of the regular representation of  $G$ .

Define

$$v = \sum_{0 \leq i \leq p^{n-2}-1} \xi^{-i} [x(\sigma^{ip}) + x(\sigma^{ip}\tau) + \cdots + x(\sigma^{ip}\tau^{p-1})].$$

Then  $\sigma^p(v) = \xi v$  and  $\tau(v) = v$ .

Define  $x_i = \sigma^i v$  for  $0 \leq i \leq p-1$ . We note that  $\sigma : x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto \xi x_0$  and  $\tau : x_i \mapsto \eta^{-i} x_i$  where  $\eta = \xi^{p^{n-3}}$ .

Applying Theorem 2.1 we find that, if  $K(x_0, x_1, \dots, x_{p-1})^G$  is rational over  $K$ , then  $K(G) = K(x(g) : g \in G)^G$  is also rational over  $K$ .

Step 2.

Define  $y_i = x_i/x_{i-1}$  for  $1 \leq i \leq p-1$ . Then  $K(x_0, x_1, \dots, x_{p-1}) = K(x_0, y_1, \dots, y_{p-1})$  and  $\sigma : x_0 \mapsto y_1 x_0, y_1 \mapsto y_2 \mapsto \dots \mapsto y_{p-1} \mapsto \xi/(y_1 \dots y_{p-1}), \tau : x_0 \mapsto x_0, y_i \mapsto \eta^{-1} y_i$ . By Theorem 2.2, if  $K(y_1, \dots, y_{p-1})^G$  is rational over  $K$ , so is  $K(x_0, y_1, \dots, y_{p-1})^G$  over  $K$ .

Define  $u_i = y_i/y_{i-1}$  for  $2 \leq i \leq p-1$ . Then  $K(y_1, \dots, y_{p-1}) = K(y_1, u_2, \dots, u_{p-1})$  and  $\sigma : y_1 \mapsto y_1 u_2, u_2 \mapsto u_3 \mapsto \dots \mapsto u_{p-1} \mapsto \xi/(y_1 y_2 \dots y_{p-2} y_{p-1}^2) = \xi/(y_1^p u_2^{p-1} u_3^{p-2} \dots u_{p-1}^2), \tau : y_1 \mapsto \eta^{-1} y_1, u_i \mapsto u_i$  for  $2 \leq i \leq p-1$ . Thus  $K(y_1, u_2, \dots, u_{p-1})^{<\tau>} = K(y_1^p, u_2, \dots, u_{p-1})$ .

Define  $u_1 = \xi^{-1} y_1^p$ . Then  $\sigma : u_1 \mapsto u_1 u_2^p, u_2 \mapsto u_3 \mapsto \dots \mapsto 1/(u_1 u_2^{p-1} \dots u_{p-1}^2) \mapsto u_1 u_2^{p-2} u_3^{p-3} \dots u_{p-2}^2 u_{p-1} \mapsto u_2$ .

Define  $w_1 = u_2, w_i = \sigma^{i-1}(u_2)$  for  $2 \leq i \leq p-1$ . Then  $K(u_1, u_2, \dots, u_{p-1}) = K(w_1, w_2, \dots, w_{p-1})$ . It follows that  $K(y_1, \dots, y_{p-1})^G = \{K(y_1, \dots, y_{p-1})^{<\tau>}\}^{<\sigma>} = K(w_1, w_2, \dots, w_{p-1})^{<\sigma>}$  and  $\sigma : w_1 \mapsto w_2 \mapsto \dots \mapsto w_{p-1} \mapsto 1/(w_1 w_2 \dots w_{p-1})$ .

Step 3.

Define  $T_0 = 1 + w_1 + w_1 w_2 + \dots + w_1 w_2 \dots w_{p-1}, T_1 = (1/T_0) - (1/p), T_{i+1} = (w_1 w_2 \dots w_i / T_0) - (1/p)$  for  $1 \leq i \leq p-1$ . Thus  $K(w_1, \dots, w_{p-1}) = K(T_1, \dots, T_p)$  with  $T_1 + T_2 + \dots + T_p = 0$  and  $\sigma : T_1 \mapsto T_2 \mapsto \dots \mapsto T_{p-1} \mapsto T_p \mapsto T_0$ .

Define  $s_i = \sum_{1 \leq j \leq p} \eta^{-ij} T_j$  for  $1 \leq i \leq p-1$ . Then  $K(T_1, T_2, \dots, T_p) = K(s_1, s_2, \dots, s_{p-1})$  and  $\sigma : s_i \mapsto \eta^i s_i$ . Clearly  $K(s_1, \dots, s_{p-1})^{<\sigma>}$  is rational over  $K$ .  $\square$

Proof of Theorem 1.8.

If  $p \geq 3$ , a non-abelian  $p$ -group of order  $p^3$  is either of exponent  $p$  or contains



a cyclic subgroup of index  $p$  (see [CK, Theorem 2.3]). The rationality of  $K(G)$  of the first group follows from Theorem 1.1 while that of the second group follows from the above Theorem 3.1. If  $p = 2$ , the rationality of  $K(G)$  is a consequence of Theorem 1.6.  $\square$

The method used in the proof of Theorem 3.1 can be applied to other groups, e.g.  $D(2^{n-1})$ ,  $Q(2^n)$ ,  $SD(2^{n-1})$ . The following results will be used in the proof of Theorem 1.7.

**Theorem 3.2.** *Let  $G = D(2^{n-1})$  or  $Q(2^n)$  with  $n \geq 4$ . If  $K$  is a field containing a primitive  $2^{n-2}$ -th root of unity, then  $K(G)$  is rational over  $K$ .*

*Proof.* Let  $\xi$  be a primitive  $2^{n-2}$ -th root of unity in  $K$ .

Let  $\bigoplus_{g \in G} K \cdot x(g)$  be the representation space of the regular representation of  $G$ .

Define

$$v = \sum_{0 \leq i \leq 2^{n-2}-1} \xi^{-i} x(\sigma^{2i}).$$

Then  $\sigma^2(v) = \xi v$ .

Define  $x_0 = v$ ,  $x_1 = \sigma \cdot v$ ,  $x_2 = \tau \cdot v$ ,  $x_3 = \tau\sigma \cdot v$ . We find that

$$\sigma : x_0 \mapsto x_1 \mapsto \xi x_0, \quad x_2 \mapsto \xi^{-1} x_3, \quad x_3 \mapsto x_2,$$

$$\tau : x_0 \mapsto x_2 \mapsto \epsilon x_0, \quad x_1 \mapsto x_3 \mapsto \epsilon x_1$$

where  $\epsilon = 1$  if  $G = D(2^{n-1})$ , and  $\epsilon = -1$  if  $G = Q(2^n)$ .

By Theorem 2.1 it suffices to show that  $K(x_0, x_1, x_2, x_3)^G$  is rational over  $K$ .

Since  $\sigma^2(x_i) = \xi x_i$  for  $i = 0, 1$ ,  $\sigma^2(x_i) = \xi^{-1} x_j$  for  $j = 2, 3$ , it follows that  $K(x_0, x_1, x_2, x_3)^{<\sigma^2>} = K(y_0, y_1, y_2, y_3)$  where  $y_0 = x_0^{2^{n-2}}$ ,  $y_1 = x_1/x_0$ ,  $y_2 = x_0 x_2$ ,  $y_3 = x_1 x_3$ . The action of  $\sigma$  and  $\tau$  are given by

$$\sigma : y_0 \mapsto y_0 y_1^{2^{n-2}}, \quad y_1 \mapsto \xi/y_1, \quad y_2 \mapsto \xi^{-1} y_3, \quad y_3 \mapsto \xi y_2,$$

$$\tau : y_0 \mapsto y_0^{-1} y_2^{2^{n-2}}, \quad y_1 \mapsto y_1^{-1} y_2^{-1} y_3, \quad y_2 \mapsto \epsilon y_2, \quad y_3 \mapsto \epsilon y_3.$$

Define

$$z_0 = y_0 y_1^{2^{n-3}} y_2^{-2^{n-4}} y_3^{-2^{n-4}}, \quad z_1 = y_1, \quad z_2 = y_2^{-1} y_3, \quad z_3 = y_2.$$

We find that

$$\sigma : z_0 \mapsto -z_0, \quad z_1 \mapsto \xi z_1^{-1}, \quad z_2 \mapsto \xi^2 z_2^{-1}, \quad z_3 \mapsto \xi^{-1} z_2 z_3,$$

$$\tau : z_0 \mapsto z_0^{-1}, \quad z_1 \mapsto z_1^{-1} z_2, \quad z_2 \mapsto z_2, \quad z_3 \mapsto \epsilon z_3.$$

By Theorem 2.2 it suffices to prove that  $K(z_0, z_1, z_2)^{<\sigma, \tau>}$  is rational over  $K$ .

Now we will apply Theorem 2.3 to find  $K(z_0, z_1, z_2)^{<\sigma>}$  with  $a = 1$  and  $b = z_2$ .

Define

$$u = \frac{z_0 - \frac{a}{z_0}}{z_0 z_1 - \frac{ab}{z_0 z_1}}, \quad v = \frac{z_1 - \frac{b}{z_1}}{z_0 z_1 - \frac{ab}{z_0 z_1}}.$$

By Theorem 2.3 we find that  $K(z_0, z_1, z_2)^{<\tau>} = K(u, v, z_2)$ . The actions of  $\sigma$  on  $u, v, z_2$  are given by

$$\begin{aligned} \sigma : z_2 &\mapsto \xi^2 z_2^{-1}, \\ u &\mapsto \frac{-z_0 + \frac{a}{z_0}}{\xi(\frac{z_1}{bz_0} - \frac{z_0}{z_1})}, \quad v \mapsto \frac{\xi(\frac{1}{z_1} - \frac{z_1}{b})}{\xi(\frac{z_1}{bz_0} - \frac{z_0}{z_1})}. \end{aligned}$$

Define  $w = u/v$ . Then  $\sigma(w) = bw/\xi = z_2 w/\xi$ .

Note that

$$\sigma(u) = \frac{-z_0 + \frac{a}{z_0}}{\xi(\frac{z_1}{bz_0} - \frac{z_0}{z_1})} = \frac{b}{\xi} \frac{z_0 - \frac{a}{z_0}}{\frac{bz_0}{z_1} - \frac{az_1}{z_0}} = \frac{bu}{\xi(bu^2 - av^2)}.$$

The last equality of the above formula is equivalent to the following identity

$$(1) \quad \frac{\frac{x - \frac{a}{x}}{bx} - \frac{ay}{y}}{\frac{ay}{y} - \frac{x}{x}} = \frac{u}{bu^2 - av^2}.$$

where  $x, y, u, v, a, b$  are the same as in Theorem 2.3. A simple way to verify Identity (1) goes as follows: The right-hand side of (1) is equal to  $(y + (b/y) - (1/u))^{-1}$  by Theorem 2.3. It is not difficult to check that the left-hand side of (1) is equal to  $(y + (b/y) - (1/u))^{-1}$ .

Thus  $\sigma(u) = bu/(\xi(bu^2 - av^2)) = z_2u/(\xi(z_2u^2 - v^2)) = z_2w^2/(\xi u(z_2w^2 - 1))$ .

Define  $T = z_2w^2/\xi$ ,  $X = w$ ,  $Y = u$ . Then  $K(u, v, z_2) = K(T, X, Y)$  and  $\sigma : T \mapsto T, X \mapsto A/X, Y \mapsto B/Y$  where  $A = T$ ,  $B = T/(\xi T - 1)$ . By Theorem 2.3 it follows that  $K(T, X, Y)^{<\sigma>}$  is rational over  $K(T)$ . In particular, it is rational over  $K$ .  $\square$

**Theorem 3.3.** *Let  $G = SD(2^{n-1})$  with  $n \geq 4$ . If  $K$  is a field containing a primitive  $2^{n-2}$ -th root of unity, then  $K(G)$  is rational over  $K$ .*

*Proof.* The case  $n = 4$  is a consequence of [CHK, Theorem 3.2]. Thus we may assume  $n \geq 5$  in the following proof.

The proof is quite similar to that of Theorem 3.2.

Define  $v, x_0, x_1, x_2, x_3$  by the same formulae as in the proof of Theorem 3.2. Then  $\sigma : x_0 \mapsto x_1 \mapsto \xi x_0, x_2 \mapsto -\xi^{-1}x_3, x_3 \mapsto -x_2, \tau : x_0 \mapsto x_2 \mapsto x_0, x_1 \mapsto x_3 \mapsto x_1$ .

Define  $y_0 = x_0^{2^{n-2}}, y_1 = x_1/x_0, y_2 = x_0x_2$ , and  $y_3 = x_1x_3$ . Then  $K(x_0, x_1, x_2, x_3)^{<\sigma^2>} = K(y_0, y_1, y_2, y_3)$  and

$$\sigma : y_0 \mapsto y_0y_1^{2^{n-2}}, y_1 \mapsto \xi/y_1, y_2 \mapsto -\xi^{-1}y_3, y_3 \mapsto -\xi y_2,$$

$$\tau : y_0 \mapsto y_0^{-1}y_2^{2^{n-2}}, y_1 \mapsto y_1^{-1}y_2^{-1}y_3, y_2 \mapsto y_2, y_3 \mapsto y_3.$$

Note that the actions of  $\sigma$  and  $\tau$  are the same as those in the proof of Theorem 3.2 except for the coefficients.

Thus we may define  $z_0, z_1, z_2, z_3$  by the same formulae as in the proof of Theorem 3.2.

Using the assumption that  $n \geq 5$ , we find

$$\begin{aligned}\sigma : z_0 &\mapsto -z_0, \quad z_1 \mapsto \xi z_1^{-1}, \quad z_2 \mapsto \xi^2 z_2^{-1}, \quad z_3 \mapsto -\xi^{-1} z_2 z_3, \\ \tau : z_0 &\mapsto z_0^{-1}, \quad z_1 \mapsto z_1^{-1} z_2, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3.\end{aligned}$$

By Theorem 2.2 it suffices to prove that  $K(z_0, z_1, z_2)^{<\sigma, \tau>}$  is rational over  $K$ . But the actions of  $\sigma, \tau$  on  $z_0, z_1, z_2$  are completely the same as those in the proof of Theorem 3.2. Hence the result.  $\square$

#### §4. Proof of Theorem 1.7

We will complete the proof of Theorem 1.7 in this section.

Let  $\zeta$  be a primitive  $p^{n-1}$ -th root of unity. If  $\zeta \in K$ , then Theorem 1.7 is a consequence of Theorem 1.2. Thus we may assume that  $[K(\zeta) : K] = p$  from now on. Let  $\text{Gal}(K(\zeta)/K) = \langle \lambda \rangle$  and  $\lambda(\zeta) = \zeta^a$  for some integer  $a$ .

If  $p \geq 3$ , it is easy to see that  $a = 1 \pmod{p^{n-2}}$  and  $\zeta^p \in K$ . By Theorem 1.9 the  $p$ -group  $G$  is isomorphic to  $M(p^n)$ . Apply Theorem 3.1. We are done.

Now we consider the case  $p = 2$ .

By Theorem 1.9  $G$  is isomorphic to  $M(2^n)$ ,  $D(2^{n-1})$ ,  $SD(2^{n-1})$  or  $Q(2^n)$ . If  $G$  is a non-abelian group of order 8, the rationality of  $K(G)$  is guaranteed by Theorem 1.6. Thus it suffices to consider the case  $G$  is a 2-group of order  $\geq 16$ , i.e.  $n \geq 4$ .

Recall that  $G$  is generated by two elements  $\sigma$  and  $\tau$  such that  $\sigma^{2^{n-1}} = 1$  and  $\tau^{-1}\sigma\tau = \sigma^k$  where

- (i)  $k = -1$  if  $G = D(2^{n-1})$  or  $Q(2^n)$ ,
- (ii)  $k = 1 + 2^{n-2}$  if  $G = M(2^n)$ ,
- (iii)  $k = -1 + 2^{n-2}$  if  $G = SD(2^{n-1})$ .

As before, let  $\zeta$  be a primitive  $2^{n-1}$ -th root of unity and  $\text{Gal}(K(\zeta)/K) = \langle \lambda \rangle$  with  $\lambda(\zeta) = \zeta^a$  where  $a^2 = 1 \pmod{2^{n-1}}$ . It follows that the only possibilities of  $a \pmod{2^{n-1}}$  are  $a = -1, \pm 1 + 2^{n-2}$ .

It follows that we have four type of groups and three choices for  $\lambda(\zeta)$  and thus we should deal with 12 situations. Fortunately many situations behaves quite similar. And if we abuse the terminology, we may even say that some situations are "semi-equivariant" isomorphic (but it may not be equivariant isomorphic in the usual sense). Hence they obey the same formulae of changing the variables. After every situation is reduced to a final form we may reduce the rationality problem of a group of order  $2^n$  ( $n \geq 4$ ) to that of a group of order 16.

Let  $\bigoplus_{g \in G} K \cdot x(g)$  be the representation space of the regular representation of  $G$ . We will extend the actions of  $G$  and  $\lambda$  to  $\bigoplus_{g \in G} K(\zeta) \cdot x(g)$  by requiring  $\rho(\zeta) = \zeta$  and  $\lambda(x(g)) = x(g)$  for any  $\rho \in G$ . Note that  $K(G) = K(x(g) : g \in G)^G = \{K(\zeta)(x(g) : g \in G)^{\langle \lambda \rangle}\}^G = K(\zeta)(x(g) : g \in G)^{\langle G, \lambda \rangle}$ .

We will find a faithful subspace  $\bigoplus_{0 \leq i \leq 3} K(\zeta) \cdot x_i$  of  $\bigoplus_{g \in G} K(\zeta) \cdot x(g)$  such that  $K(\zeta)(x_0, x_1, x_2, x_3)^{\langle G, \lambda \rangle}(y_1, \dots, y_{12})$  is rational over  $K$  where each  $y_i$  is fixed by  $G$  and  $\lambda$ . By Theorem 2.1,  $K(\zeta)(x(g) : g \in G)^{\langle G, \lambda \rangle} = K(\zeta)(x_0, x_1, x_2, x_3)^{\langle G, \lambda \rangle}(X_1, \dots, X_N)$  where  $N = 2^n - 4$  and each  $X_i$  is fixed by  $G$  and  $\lambda$ . It follows that  $K(G)$  is rational provided that  $K(\zeta)(x_0, x_1, x_2, x_3)^{\langle G, \lambda \rangle}(y_1, \dots, y_{12})$  is rational over  $K$ .

Define

$$v_1 = \sum_{0 \leq j \leq 2^{n-1}-1} \zeta^{-j} x(\sigma^j), \quad v_2 = \sum_{0 \leq j \leq 2^{n-1}-1} \zeta^{-aj} x(\sigma^j)$$

where  $a$  is the integer with  $\lambda(\zeta) = \zeta^a$ .

We find that  $\sigma : v_1 \mapsto \zeta v_1, v_2 \mapsto \zeta^a v_2, \lambda : v_1 \mapsto v_2 \mapsto v_1$ .

Define  $x_0 = v_1, x_1 = \tau \cdot v_1, x_2 = v_2, x_3 = \tau \cdot v_2$ .

It follows that

$$\sigma : x_0 \mapsto \zeta x_0, \ x_1 \mapsto \zeta^k x_1, \ x_2 \mapsto \zeta^a x_2, \ x_3 \mapsto \zeta^{ak} x_3,$$

$$\lambda : x_0 \mapsto x_2 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1, \ \zeta \mapsto \zeta^a,$$

$$\tau : x_0 \mapsto x_1 \mapsto \epsilon x_0, \ x_2 \mapsto x_3 \mapsto \epsilon x_2,$$

$$\tau\lambda : x_0 \mapsto x_3 \mapsto \epsilon x_0, \ x_1 \mapsto \epsilon x_2, \ x_2 \mapsto x_1, \ \zeta \mapsto \zeta^a$$

where (i)  $\epsilon = 1$  if  $G \neq Q(2^n)$ , and (ii)  $\epsilon = -1$  if  $G = Q(2^n)$ .

Case 1.  $k = -1$ , i.e.  $G = D(2^{n-1})$  or  $Q(2^n)$ .

Throughout the discussion of this case, we will adopt the convention that  $\epsilon = 1$  if  $G = D(2^{n-1})$ , while  $\epsilon = -1$  if  $G = Q(2^n)$ .

Subcase 1.1.  $a = -1$ , i.e.  $\lambda(\zeta) = \zeta^{-1}$ .

It is easy to find that  $K(\zeta)(x_0, x_1, x_2, x_3)^{<\sigma>} = K(\zeta)(x_0^{2^{n-1}}, x_0x_1, x_0x_2, x_1x_3)$ .

Define

$$y_0 = x_0^{2^{n-1}}, \ y_1 = x_0x_1, \ y_2 = x_0x_2, \ y_3 = x_1x_3.$$

It follows that

$$\lambda : y_0 \mapsto y_0^{-1}y_2^{2^{n-1}}, \ y_1 \mapsto y_1^{-1}y_2y_3, \ y_2 \mapsto y_2, \ y_3 \mapsto y_3, \ \zeta \mapsto \zeta^{-1},$$

$$\tau : y_0 \mapsto y_0^{-1}y_1^{2^{n-1}}, \ y_1 \mapsto \epsilon y_1, \ y_2 \mapsto y_3 \mapsto y_2.$$

Define

$$z_0 = y_0y_1^{-2^{n-2}}y_2^{-2^{n-3}}y_3^{2^{n-3}}, \ z_1 = y_2y_3, \ z_2 = y_2, \ z_3 = y_1.$$

We find that

$$\lambda : z_0 \mapsto 1/z_0, \ z_1 \mapsto z_1, \ z_2 \mapsto z_2, \ z_3 \mapsto z_1/z_3, \ \zeta \mapsto \zeta^{-1},$$

$$\tau : z_0 \mapsto 1/z_0, \ z_1 \mapsto z_1, \ z_2 \mapsto z_1/z_2, \ z_3 \mapsto \epsilon z_3.$$

It turns out the parameter  $n$  does not come into play in the actions of  $\lambda$  and  $\tau$  on  $z_0, z_1, z_2, z_3$ .

By Theorem 2.1  $K(G) = K(\zeta)(z_0, z_1, z_2, z_3)^{<\lambda, \tau>}(X_1, \dots, X_N)$  where  $N = 2^n - 4$  and  $\lambda(X_i) = \tau(X_i) = X_i$  for  $1 \leq i \leq N$ .

By Lemma 2.4  $K(\zeta) = K(\zeta_4)$  where  $\lambda(\zeta_4) = \zeta_4^{-1}$ . Thus  $K(G) = K(\zeta_4)(z_0, z_1, z_2, z_3)^{<\lambda, \tau>}(X_1, \dots, X_N)$

Denote  $G_4 = D(8)$  or  $Q(16)$ . Then  $K(G_4) = K(\zeta_4)(z_0, z_1, z_2, z_3)^{<\lambda, \tau>}(X_1, \dots, X_{12})$ . Since  $K(G_4)$  is rational over  $K$  by Theorem 1.6 (see [Ka1, Theorem 1.3]), it follows that  $K(\zeta_4)(z_0, \dots, z_3)^{<\lambda, \tau>}(X_1, \dots, X_{12})$  is rational over  $K$ . Thus  $K(\zeta_4)(z_0, \dots, z_3)^{<\lambda, \tau>}(X_1, \dots, X_N)$  is rational over  $K$  for  $N = 2^n - 4$ . The last field is nothing but  $K(G)$ . Done.

Subcase 1.2.  $a = -1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta^{-1}$ .

The actions of  $\sigma$ ,  $\tau$ ,  $\lambda$ ,  $\tau\lambda$  are given by

$$\sigma : x_0 \mapsto \zeta x_0, \quad x_1 \mapsto \zeta^{-1} x_1, \quad x_2 \mapsto -\zeta^{-1} x_2, \quad x_3 \mapsto -\zeta x_3,$$

$$\lambda : x_0 \mapsto x_2 \mapsto x_0, \quad x_1 \mapsto x_3 \mapsto x_1, \quad \zeta \mapsto -\zeta^{-1},$$

$$\tau : x_0 \mapsto x_1 \mapsto \epsilon x_0, \quad x_2 \mapsto x_3 \mapsto \epsilon x_2,$$

$$\tau\lambda : x_0 \mapsto x_3 \mapsto \epsilon x_0, \quad x_1 \mapsto \epsilon x_2, \quad x_2 \mapsto x_1, \quad \zeta \mapsto -\zeta^{-1}$$

Define  $y_0 = x_0^{2^{n-1}}$ ,  $y_1 = x_0 x_1$ ,  $y_2 = x_2 x_3$ ,  $y_3 = x_0^{-1-2^{n-2}} x_3$ . Then  $K(\zeta)(x_0, \dots, x_3)^{<\sigma>} = K(\zeta)(y_0, \dots, y_3)$ . Consider the actions of  $\tau\lambda$  and  $\tau$  on  $K(\zeta)(y_0, \dots, y_3)$ . We find that

$$\tau\lambda : y_0 \mapsto y_0^{1+2^{n-2}} y_3^{2^{n-1}}, \quad y_1 \mapsto \epsilon y_2 \mapsto y_1, \quad y_3 \mapsto \epsilon y_0^{-1-2^{n-3}} y_3^{-1-2^{n-2}}, \quad \zeta \mapsto -\zeta^{-1},$$

$$\tau : y_0 \mapsto y_0^{-1} y_1^{2^{n-1}}, \quad y_1 \mapsto \epsilon y_1, \quad y_2 \mapsto \epsilon y_2, \quad y_3 \mapsto \epsilon y_1^{-1-2^{n-2}} y_2 y_3^{-1}.$$

Define

$$z_0 = y_1, \quad z_1 = y_1^{-1} y_2, \quad z_2 = y_0 y_1 y_2^{-1} y_3^2, \quad z_3 = y_0^{1+2^{n-4}} y_1^{-2^{n-4}} y_2^{-2^{n-4}} y_3^{1+2^{n-3}}.$$

We find

$$\tau\lambda : z_0 \mapsto \epsilon z_0 z_1, \quad z_1 \mapsto 1/z_1, \quad z_2 \mapsto 1/z_2, \quad z_3 \mapsto \epsilon z_1^{-1} z_2^{-1} z_3, \quad \zeta \mapsto -\zeta^{-1},$$

$$\tau : z_0 \mapsto \epsilon z_0, \quad z_1 \mapsto z_1, \quad z_2 \mapsto 1/z_2, \quad z_3 \mapsto \epsilon z_1/z_3.$$

By Lemma 2.4 we may replace  $K(\zeta)$  in  $K(\zeta)(z_0, z_1, z_2, z_3)^{<\tau\lambda, \tau>}$  by  $K(\zeta_4)$  where  $\tau\lambda(\zeta_4) = \zeta_4^{-1}$ . Then we may proceed as in Subcase 1.1. The details are omitted.

Subcase 1.3.  $a = 1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta$ .

Note that  $\zeta^2 \in K$  and  $\zeta^2$  is a primitive  $2^{n-2}$ -th root of unity. Thus we may apply Theorem 3.2. Done

Case 2.  $k = 1 + 2^{n-2}$ , i.e.  $G = M(2^n)$ .

Subcase 2.1.  $a = -1$ , i.e.  $\lambda(\zeta) = \zeta^{-1}$ .

The actions of  $\sigma$ ,  $\tau$ ,  $\lambda$ ,  $\tau\lambda$  are given by

$$\sigma : x_0 \mapsto \zeta x_0, \ x_1 \mapsto -\zeta x_1, \ x_2 \mapsto \zeta^{-1} x_2, \ x_3 \mapsto -\zeta^{-1} x_3,$$

$$\lambda : x_0 \mapsto x_2 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1, \ \zeta \mapsto \zeta^{-1},$$

$$\tau : x_0 \mapsto x_1 \mapsto x_0, \ x_2 \mapsto x_3 \mapsto x_2,$$

$$\tau\lambda : x_0 \mapsto x_3 \mapsto x_0, \ x_1 \mapsto x_2 \mapsto x_1, \ \zeta \mapsto \zeta^{-1}.$$

Define  $X_0 = x_0$ ,  $X_1 = x_2$ ,  $X_2 = x_3$ ,  $X_3 = x_1$ . Then the actions of  $\sigma$ ,  $\tau$ ,  $\lambda$  on  $X_0$ ,  $X_1$ ,  $X_2$ ,  $X_3$  are the same as those of  $\sigma$ ,  $\tau\lambda$ ,  $\tau$ , on  $x_0$ ,  $x_1$ ,  $x_2$ ,  $x_3$  in Subcase 1.2 for  $D(2^{n-1})$  except on  $\zeta$ . Thus we may consider  $K(\zeta)(X_0, X_1, X_2, X_3)^{<\sigma, \tau, \lambda>}(Y_1, \dots, Y_{12})$ . Hence the same formulae of changing the variables in Subcase 1.2 can be copied and the same method can be used to prove that  $K(\zeta)(X_0, X_1, X_2, X_3)^{<\sigma, \tau, \lambda>}(Y_1, \dots, Y_{12})$  is rational over  $K$ .

Subcase 2.2.  $a = -1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta^{-1}$ .

The actions of  $\sigma$ ,  $\tau$ ,  $\lambda$ ,  $\tau\lambda$  are given by

$$\sigma : x_0 \mapsto \zeta x_0, \ x_1 \mapsto -\zeta x_1, \ x_2 \mapsto -\zeta^{-1} x_2, \ x_3 \mapsto \zeta^{-1} x_3,$$

$$\lambda : x_0 \mapsto x_2 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1, \ \zeta \mapsto -\zeta^{-1},$$

$$\tau : x_0 \mapsto x_1 \mapsto x_0, \ x_2 \mapsto x_3 \mapsto x_2,$$

$$\tau\lambda : x_0 \mapsto x_3 \mapsto x_0, \ x_1 \mapsto x_2 \mapsto x_1, \ \zeta \mapsto -\zeta^{-1}.$$



Define  $X_0 = x_0$ ,  $X_1 = x_3$ ,  $X_2 = x_2$ ,  $X_3 = x_1$ . Then the actions of  $\sigma$ ,  $\tau$ ,  $\tau\lambda$  on  $X_0$ ,  $X_1$ ,  $X_2$ ,  $X_3$  are the same as those of  $\sigma$ ,  $\tau\lambda$ ,  $\tau$ , on  $x_0$ ,  $x_1$ ,  $x_2$ ,  $x_3$  in Subcase 1.2 for  $D(2^{n-1})$ . Hence the result.

Subcase 2.3.  $a = 1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta$ .

Apply Theorem 3.1.

Case 3.  $k = -1 + 2^{n-2}$ , i.e.  $G = SD(2^{n-1})$ .

Subcase 3.1.  $a = -1$ , i.e.  $\lambda(\zeta) = \zeta^{-1}$ .

The actions of  $\sigma$ ,  $\tau$ ,  $\lambda$ ,  $\tau\lambda$  are given by

$$\sigma : x_0 \mapsto \zeta x_0, x_1 \mapsto -\zeta^{-1}x_1, x_2 \mapsto \zeta^{-1}x_2, x_3 \mapsto -\zeta x_3,$$

$$\lambda : x_0 \mapsto x_2 \mapsto x_0, x_1 \mapsto x_3 \mapsto x_1, \zeta \mapsto \zeta^{-1},$$

$$\tau : x_0 \mapsto x_1 \mapsto x_0, x_2 \mapsto x_3 \mapsto x_2,$$

$$\tau\lambda : x_0 \mapsto x_3 \mapsto x_0, x_1 \mapsto x_2 \mapsto x_1, \zeta \mapsto \zeta^{-1}.$$

Define  $X_0 = x_0$ ,  $X_1 = x_2$ ,  $X_2 = x_1$ ,  $X_3 = x_3$ . Then the actions of  $\sigma$ ,  $\tau\lambda$ ,  $\lambda$  on  $X_0$ ,  $X_1$ ,  $X_2$ ,  $X_3$  are the same as those of  $\sigma$ ,  $\tau\lambda$ ,  $\tau$ , on  $x_0$ ,  $x_1$ ,  $x_2$ ,  $x_3$  in Subcase 1.2 for  $D(2^{n-1})$  except on  $\zeta$ . Done.

Subcase 3.2.  $a = -1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta^{-1}$ .

Define  $y_0 = x_0^{2^{n-1}}$ ,  $y_1 = x_0^{1+2^{n-2}}x_1$ ,  $y_2 = x_1^{-1}x_2$ ,  $y_3 = x_0^{-1}x_3$ . Then  $K(\zeta)(x_0, x_1, x_2, x_3)^{<\sigma>} = K(\zeta)(y_0, y_1, y_2, y_3)$  and

$$\tau : y_0 \mapsto y_0^{-1-2^{n-2}}y_1^{2^{n-1}}, y_1 \mapsto y_0^{-1-2^{n-3}}y_1^{1+2^{n-2}}, y_2 \mapsto y_3 \mapsto y_2,$$

$$\tau\lambda : y_0 \mapsto y_0y_3^{2^{n-1}}, y_1 \mapsto y_1y_2y_3^{1+2^{n-2}}, y_2 \mapsto y_2^{-1}, y_3 \mapsto y_3^{-1}, \zeta \mapsto -\zeta^{-1}.$$

Define  $z_0 = y_0^{1+2^{n-3}}y_1^{-2^{n-2}}y_2^{-2^{n-3}}y_3^{2^{n-3}}$ ,  $z_1 = y_0^{2^{n-4}}y_1^{1-2^{n-3}}y_2^{-2^{n-4}}y_3^{2^{n-4}}$ ,  $z_2 = y_2$ ,  $z_3 = y_2^{-1}y_3$ . It follows that  $K(\zeta)(y_0, y_1, y_2, y_3) = K(\zeta)(z_0, z_1, z_2, z_3)$  and

$$\tau : z_0 \mapsto 1/z_0, z_1 \mapsto z_1/z_0, z_2 \mapsto z_2z_3, z_3 \mapsto 1/z_3,$$

$$\tau\lambda : z_0 \mapsto z_0, z_1 \mapsto z_1z_2^2z_3, z_2 \mapsto 1/z_2, z_3 \mapsto 1/z_3, \zeta \mapsto -\zeta^{-1}.$$

Thus we can establish the rationality because we may replace  $K(\zeta)$  by  $K(\zeta_4)$  as in Subcase 1.2.

Subcase 3.3.  $a = 1 + 2^{n-2}$ , i.e.  $\lambda(\zeta) = -\zeta$ .

Apply Theorem 3.3.

Thus we have finished the proof of Theorem 1.7.  $\square$

## REFERENCES

- [AHK] H. Ahmad, M. Hajja and M. Kang, *Rationality of some projective linear actions*, J. Algebra **228**(2000) 643-658.
- [Bo] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Math. USSR Izv. **30**(1988) 455-485.
- [CHK] H. Chu, S. J. Hu and M. Kang, *Noether's problem for dihedral 2-groups*, Comment. Math. Helv. **79**(2004) 147-159.
- [CK] H. Chu and M. Kang, *Rationality of  $p$ -group actions*, J. Algebra **237**(2001) 673-690.
- [DM] F. DeMeyer and T. McKenzie, *On generic polynomials*, J. Alg. **261** (2003) 327-333.
- [GMS] S. Garibaldi, A. Merkurjev and J. P. Serre, *Cohomological invariants in Galois cohomology*, AMS Univ. Lecture Series vol. 28, Amer. Math. Soc., Providence, 2003.
- [Ka1] M. Kang, *Noether's problem for dihedral 2-groups II*, to appear in "Pacific J. Math."
- [Ka2] M. Kang, *Noether's problem for metacyclic  $p$ -groups*, to appear in "Advances in Math."
- [Sa1] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984) 71-84.
- [Sa2] D. J. Saltman, *Galois groups of order  $p^3$* , Comm. Alg. **15** (1987) 1365-1373.
- [Su] M. Suzuki, *Group theory II*, Grund. math. Wiss. Vol. 248, Springer-Verlag, Berlin, 1986.
- [Sw] R. G. Swan, *Noether's problem in Galois theory*, in "Emmy Noether in Bryn Mawr", edited by B. Srinivasan and J. Sally, Springer-Verlag, Berlin, 1983.