

# DOUBLE SPACES WITH ISOLATED SINGULARITIES

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ABSTRACT. We prove the non-rationality of a double cover of  $\mathbb{P}^n$  branched over a hypersurface  $F \subset \mathbb{P}^n$  of degree  $2n$  having isolated singularities such that  $n \geq 4$  and every singular point of the hypersurface  $F$  is ordinary, i.e. the projectivization of its tangent cone is smooth, whose multiplicity does not exceed  $2(n-2)$ .

## 1. INTRODUCTION.

For a given algebraic variety, it is one of the most substantial questions whether it is rational<sup>1</sup> or not. Global holomorphic differential forms are natural birational invariants of a smooth algebraic variety that solve the rationality problem for algebraic curves and surfaces (see [61]). However, there are only four known methods to prove the non-rationality of a rationally connected higher-dimensional (see [34]). In the following we assume that all varieties are projective, normal, and defined over  $\mathbb{C}$ .

The non-rationality of a smooth quartic 3-fold was proved in [35] using the group of birational automorphisms as a birational invariant. The non-rationality of a smooth cubic 3-fold was proved in [19] through the study of its intermediate Jacobian. The birational invariance of the torsion subgroup of the group  $H^3(\mathbb{Z})$  was used in [4] to prove the non-rationality of some unirational varieties. The non-rationality of a wide class of rationally connected varieties was proved in [41] using the degeneration technique and the reduction into positive characteristic (see [42], [44], [18]).

**Definition 1.** A terminal  $\mathbb{Q}$ -factorial Fano variety  $V$  of Picard rank 1 is called birationally super-rigid if the following three conditions hold: the variety  $V$  is not birational to a fibration<sup>2</sup> whose general fiber has Kodaira dimension  $-\infty$ ; the variety  $V$  is not birational to a terminal  $\mathbb{Q}$ -factorial Fano variety of Picard rank 1 that is not biregular to  $V$ ; the groups  $\text{Bir}(V)$  and  $\text{Aut}(V)$  coincide.

The notion of birational super-rigidity goes back to [35]. For example, the paper [35] implicitly proves that any smooth quartic 3-fold in  $\mathbb{P}^4$  is birationally super-rigid (see [20]).

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<sup>1</sup>A variety is called rational when it is birationally isomorphic to  $\mathbb{P}^n$ , i.e. when its field of rational functions is a purely transcendental extension of the base field.

<sup>2</sup>For a fibration  $\tau : Y \rightarrow Z$  we assume  $\dim(Y) > \dim(Z) \neq 0$  and  $\tau_*(\mathcal{O}_Y) = \mathcal{O}_Z$ .

*Remark 2.* A birationally super-rigid Fano variety is not rational and not birational to a conic bundle. However, there are non-rational Fano varieties that are not birationally super-rigid, e.g. a smooth cubic 3-fold.

Let  $\pi : X \rightarrow \mathbb{P}^n$  be a double cover branched over a hypersurface  $F$  of degree  $2n$  with isolated singularities. Then  $K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))$ . So, the variety  $X$  is a Fano variety.

*Remark 3.* The variety  $X$  is known to be birationally super-rigid in the following three cases:  $n \geq 3$  and  $F$  is smooth (see [33], [48]);  $n \geq 3$  and the hypersurface  $F$  has one ordinary singular point of even multiplicity that does not exceed  $2(n-2)$  (see [50]);  $n = 3$  and the variety  $X$  is nodal and  $\mathbb{Q}$ -factorial (see [17]). For  $n \geq 3$  the non-rationality of a double cover of  $\mathbb{P}^n$  ramified in a very general hypersurface of degree greater than  $\frac{n+1}{2}$  is proved in [42].

The main purpose of this paper is to prove the following result.

**Theorem 4.** *Suppose that  $n \geq 4$  and every singular point  $O$  of  $F$  is ordinary, i.e. the projectivization of a tangent cone to  $F$  at  $O$  is smooth, such that  $\text{mult}_O(F) \leq 2(n-2)$ . Then  $X$  is birationally super-rigid.*

**Corollary 5.** *In the conditions of Theorem 4, the group  $\text{Bir}(X)$  is finite.*

**Corollary 6.** *A double cover of  $\mathbb{P}^n$  branched over a nodal hypersurface of degree  $2n$  with any number of ordinary double points is not birationally equivalent to any elliptic fibration for  $n \geq 4$ .*

*Example 7.* Let  $n = 2k$  for  $k \in \mathbb{N}$  and  $F \subset \mathbb{P}^{2k}$  be a sufficiently general hypersurface of degree  $4k$  passing through a linear subspace  $\Pi \subset \mathbb{P}^{2k}$  of dimension  $k$ . The variety  $X$  can be given by the equation

$$y^2 = \sum_{i=1}^k a_i(x_0, \dots, x_{2k+1})x_i \subset \mathbb{P}(1^{2k+1}, 2k) \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_{2k+1}, y]),$$

where  $a_i$  is a homogeneous polynomial of degree  $4k-1$ , and the linear subspace  $\Pi \subset \mathbb{P}^n$  is given by  $x_1 = \dots = x_k = 0$ . The hypersurface  $F$  is nodal, it has  $(4k-1)^k$  ordinary double points given by the equations

$$a_1 = \dots = a_k = x_1 = \dots = x_k = 0,$$

and  $X$  is non-rational for  $k \geq 2$  by Corollary 6.

*Example 8.* Let  $n = 2k+1$  for  $k \in \mathbb{N}$  and  $F \subset \mathbb{P}^{2k+1}$  be a sufficiently general hypersurface of degree  $4k+2$  that is given by the equation

$$g^2(x_0, \dots, x_{2k+2}) = \sum_{i=1}^k a_i(x_0, \dots, x_{2k+2})b_i(x_0, \dots, x_{2k+2}),$$

where  $g$ ,  $a_i$  and  $b_i$  are homogeneous polynomials of degree  $2k+1$ , and  $x_i$  is a homogeneous coordinate on  $\mathbb{P}^n$ . The hypersurface  $F$  is nodal, and it has  $(2k+1)^{2k+1}$  ordinary double points given by the equations

$$g = a_1 = \dots = a_k = b_1 = \dots = b_k = 0,$$

and the double cover  $\pi : X \rightarrow \mathbb{P}^{2k+1}$  branched over  $F$  is non-rational and birationally super-rigid for  $k \geq 2$  by Theorem 4. In the case  $k = 1$  one can unproject (see [52]) the variety  $X$  into a fibration of cubic surfaces, i.e. the variety  $X$  is not birationally super-rigid. In the latter case it is unknown whether  $X$  is rational or not.

*Remark 9.* In the conditions of Theorem 4, the best known upper bound of the number of ordinary singular points of the hypersurface  $F \subset \mathbb{P}^n$  is due to [57]. Namely,  $|\text{Sing}(F)| \leq A_n(2n)$ , where  $A_n(2n)$  is a number of integer points  $(a_1, \dots, a_n) \in \mathbb{R}^n$  such that  $(n-1)^2 < \sum_{i=1}^n a_i \leq n^2$  and all  $a_i \in (0, 2n)$ . Hence,  $|\text{Sing}(X)|$  does not exceed 68, 1190 and 27237 when  $n = 3, 4$  and  $5$  respectively. It is expected that this bound is far from being sharp for  $n \gg 0$  (cf. [56]). In the case  $n = 3$  there is a sharp bound  $|\text{Sing}(X)| \leq 65$  (see [55], [10], [5], [37], [58]).

The condition  $\text{mult}_O(F) \leq 2(n-2)$  in Theorem 4 can not be omitted.

*Example 10.* Let  $O$  be a singular point of  $F$  such that  $\text{mult}_O(F) = 2(n-1)$ , and  $\gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$  be a projection from  $O$ . Then the normalization of the general fiber of  $\gamma \circ \pi$  is a smooth rational curve, i.e.  $X$  is birationally isomorphic to a conic bundle.

The condition  $n \geq 4$  in Theorem 4 can not be omitted.

*Example 11.* Let  $n = 3$  and  $F$  be a Barth sextic (see [5]) given by

$$4(\alpha^2 x^2 - y^2)(\alpha^2 y^2 - z^2)(\alpha^2 z^2 - x^2) - t^2(1 + 2\alpha)(x^2 + y^2 + z^2 - t^2)^2 = 0$$

in  $\mathbb{P}^3 \cong \text{Proj}(\mathbb{C}[x, y, z, t])$ , where  $\alpha = \frac{1+\sqrt{5}}{2}$ . Then  $X$  has only ordinary double points,  $|\text{Sing}(X)| = 65$ , and  $X$  is birational to a determinantal quartic 3-fold in  $\mathbb{P}^4$  with 42 nodes (see [26], [46]). Thus,  $X$  is rational.

The claim of Theorem 4 holds for  $n = 3$  in the additional assumption that  $X$  is  $\mathbb{Q}$ -factorial, which is always the case when the number of nodes of  $X$  does not exceed 14 due to [17]. On the other hand, there are nodal double covers of  $\mathbb{P}^3$  with 15 nodes that are not  $\mathbb{Q}$ -factorial and not birationally super-rigid.

The nature of Theorem 4 is a reminiscence of the Noether theorem on the structure of the group  $\text{Bir}(\mathbb{P}^2)$  (see [45], [33], [20]). The relevant problem is to classify pencils of plane elliptic curves up to the action of the group  $\text{Bir}(\mathbb{P}^2)$ . It was studied in [6]. The ideas of [6] were recovered later in [24], where it was proved that any pencil of plane elliptic curves can be birationally transformed into a special elliptic pencil, so-called Halphen pencil (see [29] and §5.6 of [23]). The similar problem can be considered for the variety  $X$  as well. Namely, we prove the following result.

**Theorem 12.** *In the conditions of Theorem 4, let  $\rho : X \dashrightarrow Z$  be a rational map such that the normalization of a general fiber of  $\rho$  is a connected elliptic curve. Then there is a point  $O$  of the hypersurface  $F$  and a birational map  $\gamma : \mathbb{P}^{n-1} \dashrightarrow Y$  such that  $\text{mult}_O(F) = 2(n-2)$  and  $\rho = \gamma \circ \beta \circ \pi$ , where  $\beta : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$  is a projection from the point  $O \in \mathbb{P}^n$ .*

*Example 13.* Let  $F \subset \mathbb{P}^n$  be a hypersurface given by the equation

$$\sum_{i=0}^4 g_{2n-i}(x_1, \dots, x_n) x_0^i = 0 \subset \mathbb{P}^n \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_n]),$$

where  $g_i$  is a general homogeneous polynomial of degree  $i$ . The hypersurface  $F$  is smooth outside a point  $(1 : 0 : \dots : 0) \in \mathbb{P}^n$ , which is an ordinary singular point of  $F$  of multiplicity  $2n - 2$ . Thus, in the case  $n \geq 4$  the variety  $X$  is birationally equivalent to a single elliptic fibration induced by the projection from the point  $O$  by Theorem 12.

**Corollary 14.** *A double cover of  $\mathbb{P}^n$  branched over a nodal hypersurface of degree  $2n$  with any number of ordinary double points is not birationally equivalent to any elliptic fibration for  $n \geq 4$ .*

The condition  $n \geq 4$  in Theorem 12 can not be omitted (see [17]).

*Example 15.* Let  $n = 3$  and  $F \subset \mathbb{P}^3$  be a nodal sextic such that  $F$  contains a line  $L \subset \mathbb{P}^3$  and the set  $\text{Sing}(F) \cap L$  consists of 4 nodes. For a sufficiently general point  $P \in X$ , there is a unique hyperplane  $H \subset \mathbb{P}^3$  passing through the point  $\pi(P)$  and the line  $L$ . For a quintic curve  $C \subset H$  given by  $F \cap H = L \cup C$ , the intersection  $L \cap (C \setminus \text{Sing}(X))$  consists of a single point  $Q$ . Take a line  $L_P \subset \mathbb{P}^3$  passing through  $\pi(P)$  and  $Q$  and define a rational map  $\Xi : X \dashrightarrow \text{Gr}(2, 4)$  by  $\Xi(P) = L_P$ . The normalization of a general fiber of the map  $\Xi$  is an elliptic curve. The rational map  $\Xi$  can not be obtained by means of the construction in Theorem 12.

Birational transformations of smooth Fano 3-folds into elliptic fibrations were used in [7], [8], [30] in the proof of the following result.

**Theorem 16.** *The set of rational points is potentially dense<sup>3</sup> on every smooth Fano 3-fold defined over a number field  $\mathbb{F}$  with a possible exception of a double cover of  $\mathbb{P}^3$  ramified in a smooth sextic surface.*

The possible exception appears in Theorem 16 because a smooth sextic double solid is the only smooth Fano 3-fold that is not birationally isomorphic to an elliptic fibration (see [11]). For results relevant to Theorem 12 see [11], [12], [13], [15], [16], and [53]. The proof of Theorem 12 implicitly gives the following result.

**Theorem 17.** *In the conditions of Theorem 4, the variety  $X$  is not birationally isomorphic to a Fano variety with canonical singularities.*

The condition  $n \geq 4$  in Theorem 17 can not be omitted (see [17]).

## 2. PRELIMINARY RESULTS.

In this chapter we consider properties of usual log-pairs and so-called movable log pairs (see [1], [11], [15]). The basic notions, notations and definitions can be found in [39], [40], [20], [21], [51], [15]. A priori we do not assume any restriction on the coefficients of the considered boundaries.

<sup>3</sup>The set of rational points of a variety  $V$  defined over a number field  $\mathbb{F}$  is called potentially dense if for a finite extension of fields  $\mathbb{K}/\mathbb{F}$  the set of  $\mathbb{K}$ -rational points of the variety  $V$  is Zariski dense.

**Theorem 18.** *Let  $X$  be a terminal  $\mathbb{Q}$ -factorial Fano variety of Picard rank 1 such that the set of centers of canonical singularities  $\mathbb{CS}(X, M_X)$  is empty for every movable log pair  $(X, M_X)$  such that  $M_X$  is effective and  $-(K_X + M_X)$  is ample. Then  $X$  is birationally super-rigid.*

*Proof.* See [20], [51] or [15].  $\square$

**Theorem 19.** *Let  $X$  be a terminal  $\mathbb{Q}$ -factorial Fano variety of Picard rank 1,  $\rho : X \dashrightarrow Y$  be a birational map,  $\tau : Y \rightarrow Z$  be a fibration whose general fiber has Kodaira dimension zero,  $H$  be a very ample divisor on  $Z$ , and  $M_X = r\rho^{-1}(|\tau^*(H)|)$  for a positive rational number  $r$  such that  $K_X + M_X \sim_{\mathbb{Q}} 0$ . Then the set of centers of canonical singularities  $\mathbb{CS}(X, M_X)$  is not empty.*

*Proof.* See [11], [15] and [17].  $\square$

**Theorem 20.** *Let  $X$  be a terminal  $\mathbb{Q}$ -factorial Fano variety of Picard rank 1,  $\rho : X \dashrightarrow Y$  be a non-biregular birational map,  $Y$  be a Fano variety with canonical singularities, and  $M_X = \frac{1}{n}\rho^{-1}(|-nK_Y|)$  for some natural number  $n \gg 0$ . Then  $K_X + M_X \sim_{\mathbb{Q}} 0$  and the set of centers of canonical singularities  $\mathbb{CS}(X, M_X)$  is not empty.*

*Proof.* See [11], [15] and [17].  $\square$

**Theorem 21.** *Let  $(X, B_X)$  be a log pair with effective  $B_X$ ,  $\mathcal{I}(X, B_X)$  be an ideal sheaf of the log canonical singularities subscheme  $\mathcal{L}(X, B_X)$ , and let  $H$  be a nef and big divisor on  $X$  such that  $K_X + B_X + H$  is a Cartier divisor. Then  $H^i(X, \mathcal{I}(X, B_X) \otimes (K_X + B_X + H)) = 0$  for  $i > 0$ .*

*Proof.* See [54], [40], [43], [2] or [15].  $\square$

**Theorem 22.** *Let  $(X, B_X)$  be a log pair,  $B_X$  be a effective boundary such that  $\lfloor B_X \rfloor = \emptyset$ , and let  $S \subset X$  be an effective irreducible divisor such that the divisor  $K_X + S + B_X$  is  $\mathbb{Q}$ -Cartier. Then  $(X, S + B_X)$  is purely log terminal if and only if  $(S, \text{Diff}_S(B_X))$  is Kawamata log terminal.*

*Proof.* See Theorem 17.6 in [40] or Theorem 7.5 in [43].  $\square$

**Corollary 23.** *Let  $(X, B_X)$  be a log pair with effective  $B_X$ ,  $H$  be an effective Cartier divisor on  $X$ ,  $Z \in \mathbb{CS}(X, B_X)$ , both  $X$  and  $H$  are smooth in the generic point of  $Z \subset H \not\subset \text{Supp}(B_X)$ . Then the set of centers of log canonical singularities  $\mathbb{LCS}(H, B_X|_H)$  is not empty.*

**Theorem 24.** *Let  $X$  be a smooth variety,  $\dim(X) \geq 3$ ,  $M_X$  be an effective movable boundary on the variety  $X$ , and the set  $\mathbb{CS}(X, M_X)$  contains a closed point  $O \in X$ . Then the inequality  $\text{mult}_O(M_X^2) \geq 4$  holds and the equality implies  $\text{mult}_O(M_X) = 2$  and  $\dim(X) = 3$ .*

*Proof.* See [35], [20], [51], [21] and [38].  $\square$

**Theorem 25.** *Let  $X$  be a variety,  $\dim(X) \geq 3$ , and  $B_X$  be an effective boundary on  $X$  such that the set  $\mathbb{CS}(X, B_X)$  contains an ordinary double point  $O$  of  $X$ . Then  $\text{mult}_O(B_X) \geq 1$  and the equality implies  $\dim(X) = 3$ .*

*Proof.* The claim is implied by Theorem 3.10 in [21] and Theorem 22.  $\square$

**Proposition 26.** *Let  $\tau : V \rightarrow \mathbb{P}^k$  be a double cover ramified in a smooth hypersurface  $S \subset \mathbb{P}^k$  of degree  $2d$  such that  $2 \leq d \leq k-1$ ,  $B_V$  be an effective boundary on  $V$  such that  $\frac{1}{r}B_V \sim_{\mathbb{Q}} \tau^*(\mathcal{O}_{\mathbb{P}^k}(1))$  for some positive rational number  $r < 1$ . Then the set of centers of log canonical singularities  $\mathbb{LCS}(V, B_V)$  is empty.*

*Proof.* Let  $C \subset V$  be an irreducible curve such that  $\tau(C) \subset S$  and the inequality  $\text{mult}_C(B_V) \geq 1$  holds. Take a point  $O$  on the curve  $\tau(C)$  and a hyperplane  $\Pi \subset \mathbb{P}^k$  that tangents  $S$  at the point  $O$ . Fix a line  $L \subset \Pi$  passing through  $O$ . Let  $\hat{L} = \tau^{-1}(L)$ . Then  $\hat{L}$  is singular at  $\hat{O} = \tau^{-1}(O)$  and a component of  $\hat{L}$  is contained in  $\text{Supp}(B_V)$ , because otherwise

$$2 > 2r = \hat{L} \cdot B_V \geq \text{mult}_{\hat{O}}(\hat{L})\text{mult}_C(B_V) \geq 2$$

which is a contradiction. On the other hand,  $\Pi$  tangents  $S$  in finitely many points (see [27], [36], [49], [59]). Hence, the curve  $\hat{L}$  spans  $V$  when we vary the point  $O$  on the curve  $\tau(C)$  and the line  $L \subset \Pi$ . The latter is a contradiction, because  $\hat{L} \subset \text{Supp}(B_V)$ .

Suppose that  $\mathbb{LCS}(V, B_V)$  contains a subvariety  $Z \subset V$  of dimension at least two. Then  $\text{mult}_Z(B_V) \geq 1$  and the set  $Z \cap \tau^{-1}(S)$  contains some curve  $\hat{C} \subset V$ . Then  $\text{mult}_{\hat{C}}(B_V) \geq 1$  and  $\tau(\hat{C}) \subset S$ , but we already prove that this is impossible. Hence, the set  $\mathbb{LCS}(V, B_V)$  does not contain subvarieties of dimension at least two.

Suppose that the set  $\mathbb{LCS}(V, B_V)$  contains a curve on  $V$ . Consider a union  $T \subset V$  of all curves in the set  $\mathbb{LCS}(V, B_V)$ . We may consider  $T$  as a possibly reducible curve on  $V$ . Let  $Y$  be a sufficiently general divisor in the linear system  $|\tau^*(\mathcal{O}_{\mathbb{P}^k}(1))|$ ,  $\gamma = \tau|_Y$  and  $B_Y = B_V|_Y$ . Then the variety  $Y$  is smooth,  $Y \not\subset \text{Supp}(B_V)$ , and  $\gamma : Y \rightarrow \mathbb{P}^{k-1}$  is a double cover branched over a smooth hypersurface of degree  $2d$ . The generality in the choice of  $Y$  implies that the set  $\mathbb{LCS}(Y, B_Y)$  does not contain subvarieties of  $Y$  of positive dimension. Moreover, the set  $\mathbb{LCS}(Y, B_Y)$  is not empty, i.e. it contains all points of  $T \cap Y$ . Consider a Cartier divisor

$$F = K_Y + B_Y + (1-r)H \sim (d-k-1)H$$

where  $H = \gamma^*(\mathcal{O}_{\mathbb{P}^{k-1}}(1))$ . The sequence of groups

$$H^0(\mathcal{O}_Y(F)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(Y, B_Y)}(F)) \rightarrow 0$$

is exact by Theorem 21 where  $\mathcal{L}(Y, B_Y)$  is a log canonical singularities subscheme of  $(Y, B_Y)$ . On the other hand,  $\text{Supp}(\mathcal{L}(Y, B_Y))$  consists of all points in  $T \cap Y$ . Hence,  $H^0(\mathcal{O}_{\mathcal{L}(Y, B_Y)}(F)) = H^0(\mathcal{O}_{T \cap Y}(F))$ . The latter contradicts  $d < k-1$ , because  $H^0(\mathcal{O}_Y(F)) = 0$  for  $d < k-1$ . However, in the case when  $d = k-1$  the latter implies that the set  $T \cap Y$  consists of a single point, because  $H^0(\mathcal{O}_Y(F)) = \mathbb{C}$  for  $d = k-1$ .

We proved that the assumption that the set  $\mathbb{LCS}(V, B_V)$  contains some curve on  $V$  implies that  $d = k-1$ , the set  $\mathbb{LCS}(V, B_V)$  contains a single curve  $\bar{C} \subset V$  such that  $\tau(\bar{C}) \subset \mathbb{P}^k$  is a line,  $\tau|_{\bar{C}}$  is an isomorphism, and the inequality  $\text{mult}_{\bar{C}}(B_V) \geq 1$  holds. On the other hand, we already

proved that the latter implies  $\tau(\bar{C}) \not\subset S$ . Therefore, there is an irreducible reduced curve  $\tilde{C} \subset V$  such that  $\bar{C} \neq \tilde{C}$  and  $\tau(\bar{C}) = \tau(\tilde{C})$ .

Let  $D_1, \dots, D_{k-2}$  be sufficiently general divisors in  $|\tau^*(\mathcal{O}_{\mathbb{P}^k}(1))|$  passing through the curves  $\bar{C}$  and  $\tilde{C}$ . Put  $D = \cap_{i=1}^{k-2} D_i$ . Then  $D$  is a smooth surface, and both  $\bar{C}$  and  $\tilde{C}$  are smooth rational curves on  $D$ . By the adjunction formula the self-intersections of the curves  $\bar{C}$  and  $\tilde{C}$  on the surface  $D$  are equal to  $1 - d$ . Therefore,  $\bar{C}^2 = \tilde{C}^2 < 0$  due to  $d > 2$ .

By construction we have  $D \not\subset \text{Supp}(B_V)$ . Therefore, we can consider a boundary  $B_D = B_V|_D$ . The generality in the choice of  $D$  implies

$$B_D = \text{mult}_{\bar{C}}(B_V)\bar{C} + \text{mult}_{\tilde{C}}(B_V)\tilde{C} + \Delta$$

where  $\Delta$  is an effective divisor on the surface  $D$  such that  $\text{Supp}(\Delta)$  does not contain both curves  $\bar{C}$  and  $\tilde{C}$ . On the other hand, the equivalence

$$B_D \sim_{\mathbb{Q}} r(\bar{C} + \tilde{C})$$

holds. In particular, the equivalence

$$(r - \text{mult}_{\tilde{C}}(B_V))\tilde{C} \sim_{\mathbb{Q}} (\text{mult}_{\bar{C}}(B_V) - r)\bar{C} + \Delta$$

holds. Therefore,  $\text{mult}_{\tilde{C}}(B_V) \geq r$  due to  $\tilde{C}^2 < 0$ . Thus, the equivalence

$$-\Delta \sim_{\mathbb{Q}} (\text{mult}_{\bar{C}}(B_V) - r)\bar{C} + (\text{mult}_{\tilde{C}}(B_V) - r)\tilde{C}$$

implies  $\Delta = \emptyset$  and  $\text{mult}_{\tilde{C}}(B_V) = \text{mult}_{\bar{C}}(B_V) = r$ . The latter is impossible, because  $\text{mult}_{\bar{C}}(B_V) \geq 1$  and  $r < 1$ . Therefore, the set  $\mathbb{LCS}(V, B_V)$  does not contain subvarieties of positive dimension.

Suppose that  $\mathbb{LCS}(V, B_V)$  contains a closed point  $O$  on  $V$ . Let

$$E = K_V + B_V + (1 - r)H$$

where  $H = \tau^*(\mathcal{O}_{\mathbb{P}^k}(1))$ . Then  $E$  is a Cartier divisor and  $H^0(\mathcal{O}_V(E)) = 0$ , because  $E \sim (d - k)H$  and  $d \leq k - 1$ . However, the sequence

$$H^0(\mathcal{O}_V(E)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(E)) \rightarrow 0$$

is exact by Theorem 21 where  $\mathcal{L}(V, B_V)$  is a log canonical singularities subscheme of  $(V, B_V)$ . On the other hand,  $\text{Supp}(\mathcal{L}(V, B_V))$  consists of finite number of points of  $V$ . Hence,  $H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(E)) = H^0(\mathcal{O}_{\mathcal{L}(V, B_V)})$  which is a contradiction. Thus, the set  $\mathbb{LCS}(V, B_V)$  is empty.  $\square$

**Proposition 27.** *Let  $S \subset \mathbb{P}^n$  be a smooth hypersurface of degree  $d \geq 2$  and  $B$  be an effective boundary on  $\mathbb{P}^n$  such that  $\frac{1}{r}B \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^n}(1)$  for a positive rational number  $r < 1$ . Then the set of centers of log canonical singularities  $\mathbb{LCS}(\mathbb{P}^n, B + \frac{1}{2}S)$  is empty for  $d \leq 2(n - 1)$ .*

*Proof.* Let  $Z \in \mathbb{LCS}(\mathbb{P}^n, B + \frac{1}{2}S)$  be a center of maximal dimension. Then

$$r + \frac{1}{2} \geq \text{mult}_Z(B) + \frac{1}{2}\text{mult}_Z(S) \geq \text{mult}_Z(B + \frac{1}{2}S) \geq 1$$

which implies  $Z \subset S$  and  $Z \neq S$ . Hence,  $\dim(Z) < n - 1$ .

Suppose that  $Z$  is a closed point. Let

$$E = K_{\mathbb{P}^n} + B + \frac{1}{2}S + (n - \frac{d}{2} - r)H$$

where  $H \sim \mathcal{O}_{\mathbb{P}^n}(1)$ . Then  $E$  is a Cartier divisor,  $n - \frac{d}{2} - r > 0$  and the equivalence  $E \sim -H$  holds. Thus,  $H^0(\mathcal{O}_{\mathbb{P}^n}(E)) = 0$ . The sequence

$$H^0(\mathcal{O}_{\mathbb{P}^n}(E)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(\mathbb{P}^n, B + \frac{1}{2}S)}(E)) \rightarrow 0$$

is exact by Theorem 21 where  $\mathcal{L}(\mathbb{P}^n, B + \frac{1}{2}S)$  is a log canonical singularities subscheme of  $(\mathbb{P}^n, B + \frac{1}{2}S)$ . However,  $\text{Supp}(\mathcal{L}(\mathbb{P}^n, B + \frac{1}{2}S))$  consists of finite number of closed points of  $\mathbb{P}^n$ . Hence,

$$H^0(\mathcal{O}_{\mathcal{L}(\mathbb{P}^n, B + \frac{1}{2}S)}(E)) = H^0(\mathcal{O}_{\mathcal{L}(\mathbb{P}^n, B + \frac{1}{2}S)})$$

which is a contradiction. Thus,  $\dim(Z) > 0$ .

Rewrite  $B + \frac{1}{2}S$  as  $D + \lambda S$  for an effective boundary  $D$  on  $\mathbb{P}^n$  and a positive rational  $\lambda$  such that  $S \not\subset \text{Supp}(D)$ . Then  $\lambda < 1$  and  $D \sim_{\mathbb{Q}} \mu H$  for a positive rational number  $\mu < 1$ . In particular,  $Z \subset S$  is a center of log canonical singularities of log pair  $(\mathbb{P}^n, D + S)$ . Thus, Theorem 22 implies  $\mathbb{LCS}(S, D|_S) \neq \emptyset$ . Moreover, Theorem 22 implies the existence of a subvariety  $T \subset S$  such that  $T \in \mathbb{LCS}(S, D|_S)$  and  $Z \subseteq T$ . In particular, the inequalities  $\dim(T) \geq 1$  and  $\text{mult}_T(D|_S) \geq 1$  hold, where  $S$  is smooth by assumption. The latter is impossible due to [49]. Namely, let  $C$  be a curve in  $T$ ,  $Y \subset \mathbb{P}^n$  be a general cone over  $C$  and  $\tilde{C} \subset S$  be a residual curve to the curve  $C$  defined as  $C \cup \tilde{C} = Y \cap S$ . Then  $\text{mult}_C(D|_S) \geq 1$ , the intersection  $C \cap \tilde{C}$  consists of  $(\deg(S) - 1)\deg(C)$  different points in a set-theoretic sense, and  $\tilde{C} \not\subset \text{Supp}(D)$ . In particular,

$$\deg(D|_{\tilde{C}}) \geq (\deg(S) - 1)\deg(C)\text{mult}_C(D|_S) \geq (\deg(S) - 1)\deg(C),$$

but  $\deg(D|_{\tilde{C}}) = \mu(\deg(S) - 1)\deg(C)$ , which is a contradiction.  $\square$

### 3. THE PROOF OF THEOREM 4.

Let  $\pi : X \rightarrow \mathbb{P}^n$  be a double cover ramified in a hypersurface  $F \subset \mathbb{P}^n$  of degree  $2n$  with isolated singularities such that  $n \geq 4$  and every singular point  $O$  of  $F$  is an ordinary singular point and  $\text{mult}_O(F) \leq 2(n - 2)$ .

**Lemma 28.** *The variety  $V$  is a Fano variety with terminal  $\mathbb{Q}$ -factorial singularities such that  $\text{Cl}(X) \cong \text{Pic}(X) \cong \mathbb{Z}$ .*

*Proof.* The ampleness of the divisor  $-K_X$  and the terminality of  $X$  are obvious. Consider a Weil divisor  $D$  on the variety  $X$ . To prove the claim it is enough to show that  $D \sim \pi^*(\mathcal{O}_{\mathbb{P}^n}(r))$  for some  $r \in \mathbb{Z}$ .

Let  $H$  be a general divisor in  $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(k))|$  for  $k \gg 0$ . Then  $H$  is a smooth complete intersection in  $\mathbb{P}(1^{n+1}, n)$  and  $\dim(X) \geq 3$ . Therefore, the group  $\text{Pic}(H)$  is generated by  $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|_H$  by Théorème 3.13 of Exp. XI in [28] (see Lemma 3.2.2 in [25], Lemma 3.5 in [22] or [9]). Thus, there is an integer  $r$  such that  $D|_H \sim \pi^*(\mathcal{O}_{\mathbb{P}^n}(r))|_H$ .

Let  $\Delta = D - \pi^*(\mathcal{O}_{\mathbb{P}^n}(r))$ . The sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(\Delta) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-k)) \rightarrow \mathcal{O}_X(\Delta) \rightarrow \mathcal{O}_H \rightarrow 0$$

is exact, because the sheaf  $\mathcal{O}_X(\Delta)$  is locally free in the neighborhood of the divisor  $H$ . Therefore, the sequence of groups

$$0 \rightarrow H^0(\mathcal{O}_X(\Delta)) \rightarrow H^0(\mathcal{O}_H) \rightarrow H^1(\mathcal{O}_X(\Delta) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-k)))$$



is exact. On the other hand, there is an exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X(\Delta) \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

where  $\mathcal{E}$  is a locally free sheaf and  $\mathcal{F}$  is a torsion free sheaf, because the sheaf  $\mathcal{O}_X(\Delta)$  is reflexive (see [31]). Hence, the sequence of groups

$$H^0(\mathcal{F} \otimes \mathcal{O}_X(-H)) \rightarrow H^1(\mathcal{O}_X(\Delta - H)) \rightarrow H^1(\mathcal{E} \otimes \mathcal{O}_X(-H))$$

is exact. However,  $H^0(\mathcal{F} \otimes \mathcal{O}_X(-H)) = 0$ , because the sheaf  $\mathcal{F}$  has no torsion, and  $H^1(\mathcal{E} \otimes \mathcal{O}_X(-H)) = 0$  by the lemma of Enriques-Severi-Zariski (see [60]). Thus, we have

$$H^1(\mathcal{O}_X(\Delta) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-k))) = 0$$

and  $H^0(\mathcal{O}_X(\Delta)) = \mathbb{C}$ . The same method gives  $H^0(\mathcal{O}_X(-\Delta)) = \mathbb{C}$ , i.e. the divisor  $\Delta$  is rationally equivalent to zero.  $\square$

Suppose that  $X$  is not birationally super-rigid. Then there is a movable log pair  $(X, M_X)$  such that  $M_X$  is effective, the set of centers of canonical singularities  $\mathbb{CS}(X, M_X)$  is not empty and the divisor  $-(K_X + M_X)$  is ample by Theorem 18. Let  $Z$  be an element of the set  $\mathbb{CS}(X, M_X)$ .

**Lemma 29.** *The subvariety  $Z \subset X$  is not a smooth point of  $X$ .*

*Proof.* Let  $Z$  be a smooth point of  $X$ . Then  $\text{mult}_Z(M_X^2) > 4$  by Theorem 24. Consider  $n-2$  general divisors  $H_1, \dots, H_{n-2}$  in  $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$  that pass through the point  $Z$ . Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geq \text{mult}_Z(M_X^2) \text{mult}_Z(H_1) \cdots \text{mult}_Z(H_{n-2}) > 4$$

which is a contradiction.  $\square$

**Lemma 30.** *The subvariety  $Z \subset X$  is not a singular point of  $X$ .*

*Proof.* The variety  $X$  can be given as a hypersurface

$$y^2 = f_{2n}(x_0, \dots, x_n) \subset \mathbb{P}(1^{n+1}, n) \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_n, y])$$

where  $f_{2n}$  is a homogeneous polynomial of degree  $2n$ . Suppose that  $Z$  is a singular point of  $X$ . Then  $O = \pi(Z)$  is an ordinary singular point on the hypersurface  $F \subset \mathbb{P}^n$ . There are two possible cases, i.e.  $\text{mult}_O(F)$  is even or odd. We handle them separately.

Suppose  $\text{mult}_O(F) = 2m \geq 2$  for some  $m \in \mathbb{N}$ . By the initial assumption  $m \leq n-2$ . There is a weighted blow up  $\beta : U \rightarrow \mathbb{P}(1^{n+1}, n)$  of the point  $Z$  with weights  $(m, 1^n)$  such that the proper transform  $V \subset U$  of the variety  $X$  is non-singular in the neighborhood of the  $\beta$ -exceptional divisor  $E$ . The morphism  $\beta$  induces a birational morphism  $\alpha : V \rightarrow X$  with an exceptional divisor  $G \subset V$ . Then  $E|_V = G$  and  $G$  is a smooth hypersurface in  $E \cong \mathbb{P}(1^n, m)$  which can be given by

$$z^2 = g_{2m}(t_1, \dots, t_n) \subset \mathbb{P}(1^n, m) \cong \text{Proj}(\mathbb{C}[t_1, \dots, t_n, z])$$

where  $g_{2m}$  is a homogeneous polynomial of degree  $2m$ .

Let  $M_V = \alpha^{-1}(M_X)$  and  $\text{mult}_Z(M_X)$  be a positive rational number such that  $M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \text{mult}_Z(M_X)G$ . Then the equivalence

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^*(K_X + M_X) + (n-1-m-\text{mult}_Z(M_X))G$$

holds. However, the linear system  $|\alpha^*(-K_X) - G|$  is free and gives a fibration  $\psi : V \rightarrow \mathbb{P}^{n-1}$  such that  $\psi = \chi \circ \pi \circ \alpha$  where  $\chi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$  is a projection from  $O$ . Let  $C$  be a general fiber of  $\psi$ . Then

$$M_V \cdot C = 2(1 - \text{mult}_Z(M_X)) + \alpha^*(K_X + M_X) \cdot C < 2(1 - \text{mult}_Z(M_X))$$

because  $-(K_X + M_X)$  is ample. Thus,  $\text{mult}_Z(M_X) < 1$ . The latter contradicts Theorem 25 in the case of  $m = 1$ . Thus,  $m > 1$ . On the other hand, the inequality  $(n - 1 - m - \text{mult}_Z(M_X)) > 0$  implies the existence of a center  $\Delta \in \mathbb{CS}(V, M_V)$  such that  $\Delta \subset G$ . Hence,  $\mathbb{LCS}(G, M_V|_G) \neq \emptyset$  by Corollary 23. The latter contradicts Proposition 26.

Therefore,  $\text{mult}_O(F) = 2k + 1 \geq 3$  for  $k \in \mathbb{N}$ . Then  $k \leq n - 3$  by the initial assumption. Let  $\lambda : W \rightarrow \mathbb{P}^n$  be a blow up of  $O$ ,  $\Lambda$  be an exceptional divisor of the birational morphism  $\lambda$ , and  $\tilde{F} \subset W$  be a proper transform of the hypersurface  $F$ . Then  $\tilde{F}$  is smooth in the neighborhood of the exceptional divisor  $\Lambda$  and  $S = \tilde{F} \cap \Lambda \subset \Lambda \cong \mathbb{P}^{n-1}$  is a smooth hypersurface of degree  $2k + 1$ . Let  $\tilde{\pi} : \tilde{X} \rightarrow W$  be a double cover ramified in the effective divisor

$$\tilde{F} \cup \Lambda \sim 2(\lambda^*(\mathcal{O}_{\mathbb{P}^n}(n)) - k\Lambda)$$

which is singular only in  $S$ . Then  $W$  is smooth outside of  $\tilde{S} = \tilde{\pi}^{-1}(S)$  and the singularities of  $W$  along  $\tilde{S}$  is of type  $\mathbb{A}_1 \times \mathbb{C}^{n-2}$ , i.e. a two-dimensional ordinary double point along  $\tilde{S}$ . Let  $\Xi = \tilde{\pi}^{-1}(\Lambda)$ . Then  $\Xi \cong \mathbb{P}^{n-1}$  and there is a birational morphism  $\xi : \tilde{X} \rightarrow X$  contracting  $\Xi$  to the point  $Z$  such that  $\pi \circ \xi = \lambda \circ \tilde{\pi}$ . The birational morphism  $\xi$  is a restriction of the weighted blow up of  $\mathbb{P}(1^{n+1}, n)$  at  $Z$  with weights  $(2k + 1, 2^n)$ .

Let  $M_{\tilde{X}} = \xi^{-1}(M_X)$  and  $\text{mult}_Z(M_X)$  be a positive rational number such that  $M_{\tilde{X}} \sim_{\mathbb{Q}} \xi^*(M_X) - \text{mult}_Z(M_X)\Xi$ . Then the equivalence

$$K_{\tilde{X}} + M_{\tilde{X}} \sim_{\mathbb{Q}} \xi^*(K_X + M_X) + (2(n - 1 - k) - \text{mult}_Z(M_X))\Xi$$

holds. On the other hand, the linear system  $|\xi^*(-K_X) - 2\Xi|$  is free and gives a fibration  $\omega : \tilde{X} \rightarrow \mathbb{P}^{n-1}$  such that  $\omega = \chi \circ \pi \circ \xi$ , where  $\chi$  is a projection of  $\mathbb{P}^n$  to  $\mathbb{P}^{n-1}$  from  $O$ . Intersecting  $M_{\tilde{X}}$  with a general fiber of  $\omega$  we get  $\text{mult}_Z(M_X) < 2$ . Thus,  $(2(n - 1 - k) - \text{mult}_Z(M_X)) > 0$  which implies the existence of a center

$$Z \in \mathbb{CS}(\tilde{X}, M_{\tilde{X}} - (2(n - 1 - k) - \text{mult}_Z(M_X))\Xi)$$

such that  $Z \subset G$ . Hence,

$$Z \in \mathbb{LCS}(\tilde{X}, M_{\tilde{X}} - (2(n - 1 - k) - \text{mult}_Z(M_X))\Xi + 2\Xi)$$

because  $2\Xi$  is a Cartier divisor. However,

$$\mathbb{LCS}(\tilde{X}, M_{\tilde{X}} - (2(n - 2 - k) - \text{mult}_Z(M_X))\Xi) \subset \mathbb{LCS}(\tilde{X}, M_{\tilde{X}} + \Xi)$$

due to  $2k + 1 \leq 2(n - 2)$ , which implies

$$\mathbb{LCS}(\Xi, \text{Diff}_{\Xi}(M_{\tilde{X}})) = \mathbb{LCS}(\Xi, M_{\tilde{X}}|_{\Xi} + \text{Diff}_{\Xi}(0)) \neq \emptyset$$

by Theorem 22. However,  $\text{Diff}_{\Xi}(0) = \frac{1}{2}\tilde{S}$  (see [40], [47]) and

$$M_{\tilde{X}}|_{\Xi} \sim_{\mathbb{Q}} -\text{mult}_Z(M_X)\Xi|_{\Xi} \sim_{\mathbb{Q}} \frac{\text{mult}_Z(M_X)}{2}H,$$

where  $H$  is a hyperplane on  $\Xi \cong \mathbb{P}^{n-1}$ . Therefore, the set of log canonical singularities  $\mathbb{LCS}(\Xi, M_{\tilde{X}}|_{\Xi} + \frac{1}{2}\tilde{S})$  is empty by Proposition 27, which is a contradiction.  $\square$

**Lemma 31.** *The inequality  $\text{codim}(Z \subset X) > 2$  is impossible.*

*Proof.* Suppose that  $\text{codim}(Z \subset X) > 2$ . Then  $\dim(Z) \neq 0$  by Lemmas 29 and 30. Thus,  $\text{mult}_Z(M_X^2) \geq 4$  by Theorem 24. Take a point  $O$  on  $Z$  and sufficiently general divisors  $H_1, \dots, H_{n-2}$  in  $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$  that pass through the point  $O$ . Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geq \text{mult}_Z(M_X^2) \geq 4$$

which is a contradiction.  $\square$

**Lemma 32.** *The equality  $\text{codim}(Z \subset X) = 2$  is impossible.*

*Proof.* Suppose  $\text{codim}(Z \subset X) = 2$ . Then  $\text{mult}_Z(M_X) \geq 1$ . Take sufficiently general divisors  $H_1, \dots, H_{n-2}$  in  $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$ . Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geq \text{mult}_Z^2(M_X) Z \cdot H_1 \cdots H_{n-2} \geq Z \cdot H_1 \cdots H_{n-2},$$

because  $-(K_X + M_X)$  is ample and  $K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))$ . Thus,  $\pi(Z)$  is a linear subspace in  $\mathbb{P}^n$  of dimension  $n - 2$  and  $\pi|_Z$  is an isomorphism.

Let  $V = \cap_{i=1}^{n-3} H_i$ ,  $C = Z \cap V$ ,  $M_V = M_X|_V$  and  $\tau = \pi|_V$ . Then  $V$  is a smooth 3-fold,  $C \subset V$  is a curve,  $M_V$  is movable,  $\tau : V \rightarrow \mathbb{P}^3$  is a double cover branched over a smooth hypersurface  $S \subset \mathbb{P}^3$  of degree  $2n$ ,  $\tau(C)$  is a line in  $\mathbb{P}^3$ ,  $\tau|_C$  is an isomorphism. Moreover,  $\tau^*(\mathcal{O}_{\mathbb{P}^3}(1)) - M_V$  is an ample divisor and  $\text{mult}_C(M_V) = \text{mult}_Z(M_X)$ .

Suppose  $\tau(C) \not\subset S$ . Then there is an irreducible curve  $\tilde{C} \subset V$  such that  $C \neq \tilde{C}$  and  $\tau(C) = \tau(\tilde{C})$ . Take a general divisor  $D \in |\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$  passing through  $C$ . Then  $D$  is a smooth surface,  $C$  and  $\tilde{C}$  are smooth rational curves. By the adjunction formula  $C^2 = \tilde{C}^2 = 1 - n < 0$  on the surface  $D$ . Consider a boundary  $M_D = M_V|_D$ . The boundary  $M_D$  is no longer movable. However, the generality in the choice of  $D$  implies

$$M_D = \text{mult}_C(M_V)C + \text{mult}_{\tilde{C}}(M_V)\tilde{C} + \Delta$$

where  $\Delta$  is a movable boundary on  $D$ . However,  $M_V \sim_{\mathbb{Q}} rD$  for some rational number  $r < 1$ . Hence, the equivalence

$$(r - \text{mult}_{\tilde{C}}(M_V))\tilde{C} \sim_{\mathbb{Q}} (\text{mult}_C(M_V) - r)C + \Delta$$

holds. The inequality  $\tilde{C}^2 < 0$  implies  $\text{mult}_{\tilde{C}}(M_V) \geq r$ . Let  $H$  be a sufficiently general divisor in the linear system  $|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ . Then

$$2r^2 = M_V^2 \cdot H \geq \text{mult}_C^2(M_V) + \text{mult}_{\tilde{C}}^2(M_V) \geq 1 + r^2$$

which contradicts the inequality  $r < 1$ .

Suppose that  $\tau(C) \subset S$ . Let  $O$  be a general point on  $\tau(C)$  and  $T$  be a hyperplane in  $\mathbb{P}^3$  that tangents  $S$  at the point  $O$ . Consider a sufficiently general line  $L \subset T$  passing through  $O$ . Let  $\hat{L} = \tau^{-1}(L)$ . Then  $\hat{L}$  is singular at the point  $\hat{O} = \tau^{-1}(O)$ . Therefore,  $\hat{L} \subset \text{Supp}(M_V)$ , because otherwise

$$2 > \hat{L} \cdot M_V \geq \text{mult}_{\hat{O}}(\hat{L})\text{mult}_C(M_V) \geq 2$$

which is a contradiction. On the other hand, the curve  $\hat{L}$  spans a divisor in the variety  $V$  when we vary the line  $L \subset T$ . The latter contradicts the movability of the boundary  $M_V$ .  $\square$

Therefore, Theorem 4 is proved.

#### 4. THE PROOF OF THEOREMS 12 AND 17.

Let  $\pi : X \rightarrow \mathbb{P}^n$  be a double cover branched over an hypersurface  $F$  of degree  $2n$  with isolated singularities,  $n = \dim(X) \geq 4$  and every singular point  $O$  of the hypersurface  $F$  is an ordinary singular point of multiplicity  $\text{mult}_O(F) \leq 2(n-2)$ . Let  $\rho : X \dashrightarrow Y$  be a birational map and  $\tau : Y \rightarrow Z$  be an elliptic fibration. Take a very ample divisor  $H$  on the variety  $Z$  and consider a linear system  $\mathcal{M} = \rho^{-1}(|\pi^*(H)|)$ .

*Remark 33.* The linear system  $\mathcal{M}$  is not composed from a pencil.

Due to Lemma 28 there is a positive rational number  $r$  such that the equivalence  $K_X + r\mathcal{M} \sim_{\mathbb{Q}} 0$  holds. Let  $M_X = r\mathcal{M}$ . Then  $\text{CS}(X, M_X) \neq \emptyset$  by Theorem 19. Let  $Z$  be an element of the set  $\text{CS}(X, M_X)$ .

**Lemma 34.** *The subvariety  $Z \subset X$  is not a smooth point of  $X$ .*

*Proof.* See the proof of Lemma 29.  $\square$

**Lemma 35.** *Let  $Z$  be a singular point of  $X$ . Then  $\text{mult}_O(F) = 2(n-2)$  and  $\tau \circ \rho = \gamma \circ \beta \circ \pi$ , where  $O = \pi(Z)$ ,  $\beta : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$  is a projection from the point  $O$ , and  $\gamma : \mathbb{P}^{n-1} \dashrightarrow Y$  is a birational map.*

*Proof.* The point  $O$  is an ordinary singular point of  $F \subset \mathbb{P}^n$  such that the inequality  $\text{mult}_O(F) \leq 2(n-2)$  holds. Suppose that the multiplicity of the hypersurface  $F$  at the point  $O$  is even, i.e.  $\text{mult}_O(F) = 2m \geq 2$  for  $m \in \mathbb{N}$ . The variety  $X$  is a hypersurface in  $\mathbb{P}(1^{n+1}, n)$  of degree  $2n$ , and there is a weighted blow up  $\beta : U \rightarrow \mathbb{P}(1^{n+1}, n)$  of the point  $Z$  with weights  $(m, 1^n)$  such that the proper transform  $V \subset U$  of  $X$  is smooth near the exceptional divisor  $E$  of  $\beta$ . The birational morphism  $\beta$  induces the birational morphism  $\alpha : V \rightarrow X$ . Let  $G$  be an exceptional divisor of  $\alpha$ . Then  $E|_V = G$  and  $G$  is a double cover of  $\mathbb{P}^{n-1}$  branched over a smooth hypersurface of degree  $2m$ .

Let  $M_V = \alpha^{-1}(M_X)$  and  $\text{mult}_Z(M_X)$  be a positive rational number such that  $M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \text{mult}_Z(M_X)G$ . Then the equivalence

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^*(K_X + M_X) + (n-1-m-\text{mult}_Z(M_X))G$$

holds. On the other hand, the linear system  $|\alpha^*(-K_X) - G|$  is free and gives a fibration  $\psi : V \rightarrow \mathbb{P}^{n-1}$  such that  $\psi = \chi \circ \pi \circ \alpha$  where  $\chi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$  is a projection from the point  $O$ . Let  $C$  be a general fiber of  $\psi$ . Then

$$M_V \cdot C = 2(1 - \text{mult}_Z(M_X))$$

and  $g(C) = n - m + 1$ . Thus,  $\text{mult}_Z(M_X) \leq 1$ . On the other hand, the equality  $\text{mult}_Z(M_X) = 1$  implies that  $\psi$  and  $\tau$  are birationally equivalent fibrations, i.e. there is a birational map that maps the generic fiber of  $\psi$  into the generic fiber of  $\tau$ . The latter is impossible in the case of  $m < n-2$ , because  $g(C) \neq 1$ . In the case of  $m = n-2$  the equivalence of  $\tau$  and  $\psi$

implies the claim of the lemma. Thus, we may assume  $\text{mult}_Z(M_X) < 1$  and proceed as in the proof of Lemma 30 to get a contradiction.

Hence, we may assume that the multiplicity of the hypersurface  $F$  at the point  $O$  is odd. In this case the arguments above together with the proof of Lemma 30 give a contradiction.  $\square$

**Lemma 36.** *The inequality  $\text{codim}(Z \subset X) > 2$  is impossible.*

*Proof.* See the proof of Lemma 31.  $\square$

**Lemma 37.** *The equality  $\text{codim}(Z \subset X) = 2$  is impossible.*

*Proof.* Suppose  $\text{codim}(Z \subset X) = 2$ . Then  $\text{mult}_Z(M_X) \geq 1$ . Take sufficiently general divisors  $H_1, \dots, H_{n-2}$  in  $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$ . Then

$$2 = M_X^2 \cdot H_1 \cdots H_{n-2} \geq \text{mult}_Z^2(M_X) Z \cdot H_1 \cdots H_{n-2} \geq Z \cdot H_1 \cdots H_{n-2},$$

and  $k = Z \cdot H_1 \cdots H_{n-2}$  is either 1 or 2.

Suppose  $k = 2$ . Then for any two different divisors  $D_1$  and  $D_2$  in the linear system  $\mathcal{M}$  the intersection  $D_1 \cap D_2$  coincide with  $Z$  in the set-theoretic sense. Let  $p \notin Z$  be a sufficiently general point and  $\mathcal{D} \subset \mathcal{M}$  be a linear subsystem of divisors passing through the point  $p$ . Then  $\mathcal{D}$  has no base components, because  $\mathcal{M}$  is not composed from a pencil. Suppose that the divisors  $D_1$  and  $D_2$  are from  $\mathcal{D}$ . Then in the set-theoretic sense

$$p \in D_1 \cap D_2 = Z$$

which is a contradiction. Therefore,  $k = 1$ , i.e.  $\pi(Z) \subset \mathbb{P}^n$  is a linear subspace in of dimension  $n - 2$  and  $\pi|_Z$  is an isomorphism.

Suppose  $\pi(Z) \not\subset F$ . There is a subvariety  $\tilde{Z} \subset X$  of codimension two, such that  $\pi(\tilde{Z}) = \pi(Z)$  and  $\tilde{Z} \neq Z$ . The proof of Lemma 37 gives

$$\text{mult}_{\tilde{Z}}(M_X) = \text{mult}_Z(M_X) = 1$$

which leads to a contradiction as in the case of  $k = 2$ . Thus,  $\pi(Z) \subset F$ .

Consider a smooth 3-fold  $V = \bigcap_{i=1}^{n-3} H_i$ , a curve  $C = Z \cap V$ , a movable boundary  $M_V = M_X|_V$ , a linear system  $\mathcal{D} = \mathcal{M}|_V$  that has no base components, and a morphism  $\tau = \pi|_V$ . Then  $\tau : V \rightarrow \mathbb{P}^3$  is a double cover branched over a smooth hypersurface  $S \subset \mathbb{P}^3$  of degree  $2n$ ,  $\tau(C) \subset S$  is a line, and  $\tau|_C$  is an isomorphism. Moreover, the equivalence

$$M_V \sim_{\mathbb{Q}} \tau^*(\mathcal{O}_{\mathbb{P}^3}(1))$$

holds and  $\text{mult}_C(M_V) = \text{mult}_Z(M_X) \geq 1$ .

Let  $O$  be a general point on  $\tau(C)$  and  $T$  be a hyperplane in  $\mathbb{P}^3$  that tangents the hypersurface  $S$  at the point  $O$ . Consider a line  $L \subset T$  passing through the point  $O$ . Let  $\hat{L} = \tau^{-1}(L)$ . Then the curve  $\hat{L}$  is singular at the point  $\hat{O} = \tau^{-1}(O)$ . Therefore,  $\text{mult}_C(M_V) = 1$ , because

$$2 = \hat{L} \cdot M_V \geq \text{mult}_{\hat{O}}(\hat{L}) \text{mult}_C(M_V) \geq 2$$

and  $\hat{L}$  spans a divisor when we vary the line  $L \subset T$ . Let  $f : U \rightarrow V$  be a blow up of  $C$ ,  $G$  be a  $g$ -exceptional divisor,  $M_U = f^{-1}(M_V)$ ,  $D$  be a general divisor in  $|(\tau \circ f)^*(\mathcal{O}_{\mathbb{P}^3}(1)) - G|$ ,  $M_D = M_U|_D$ . Then  $D$  is smooth,

$$M_D = \text{mult}_{\tilde{C}}(M_U) \tilde{C} + \Delta$$

where  $\tilde{C} \subset G$  is a base curve of  $|(\tau \circ f)^*(\mathcal{O}_{\mathbb{P}^3}(1)) - G|$  and  $\Delta$  is a movable boundary on  $D$ . The curve  $\tilde{C} \subset G$  is a smooth rational curve which dominates the curve  $C$ . By the adjunction formula  $\tilde{C}^2 = 1 - n$  on the surface  $D$ . On the other hand, the equivalence

$$M_D \sim_{\mathbb{Q}} \tilde{C}$$

holds on  $D$ . The latter implies  $\text{mult}_{\tilde{C}}(M_U) = 1$  and  $\Delta = \emptyset$ . After blowing up the curve  $\tilde{C}$  we see that the linear system  $\mathcal{D}$  lies in the fibers of the rational map given by the pencil  $|(\tau \circ f)^*(\mathcal{O}_{\mathbb{P}^3}(1)) - G|$ , which is impossible because  $\mathcal{D}$  is not composed from a pencil.  $\square$

Therefore, Theorem 12 is proved. The proof of Theorem 17 is almost identical to the proof of Theorem 12. The only difference is that one must use Theorem 20 instead of Theorem 19.

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