FIELDS OF INVARIANTS OF FINITE LINEAR GROUPS

YURI G. PROKHOROV

1. Introduction

Let G be a finite group of order n and let $\Bbbk$ be a field. Consider a rational (i.e., pure transcendental) extension $K/\Bbbk$ of transcendence degree n. We may assume that $K = \Bbbk(\{x_g\})$, where g runs through all the elements of G. The group G naturally acts on K via $h(x_g) = x_{hg}$. E. Noether [43] asked whether the field of invariants K^G is rational over $\Bbbk$ or not. On the language of algebraic geometry, this is a question about the rationality of the quotient variety $\mathbb{A}^n/G$.

The most complete answer on this question is known for Abelian groups. Thus, if G is Abelian of exponent e, $\operatorname{char}\Bbbk$ does not divide e and $\Bbbk$ contains a primitive eth roots of unity then $\mathbb{A}^n/G$ is rational [16]. On the other hand, over an arbitrary field $\Bbbk$ the rationality of $\mathbb{A}^n/G$ is related to some number theoretic questions. In this case $\mathbb{A}^n/G$ can be non-rational [53] (see also [58], [13], [33]).

Noether's question can be generalized as follows.

Problem 1.1. Let G be a finite group, let V be a finite-dimensional vector space over an algebraically closed field $\Bbbk$, and let $\rho\colon G \to GL(V)$ be a representation. Whether the quotient variety V/G is rational over $\Bbbk$?

Note that V/G has a natural birational structure of $\mathbb{P}^1$-fibration over $\mathbb{P}(V)/G$ admitting a section. So we have the following easy

Proposition 1.2 ([38], [27])**.** *V/G is birationally equivalent to $\mathbb{P}(V)/G \times \mathbb{P}^1$.*

Therefore the rationality of $\mathbb{P}(V)/G$ implies the rationality of V/G. The inverse implication is not known (cf. [2]). The affirmative answer to the Lüroth problem gives us the rationality of $\mathbb{P}(V)/G$ (and therefore V/G) for $\dim V \le 3$ over any algebraically closed field, cf. [6, Ch. 17]. Thus $\dim V = 4$ is the first nontrivial case.

Most of this survey is devoted to reviewing the current status of this problem in the special case where V is of dimension four. We refer to surveys [17], [12], [27], [21] for other aspects of 1.1.

If the representation $G \hookrightarrow GL(V)$ is not irreducible, then the decomposition $V = V_1 \oplus V_2$ gives us a G-equivariant $\mathbb{P}^1$-bundle structure $\widetilde{\mathbb{P}(V)} \to \mathbb{P}(V_1) \times \mathbb{P}(V_2)$, where $\widetilde{\mathbb{P}(V)}$ is the blowup of $\mathbb{P}(V)$ along $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$.

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Obviously, this $\mathbb{P}^1$-bundle has a G-invariant sections, exceptional divisors over $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$. Therefore, $\mathbb{P}(V)/G \approx (\mathbb{P}(V_1) \times \mathbb{P}(V_2))/G \times \mathbb{P}^1$. If $\dim V = 4$, this implies the rationality of $\mathbb{P}(V)/G$. Thus, in the rationality question of $\mathbb{P}^3/G$, we may assume that $G \subset GL(V)$ is irreducible.

Definition 1.3. Let $G \subset GL(V)$ be a finite irreducible subgroup. We say that G is *imprimitive of type* (m^k) if V is a nontrivial direct sum $V = V_1 \oplus \cdots \oplus V_k$ with $\dim V_i = m$ such that $gV_i = V_j$ for all $g \in G$. If such a direct splitting does not exists, G is called *primitive*.

As a consequence of Jordan's theorem [25] one can see that modulo scalar matrices in any dimension n there is only a finite number of primitive finite subgroups $G \subset GL(n, \mathbb{C})$. Lower-dimensional primitive groups have been completely classified, see references in [14, §8.5].

1.4. Let $G \subset GL(4, \mathbb{C})$ be a finite subgroup. By 1.2 we may regard G modulo scalar matrices. Following [4] we reproduce the list of all primitive subgroups $G \subset GL(4, \mathbb{C})$ modulo scalar multiplications. In particular, we assume that $G \subset SL(4, \mathbb{C})$. Notation are taken from [14, §8.5] with small modifications. Here o is the order of the group and z is the order of its center, $\widetilde{G}$ denotes some central extension of G.

- (I) $A \times B/Z$, where A, B are two-dimensional primitive subgroups in $SL(2, \mathbb{C})$ and Z is the central subgroup of order 2 which is contained in neither A nor B,
- (V) $\mathfrak{A}_5$, $\mathfrak{S}_5$, $\quad o = 60,\ 120,\ z = 1$,
- (VI) $\widetilde{\mathfrak{A}}_6$, $\widetilde{\mathfrak{S}}_6$, $\quad o = 360z,\ 720z,\ z = 2$,
- (VII) $\widetilde{\mathfrak{A}}_7$, $\quad o = (\frac{1}{2}7!)z,\ z = 2$,
- (VIII) $SL(2, \mathbb{F}_5)$, $\widetilde{\mathfrak{S}}_5$ $\quad o = 60z,\ 120z,\ z = 2$,
- (IX) $SL(2, \mathbb{F}_7)$, $\quad o = 168z,\ z = 2$,
- (X) G_{25920}, $\quad o = 25920z,\ z = 2$,
- (XI) G is such that $N \subset G \subset M \subset SL(4, \mathbb{C})$, where N a special imprimitive subgroup of order 32 and M is its extension by the automorphism group, $|M| = 32 \cdot 6!$.

Note that (XI) is the biggest class. It includes also three groups from (I). We discuss the rationality problem of $\mathbb{P}^3/G$ for primitive and imprimitive subgroups in $GL(4, \mathbb{C})$ in case by case manner.

Notation. Throughout this paper we will use the following notation unless otherwise specified.

- $X \approx Y$ denotes the birational equivalence of algebraic varieties X and Y.
- $\mathbb{P}(a_1, \ldots, a_n)$ denotes the weighted projective space.
- $Z(G)$ usually denotes the center of a group G.
- $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric alternating groups on n letters, respectively. Let $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by permuting the coordinates x_i.

The restriction of this representation to the invariant hyperplane $\sum x_i = 0$ we call the *standard* representation.

- $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I} \subset SL(2, \mathbb{C})$ are binary tetrahedral, octahedral, and icosahedral groups, respectively. There are well-known isomorphisms $\mathbb{T} \simeq SL(2, \mathbb{F}_3)$, $\mathbb{O} \simeq GL(2, \mathbb{F}_3)$, and $\mathbb{I} \simeq SL(2, \mathbb{F}_5)$.
- If V is a vector space, then $\mathbb{P}(V)$ denotes its projectivization and for an element $x \in V$, $[x]$ denotes the corresponding point on $\mathbb{P}(V)$.

For the sake of simplicity, we work over the complex number field $\mathbb{C}$. However some results can be extended for an arbitrary algebraically closed field (at least if the characteristic is sufficiently large).

2. Groups of type (I)

In this section we follow [30]. Regard $V = \mathbb{C}^4$ as the space of 2×2-matrices. The group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ acts on V from the left and the right. Therefore, for any two subgroups G_1, $G_2 \subset SL(2, \mathbb{C})$ there is a natural representation $\Psi \colon G_1 \times G_2 \to SL(V)$. Denote $G := \Psi(G_1, G_2) \subset SL(V)$. For $\Psi(\mathbb{T}, \mathbb{T})$, $\Psi(\mathbb{T}, \mathbb{O})$, $\Psi(\mathbb{T}, \mathbb{I})$, $\Psi(\mathbb{O}, \mathbb{O})$, $\Psi(\mathbb{O}, \mathbb{I})$, and $\Psi(\mathbb{I}, \mathbb{I})$ we get primitive groups of type (I) (they are 1^o, 3^o, 4^o, 5^o, 6^o, and 7^o in Blichfeldt's classification [4]).

Theorem 2.1 ([30]). *The variety $\mathbb{P}^3/G$ is rational for $G = \Psi(\mathbb{T}, \mathbb{T})$, $\Psi(\mathbb{T}, \mathbb{O})$, $\Psi(\mathbb{T}, \mathbb{I})$, $\Psi(\mathbb{O}, \mathbb{O})$, $\Psi(\mathbb{I}, \mathbb{I})$.*

Another proof of the rationality of Theorem 2.1 for $G = \Psi(\mathbb{T}, \mathbb{T})$, $\Psi(\mathbb{T}, \mathbb{O})$, $\Psi(\mathbb{O}, \mathbb{O})$ will be given in §3.

Proof. We have

$$\mathbb{P}(V)/G \approx V/(G \cdot \mathbb{C}^*) \approx SL(2, \mathbb{C})/G \approx G_1 \backslash SL(2, \mathbb{C})/G_2.$$

The affine variety $G_1 \backslash SL(2, \mathbb{C})$ is a homogeneous space under the action of $SL(2, \mathbb{C})$. In all cases $G_1 = \mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$, there is a natural non-singular projective compactification W of $G_1 \backslash SL(2, \mathbb{C})$ with Picard number one. We describe this construction below.

Denote by M_n the space of binary forms of degree n. In the case $G_1 = \mathbb{T}$, the group $\mathbb{T}$ has two semi-invariants

$$x_4,\ x_4' = t_1^4 \pm 2\sqrt{-3}\ t_1^2 t_2^2 + t_2^4 \in M_4$$

(see [59, vol. 2], [52, §4.5]). Let $W_2 := \overline{SL(2, \mathbb{C}) \cdot [x_4]} \subset \mathbb{P}(M_4) = \mathbb{P}^4$ be the closure of the $SL(2, \mathbb{C})$-orbit of $[x_4] \in \mathbb{P}(M_4)$. This set is $SL(2, \mathbb{C})$-invariant and contains an open orbit isomorphic to $\mathbb{T} \backslash SL(2, \mathbb{C})$. On the other hand, there is a non-degenerate symmetric bilinear form $q(\ ,\ )$ on M_4 (see [59, vol. 2], [52, §3.1]). For an element $a \in \mathbb{T}$ of order 3 one has $a \cdot x_4 = \omega x_4$, where ω is a primitive 3-th root of unity. Therefore,

$$q(x_4, x_4) = q(a \cdot x_4, a \cdot x_4) = \omega^2 q(x_4, x_4),$$

so that $q(x_4, x_4) = 0$. Thus the variety W_2 is defined by the equation $q(x, x) = 0$. This shows that $W_2 \subset \mathbb{P}^4$ is a smooth quadric.

Similarly, in cases $G_1 = \mathbb{O}$ and $\mathbb{I}$ by [59, vol. 2] or [52, §4.5], the group $\mathbb{O}$ (resp. $\mathbb{I}$) has a semi-invariant

$$x_6 = t_1 t_2 (t_1^4 - t_2^4) \in M_6,$$

$$\text{(resp. invariant } x_{12} = t_1 t_2 (t_1^{10} + 11 t_1^5 t_2^5 - t_2^{10}) \in M_{12}).$$

By [40] the closure $W_5 := \overline{SL(2,\mathbb{C}) \cdot [x_6]} \subset \mathbb{P}(M_6) = \mathbb{P}^6$ (resp. $W_{12} := \overline{SL(2,\mathbb{C}) \cdot [x_{12}]} \subset \mathbb{P}(M_{12}) = \mathbb{P}^{12}$) is a smooth compactification of $\mathbb{O}\backslash SL(2,\mathbb{C})$ (resp $\mathbb{I}\backslash SL(2,\mathbb{C})$).

Remark 2.2. Varieties W_2, W_5, and W_{22} are smooth Fano threefolds with $\rho = 1$. One has $\operatorname{Pic} W = \mathbb{Z} \cdot H$ and $-K_W \sim rH$, where H is the class of hyperplane section and $r = 3$, 2, and 1, respectively. The theory of $SL(2,\mathbb{C})$-varieties with an open orbit was developed from the minimal model theoretic viewpoint in [40], [57], [42]. The main result is as follows: for any smooth $SL(2,\mathbb{C})$-variety X with an open orbit, there is a sequence of $SL(2,\mathbb{C})$-equivariant birational morphisms $X = X_1 \to \cdots \to X_n = Y$, where each X_i is smooth, $X_i \to X_{i+1}$ is the blow-up of a smooth $SL(2,\mathbb{C})$-invariant curve, and Y is a so-called minimal $SL(2,\mathbb{C})$-variety. Minimal $SL(2,\mathbb{C})$-varieties are completely classified. In case when the stabilizer of a general point is $\mathbb{I}$ (resp., $\mathbb{O}$, $\mathbb{T}$), there is only one case $Y = W_{22}$ (resp., W_5, W_2). In cases when the stabilizer is dihedral or cyclic group, the situation is more complicated: either $Y \simeq \mathbb{P}^3$ or Y has the form $\mathbb{P}_S(\mathcal{E})$, where S is a smooth surface admitting an $SL(2,\mathbb{C})$-action or $\mathbb{P}^1$ and $\mathcal{E}$ is an equivariant vector bundle of rank $4 - \dim S$. A complete description of pairs $(S, \mathcal{E})$ is given.

Now we consider the following cases.

1) $G = \Psi(\mathbb{T}, \mathbb{T})$. Then

$$\mathbb{P}(V)/\Psi(\mathbb{T},\mathbb{T}) \approx \mathbb{T}\backslash SL(2,\mathbb{C})/\mathbb{T} \approx W_2/\mathbb{T}.$$

The point $[x_4] \in W_2$ is $\mathbb{T}$-invariant. Projection from this point gives a birational isomorphism $W_2/\mathbb{T} \approx \mathbb{P}^3/\mathbb{T}$. The last variety is obviously rational (because $\mathbb{T}$ has no irreducible four-dimensional representations).

2) $G = \Psi(\mathbb{I}, \mathbb{T})$ or $\Psi(\mathbb{I}, \mathbb{I})$. As above, $\mathbb{P}(V)/G \approx W_{22}/G_2$ and the point $P := [x_{12}] \in W_{22}$ is G_2-invariant. Since the point P is contained in an open orbit, it does not lie on a line [40]. Triple projection from P (the rational map given by the linear system $|-K_W - 3P|$) gives a birational G_2-equivariant map $W_{22} \dashrightarrow \mathbb{P}^3$ [55], see also [24, §4.5]. Therefore, $\mathbb{P}(V)/G \approx W_{22}/G_2 \approx \mathbb{P}^3/G_2$. If $G_2 = \mathbb{T}$, the last variety is rational, as in case 1). In the case $G_2 = \mathbb{I}$, the rationality of $\mathbb{P}^3/G_2$ will be proved below.

3) $G = \Psi(\mathbb{O}, \mathbb{T})$ or $\Psi(\mathbb{O}, \mathbb{O})$. Then $\mathbb{P}(V)/G \approx W_6/G_2$ and the point $P := [x_6] \in W_5$ is G_2-invariant. Double projection from P (i.e., the rational map defined by the linear system $|-1/2K_{W_5} - 2P|$) gives a G_2-equivariant rational map $W_5 \dashrightarrow \mathbb{P}^2$ [30], [18]. There is the following diagram of G_2-equivariant maps

$$\begin{tikzcd} W' \arrow[r, dashed, "\chi"] \arrow[d, "\sigma"'] & W^+ \arrow[d, "\varphi"'] \\ W_5 \arrow[r, dashed, "\psi"] & \mathbb{P}^2 \end{tikzcd}$$

where σ is the blow up of P, χ is a flop, and φ is a $\mathbb{P}^1$-bundle. Let $S := \sigma^{-1}(P)$ be the exceptional divisor and let S^+ be its proper transform on W^+. Then S^+ is a G_2-invariant section. Therefore, the quotient $W^+/G_2 \to \mathbb{P}^2/G_2$ has a birational structure of $\mathbb{P}^1$-bundle. This implies that $\mathbb{P}^3/G \approx W^+/G_2$ is rational. $\square$

The rationality of $\mathbb{P}^3/\Psi(\mathbb{O}, \mathbb{I})$ is an open question.

Theorem 2.3. *Notation as above.*

(i) $\mathbb{P}^3/\Psi(\mathbb{O}, \mathbb{I}) \approx W_{22}/\mathbb{O} \approx W_5/\mathbb{I}$.

(ii) $\mathbb{P}^3/\Psi(\mathbb{O}, \mathbb{I})$ *is stably rational. More precisely,* $\mathbb{P}^3/\Psi(\mathbb{O}, \mathbb{I}) \times \mathbb{P}^2$ *is rational.*

Proof. (i) follows by the above arguments. We prove (ii). Put $V := \mathbb{C}^3$. There is a faithful three-dimensional representation $\mathfrak{A}_5 \to GL(V)$ that induces an action of $\mathfrak{A}_5$ on $\mathbb{P}^2 = \mathbb{P}(V)$. We have the following fibrations

$$\begin{tikzcd} & (W_5 \times \mathbb{P}^2)/\mathfrak{A}_5 \arrow[dl, "f"'] \arrow[dr, "g"] & \\ W_5/\mathfrak{A}_5 & & \mathbb{P}^2/\mathfrak{A}_5 \end{tikzcd}$$

where f (resp. g) is a generically $\mathbb{P}^2$ (resp. W_5)-bundle in the étale topology. Since the action of $\mathfrak{A}_5$ on $\mathbb{P}^2$ is faithful, the map f admits a section. Put $X := (W_5 \times \mathbb{P}^2)/\mathfrak{A}_5$ and $K := \mathbb{C}(\mathbb{P}^2/\mathfrak{A}_5)$. Then $X \approx W_5/\mathfrak{A}_5 \times \mathbb{P}^2$. On the other hand, $X_K := X \otimes K$ is a smooth Fano threefold of index 2 and degree 5 (see [24]) defined over a non-closed field K. A general pencil of hyperplane sections defines a structure of del Pezzo fibration of degree 5 on X_K. By [36, Ch. 4] the variety X_K is K-rational (cf. Proof of Theorem 5.2, case Γ_{12}^9). $\square$

3. The primitive group of order $64 \cdot 6!$ and the Segre cubic

In this section we prove the rationality of quotients of $\mathbb{P}^3$ by primitive groups of type (XI) with two exceptions. We also give an alternative proof of the rationality $\mathbb{P}^3/G$ for groups of type (I) for $(A, B) = (\mathbb{T}, \mathbb{T})$, $(\mathbb{T}, \mathbb{O})$, $(\mathbb{O}, \mathbb{O})$. This section is a modified and simplified version of [31] but basically follows the same idea.

The primitive group of order $64\cdot 6!$ **([4]).** Let $Q_8\subset SL(2,\mathbb{C})$ be the quaternion group of order 8. We may assume that $Q_8=\{\pm E,\pm I,\pm J,\pm K\}$, where

$$I=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\quad J=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix},\quad K=I\cdot J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Regard $V=\mathbb{C}^4$ as the space of 2×2 matrices. The group $Q_8\times Q_8$ naturally acts on V by multiplications from the left and the right. This induces a representation

$$\rho\colon Q_8\times Q_8\to SL(V).$$

The image of ρ is an imprimitive group of order 32 isomorphic to $Q_8\times Q_8/\{\pm 1\}$. Indeed, $Q_8\times Q_8$ interchanges the one-dimensional subspaces $V_i\subset V$ generated by $e_1:=E$, $e_2:=I$, $e_3:=J$, and $e_4:=K$. Denote by N the subgroup of order 64 in $SL(V)$ generated by $\rho(Q_8\times Q_8)$ and scalar multiplications by $\sqrt{-1}$. Let $M\subset SL(V)$ be the normalizer of N. We study the rationality question for different subgroups of M.

The group M naturally acts on $\wedge^2V\simeq\mathbb{C}^6$ as an imprimitive group (see [4]). This easily follows from the fact that the image of N in $GL(\Lambda^2V)$ is an Abelian group. The eigenvectors

$$\begin{array}{lll} w_{12}^+:=e_1\wedge e_2+e_3\wedge e_4, & w_{13}^+:=e_1\wedge e_3+e_2\wedge e_4, & w_{14}^+:=e_1\wedge e_4+e_2\wedge e_3,\\ w_{12}^-:=e_1\wedge e_2-e_3\wedge e_4, & w_{13}^-:=e_1\wedge e_3-e_2\wedge e_4, & w_{14}^-:=e_1\wedge e_4-e_2\wedge e_3 \end{array}$$

form a basis of $\wedge^2V$. This gives a decomposition

$$\bigwedge^2 V=\bigoplus_{i=1}^{6}W_i,\qquad W_i=\mathbb{C}\cdot w_{j,k}^{\pm} \tag{3.1}$$

such that $N\cdot W_i=W_i$. The group M/N permutes subspaces W_i and is isomorphic to the symmetric group $\mathfrak{S}_6$. Thus we have the exact sequence

$$1\longrightarrow N\longrightarrow M\xrightarrow{\pi}\mathfrak{S}_6\longrightarrow 1 \tag{3.2}$$

and any finite group containing N as a normal subgroup is uniquely determined by its image under $\pi\colon M\to\mathfrak{S}_6$.

Example 3.3. It is easy to see that the matrix $S:=\frac{1+\sqrt{-1}}{\sqrt{2}}\operatorname{Diag}(\sqrt{-1},\sqrt{-1},1,1)$ is contained in M. The action on $\wedge^2V$ is as follows

$$\begin{array}{lll} w_{12}^+\to-\sqrt{-1}\,w_{12}^-, & w_{13}^+\to-w_{13}^+, & w_{14}^+\to-w_{14}^+,\\ w_{12}^-\to-\sqrt{-1}\,w_{12}^+, & w_{13}^-\to-w_{13}^-, & w_{14}^-\to-w_{14}^-. \end{array}$$

Therefore, S is a transposition $(1,2)$ in $M/N=\mathfrak{S}_6$. Similarly, the matrix $B:=\frac{1+\sqrt{-1}}{\sqrt{2}}\operatorname{Diag}(1,1,1,-1)$ also normalizes N. The corresponding substitution in $\mathfrak{S}_6$ is $(1,2)(3,4)(5,6)$ (odd element of order 2).

Remark 3.4. The group $N/Z(N)=Q_8\times Q_8/\{\pm1,\pm1\}$ is Abelian of order 16 isomorphic to $(\mathbb{F}_2)^4$. The exact sequence (3.2) induces an embedding $\mathfrak{S}_6\hookrightarrow\operatorname{Aut}\mathbb{F}_2^4\simeq SL(4,\mathbb{F}_2)$.

The results.

Theorem 3.5 ([31])**.** *The quotient $\mathbb{P}^3/N$ is isomorphic to the Igusa quartic $\mathcal{I}_4 \subset \mathbb{P}^4$ (see below).*

Theorem 3.6 ([31])**.** *Let $G \subset SL(4,\mathbb{C})$ be a finite group having a normal imprimitive subgroup N of order 64. Then $\mathbb{P}^3/G$ is birationally isomorphic to the quotient of the Segre cubic $\mathcal{S}_3 \subset \mathbb{P}^4$ by G/N, where the action of G/N on $\mathbb{P}^4$ is given by the composition of the canonical embedding $\pi\colon G/N \hookrightarrow \mathfrak{S}_6$, an outer automorphism $\lambda\colon \mathfrak{S}_6 \to \mathfrak{S}_6$ (see, e.g., [37]), and the standard action of $\mathfrak{S}_6$ on $\mathcal{S}_3$.*

Corollary 3.7 ([31])**.** *In the above notation, the variety $\mathbb{P}^3/G$ is rational except possibly for the following two groups (we use Blichfeldt's notation [4]):*

20^o $|G| = 64\cdot 360$ *and* $G/N \simeq \mathfrak{A}_6$*,*
17^o $|G| = 64{\cdot}60$*,* $G/N \simeq \mathfrak{A}_5$*, and* $\pi(G/N) \subset \mathfrak{S}_6$ *as a transitive subgroup.*

Corollary 3.8. *In cases 20^o and 17^o above, the variety $\mathbb{P}^3/G$ is birationally isomorphic to $\mathcal{S}_3/\lambda\circ\pi(G/N)$.*

Note that in case 17^o one of the x_i is $\lambda\circ\pi(G/N)$-invariant.

Segre cubic. Regard the variety $\mathcal{S}_3$ given by the following equations

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5+x_6 &= 0,\\ x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3 &= 0. \end{aligned} \tag{3.9}$$

as a cubic hypersurface in $\mathbb{P}^4$. This cubic satisfies many remarkable properties (see [48], [49, Ch. 8], [11]) and is called the *Segre cubic*. For example, any cubic hypersurface in $\mathbb{P}^4$ has at most ten isolated singular points, this bound is sharp and achieved exactly for the Segre cubic (up to projective isomorphism). The symmetric group $\mathfrak{S}_6$ acts in $\mathcal{S}_3$ in the standard way. Moreover, it is easy to show that $\mathrm{Aut}(\mathcal{S}_3) = \mathfrak{S}_6$ (see e.g. [15]). We refer to [11] and [28] for further interesting properties of $\mathcal{S}_3$.

The singular locus of $\mathcal{S}_3$ consists of ten nodes given by

$$x_{i_1} = x_{i_2} = x_{i_3} = -x_{i_4} = -x_{i_5} = -x_{i_6}, \tag{3.10}$$

where $\{i_1,i_2,i_3,i_4,i_5,i_6\} = \{1,2,3,4,5,6\}$. We denote such a point by $P\left[\begin{smallmatrix} i_1,i_2,i_3\\ i_4,i_5,i_6\end{smallmatrix}\right]$. For example, $P\left[\begin{smallmatrix} 123\\ 345\end{smallmatrix}\right] = (1,1,1,-1,-1,-1)$. It is easy to see that $P\left[\begin{smallmatrix} i_1,i_2,i_3\\ i_4,i_5,i_6\end{smallmatrix}\right] = P\left[\begin{smallmatrix} j_1,j_2,j_3\\ j_4,j_5,j_6\end{smallmatrix}\right]$ if and only if corresponding matrices are obtained from each other by permutations of rows and elements in each row. Hence there is an $\mathfrak{S}_6$-equivariant 1-1-correspondence

$$\mathrm{Sing}(\mathcal{S}_3) \longleftrightarrow \{\text{Sylow 3-groups in } \mathfrak{S}_6\}.$$

Since the action of $\mathfrak{S}_6$ on $\mathrm{Sing}(\mathfrak{S}_3)$ is transitive, the stabilizer $\mathrm{St}(P\left[\begin{smallmatrix} i_1,i_2,i_3\\ i_4,i_5,i_6\end{smallmatrix}\right])$ of $P\left[\begin{smallmatrix} i_1,i_2,i_3\\ i_4,i_5,i_6\end{smallmatrix}\right]$ is a group of order 72. For example, $\mathrm{St}(P\left[\begin{smallmatrix} 123\\ 345\end{smallmatrix}\right])$ is generated by $\mathfrak{S}_3\times\mathfrak{S}_3$ and $(1,3)(2,4)(3,5)$.

Further, there are 15 planes on $\mathcal{S}_3$ (see e.g. [15]). Each of them is given by equations

$$x_{i_1} + x_{i_4} = x_{i_2} + x_{i_5} = x_{i_3} + x_{i_6} = 0, \tag{3.11}$$

where $\{i_1, i_2, i_3, i_4, i_5, i_6\} = \{1, 2, 3, 4, 5, 6\}$. Denote such a plane by $\Pi\left[\begin{smallmatrix} i_1,i_2,i_3 \\ i_4,i_5,i_6 \end{smallmatrix}\right]$ and the set of all the planes on $\mathcal{S}_3$ by Ω. It is easy to see that $\Pi\left[\begin{smallmatrix} i_1,i_2,i_3 \\ i_4,i_5,i_6 \end{smallmatrix}\right] = \Pi\left[\begin{smallmatrix} j_1,j_2,j_3 \\ j_4,j_5,j_6 \end{smallmatrix}\right]$ if and only if corresponding matrices are obtained from each other by permutations of columns and elements in each column. Thus there is an $\mathfrak{S}_6$-equivariant 1-1-correspondence

$$\Omega \longleftrightarrow \{(i_1, i_4)(i_2, i_5)(i_3, i_6)\} \subset \mathfrak{S}_6.$$

For $\sigma = (i_1, i_4)(i_2, i_5)(i_3, i_6)$, we often will write $\Pi(\sigma)$ instead of $\Pi\left[\begin{smallmatrix} i_1,i_2,i_3 \\ i_4,i_5,i_6 \end{smallmatrix}\right]$.

Corollary 3.12. $\mathrm{St}(\Pi(\sigma)) = Z(\sigma)$ *and* $|Z(\sigma)| = 48$.

Comparing (3.11) and (3.10) one can see the following facts.

a) Every plane $\Pi(\sigma)$ contains exactly four singular points and there are six planes passing trough every singular point.
b) $\Pi(\sigma_1) \cap \Pi(\sigma_2) = \left\{P\left[\begin{smallmatrix} i_1,i_2,i_3 \\ i_4,i_5,i_6 \end{smallmatrix}\right]\right\} \iff \sigma_1 \circ \sigma_2 = (i_1, i_2, i_3)(i_4, i_5, i_6)$.
c) $\Pi(\sigma_1) \cap \Pi(\sigma_2)$ is a line $\iff \sigma_1$ and σ_2 contain a common transposition.

Recall that there exists an outer automorphism $\lambda\colon \mathfrak{S}_6 \to \mathfrak{S}_6$ (see, e.g., [37]). For any standard embedding $\mathfrak{S}_5 \subset \mathfrak{S}_6$, the subgroup $\lambda(\mathfrak{S}_5)$ is a "non-standard" transitive subgroup isomorphic to $\mathfrak{S}_5$. We need the following simple lemmas.

Lemma 3.13 (cf. [49, pp. 169–170])**.** *Let $G \subset \mathfrak{S}_6$ be a subgroup of order* 120 (*isomorphic to $\mathfrak{S}_5$*)*. One of the following holds:*

(i) *G is not transitive, then $G \cap \mathrm{St}(\Pi(\sigma))$ is of order* 8 *and G acts on Ω transitively;*
(ii) *G is transitive, then the order of $G \cap \mathrm{St}(\Pi(\sigma))$ is either* 12 *or* 24 *and Ω splits into two G-orbits Ω' and Ω'' consisting of* 10 *and* 5 *planes, respectively. A plane $\Pi(\sigma)$ is contained in Ω' if and only if $\sigma \in G$. Moreover, every two planes from Ω'' intersect each other at a* (*singular*) *point.*

Proof. (ii) Assume that there are two planes $\Pi(\sigma_1), \Pi(\sigma_2) \in \Omega''$ such that $\Pi(\sigma_1) \cap \Pi(\sigma_2)$ is a line. Then σ_1 and σ_2 contain a common transposition and $\sigma_1, \sigma_2 \notin G$. This is equivalent to that $\sigma_1 \circ \sigma_2$ is an element of order 2. But then $\lambda(\sigma_1)$, $\lambda(\sigma_2)$ are two transpositions such that $\lambda(\sigma_1) \circ \lambda(\sigma_2)$ is an element of order 2 and both $\lambda(\sigma_1)$, $\lambda(\sigma_2)$ are not contained in a standard subgroup $\mathfrak{S}_5 \subset \mathfrak{S}_6$. Clearly, this is impossible. $\square$

Lemma 3.14. *Let $\Pi = \Pi(\sigma)$ be a plane on $\mathcal{S}_3$ and let $\mathcal{H}$ be the pencil of hyperplane sections through Π. Let $\mathcal{Q}$ be the pencil of residue quadrics to Π*

(i.e., $\mathcal{H} = \Pi + \mathcal{Q}$). Then the base locus of $\mathcal{Q}$ consists of four singular points of $\mathcal{S}_3$ contained in Π and a general member of $\mathcal{Q}$ is smooth.

Proof. It is sufficient to check statement only for one plane. In this case it can be done explicitly. $\square$

Igusa quartic. Consider the dual map $\Psi\colon \mathcal{S}_3 \dashrightarrow \mathbb{P}^{4*}$ sending a smooth point $P \in \mathcal{S}_3$ to the tangent space $T_{P,\mathcal{S}_3}$. Let $\mathcal{S}_3^* = \Psi(\mathcal{S}_3)$ be the *dual variety.*

Lemma 3.15 ([28], [11])**.** *The map $\Psi\colon \mathcal{S}_3 \dashrightarrow \mathcal{S}_3^*$ is birational and $\deg \mathcal{S}_3^* = 4$.*

Proof. Assume that Ψ is not birational. Then for a general $P \in \mathcal{S}_3$ there is at least one point $P' \neq P$ such that $\Psi(P) = \Psi(P')$. This means that tangent spaces $T_{P,\mathcal{S}_3}$ and $T_{P',\mathcal{S}_3}$ coincide. We may assume that P, P' are not contained in any plane $\Pi(\sigma)$. The line L passing through P and P' is contained in $\mathcal{S}_3$. This line meets some plane $\Pi = \Pi(\sigma) \subset \mathcal{S}_3$. Thus the linear span $\langle L, \Pi\rangle$ is a hyperplane in $\mathbb{P}^4$. It is easy to see that $\langle L, \Pi\rangle \cap \mathcal{S}_3 = \Pi \cup Q$, where Q is a two-dimensional quadric. Clearly, $T_{P,Q} = T_{P',Q}$. This is possible only if Q is a quadratic cone. By Bertini's theorem its vertex is contained in Π. This contradicts Lemma 3.14.

Therefore, Ψ is birational and a general tangent hyperplane section $T_{P,\mathcal{S}_3} \cap \mathcal{S}_3$ has exactly one node. This implies that a general pencil $\mathcal{H}$ of hyperplane sections of $\mathcal{S}_3$ is a Lefschetz pencil. Let $H_1, \dots, H_r$ be singular members of $\mathcal{H}$. Ten of them pass through singular points of $\mathcal{S}_3$. Hence the degree of $\mathcal{S}_3^* \subset \mathbb{P}^{4*}$ is equal to $r - 10$. Finally, it is easy to compute that the topological Euler number of a cubic hypersurface in $\mathbb{P}^4$ having only nodes as singularities is equal to $m - 6$, where m is the number of singular points. Using this fact one can see that $\chi_{\mathrm{top}}(\mathcal{S}_3) = 4 = 9(2 - r) + 8r$, so $r = 14$ and $\deg \mathcal{S}_3^* = 4$. $\square$

Therefore, $\mathcal{S}_3^*$ is a hypersurface of degree 4 in $\mathbb{P}^4$. This famous quartic is called the *Igusa quartic* (cf. [11], [28]) and denoted by $\mathcal{I}_4$. An interesting fact is that $\mathcal{I}_4$ is the compact moduli space of Abelian surfaces with the level two structure. By the above, there is an $\mathfrak{S}_6$-equivariant birational map $\Psi\colon \mathcal{S}_3 \dashrightarrow \mathcal{I}_4$. We need the following characterization of $\mathcal{I}_4$ in terms of the action of $\mathfrak{S}_6$.

Lemma 3.16. *The image of every plane $\Pi = \Pi(\sigma)$ is a line on $\mathcal{S}_3^*$. Therefore, $\mathcal{S}_3^*$ contains an $\mathfrak{S}_6$-invariant configuration of 15 lines such that the action on these lines is transitive. Moreover, the above 15 lines form the singular locus of $\mathcal{S}_3^*$ and the stabilizer of such a line is $Z(\sigma)$, where $\sigma = (i_1, i_2)(i_3, i_4)(i_5, i_6)$.*

Proof. The tangent space $T_{P,\mathcal{S}_3}$ at $P \in \Pi$ contains Π. Hence $T_{P,\mathcal{S}_3}$ is contained in the pencil of hyperplane sections passing through Π. The fact that $\Psi(\Pi) \subset \mathrm{Sing}(\mathcal{S}_3^*)$ can be checked by direct computations, see [28]. $\square$

Lemma 3.17. *Let $X \subset \mathbb{P}^4$ be an $\mathfrak{S}_6$-invariant quartic under the standard action of $\mathfrak{S}_6$ on $\mathbb{P}^4$. Assume that there is an odd element $\sigma \in \mathfrak{S}_6$ of order 2 such that*

(i) *the fixed point locus of σ on $\mathbb{P}^4$ is a disjointed union of a line L and a plane Π,*
(ii) *L is contained in X.*

Then X is the Igusa quartic (up to projective equivalence) and $L \subset \operatorname{Sing}(X)$.

Proof. It is clear that X must be defined by an invariant ψ of degree 4. On the other hand, modulo $\sum x_i$ there are exactly two linearly independent invariants of degree 4: s_4 and s_2^2, where $s_d := \sum x_i^d$. Hence, $\psi = \alpha s_4 + \beta s_2^2$, $\alpha, \beta \in \mathbb{C}$.

Applying a suitable automorphism of $\mathfrak{S}_6$ we may assume that $\sigma = (12)(34)(56)$. Therefore, L is given by the following equations:

$$x_1 = x_2, \quad x_3 = x_4, \quad x_5 = x_6, \quad x_1 + x_3 + x_5 = 0.$$

On the other hand, in the pencil $\psi = \alpha s_4 + \beta s_2^2$ there is exactly one quartic containing such an L. □

Invariants of N.

Lemma 3.18. *The ring of invariants $\mathbb{C}[x, y, z, u]^N$ is generated by the following elements of degree 4:*

$$f_1 = x^2u^2 + y^2z^2, \qquad f_2 = x^2z^2 + y^2u^2, \quad f_3 = x^2y^2 + z^2u^2,$$

$$f_4 = x^4 + y^4 + z^4 + u^4, \qquad f_5 = xyzu.$$

The only relation is

$$(f_4 + 2f_1 + 2f_2 + 2f_3)\Big(f_1f_2f_3 - f_4f_5^2 + 2f_5^2(f_1 + f_2 + f_3)\Big) - \\ - \big(f_1f_2 + f_2f_3 + f_3f_1 + 4f_5^2\big)^2 = 0. \tag{3.19}$$

Proof. Indeed, it is easy to see that all the f_i are N-invariants. Hence, $\mathbb{C}[f_1, \ldots, f_5] \subset \mathbb{C}[x, y, z, t]^N$. Now it is sufficient to show that

$$[\mathbb{C}(x,\, y,\, z,\, u) : \mathbb{C}(f_1,\, f_2,\, f_3,\, f_4,\, f_5)] = |N| = 64.$$

Using the following relations

$$\big[\mathbb{C}(f_1,\, f_2,\, f_3,\, f_4,\, f_5) : \mathbb{C}(f_1,\, f_2,\, f_3,\, f_4,\, f_5^2)\big] = 2,$$

$$\big[\mathbb{C}(f_1, f_2, f_3, f_4, f_5^2) : \mathbb{C}(xu + yz, xz + yu, xy + zu, x^2 + y^2 + z^2 + u^2, f_5)\big] = 16,$$

$$[\mathbb{C}(xu + yz,\, xz + yu,\, xy + zu,\, x^2 + y^2 + z^2 + u^2,\, f_5) : \\ : \mathbb{C}(xu + yz,\, xz + yu,\, xy + zu,\, x + y + z + u,\, f_5)] = 2$$

we get the desired equality. Relation (3.19) can be checked directly. It is easy to see that the quartic given by (3.19) is smooth in codimension one. Hence it is irreducible and (3.19) is the only relation between the f_i. □

Lemma 3.20. *The quartic* $X := \mathbb{P}^3/N$ *given by* (3.19) *is singular along* 15 *lines* $L_1, \dots, L_{15}$. *Moreover,* $\mathfrak{S}_6$ *acts on the* L_i *transitively.*

Proof. This can be obtained immediately from equation (3.19) but we prefer to use the quotient structure on X. The group $N/Z(N)$ is an Abelian group of order 16 isomorphic to $(\boldsymbol{\mu}_2)^4$. Every non-trivial element $a \in N/Z(N)$ fixes points on two lines $\bar{L}_a^i \subset \mathbb{P}^3$, $i = 1, 2$. Thus in $\mathbb{P}^3$ there are 30 lines whose general points have stabilizer of order 2. The images of the $\bar{L}_a^i$ are contained in the singular locus of X. $\square$

Proof of Theorem 3.5. Put $X := \mathbb{P}^3/N$ and fix the embedding $X \hookrightarrow \mathbb{P}^4$ by the f_i's. Clearly, the group $\mathfrak{S}_6 \simeq M/N$ naturally acts on X. Since $\operatorname{Pic} X \simeq \mathbb{Z} \cdot (-K_X)$, the action is induced by a linear action on $\mathbb{P}^4$. The transposition S from Example 3.3 acts on the f_i by the diagonal matrix $\operatorname{Diag}(1, 1, -1, -1, 1)$. In particular, it is not a reflection. This shows that up to scalar multiplication the action of $\mathfrak{S}_6$ on $X \subset \mathbb{P}^4$ is given by the composition of the standard action of $\mathfrak{S}_6$ on $\mathbb{P}^4$ and the outer automorphism λ. Now it is sufficient to show that $\operatorname{Sing}(X)$ contains a line L such as in Lemma 3.17. Indeed, the fixed point locus of S on $\mathbb{P}^4$ consists of disjointed union of the plane $f_3 = f_4 = 0$ and the line $f_1 = f_2 = f_5 = 0$ contained in X. Moreover, X is singular along $f_3 = f_4 = 0$. Thus Lemma 3.17 can be applied and $X \simeq \mathcal{S}_3$. $\square$

Proof of Theorem 3.6 *and Corollary* 3.7. Consider any subgroup G such that $N \subset G \subset M$. By Theorem 3.5 the quotient $\mathbb{P}^3/N$ is the Igusa quartic $\mathcal{I}_4 \subset \mathbb{P}^4$, where the action of $G/N = \mathfrak{S}_6$ on $\mathbb{P}^4$ is given by the composition of the canonical embedding $\pi\colon G/N \hookrightarrow M/N = \mathfrak{S}_6$, an outer automorphism $\lambda\colon \mathfrak{S}_6 \to \mathfrak{S}_6$, and the standard action of $\mathfrak{S}_6$ on $\mathbb{P}^4$. Thus $\mathbb{P}^3/G \approx \mathcal{I}_4/(G/N)$. By Lemma 3.15 there is an $\mathfrak{S}_6$-equivariant birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$. Hence $\mathbb{P}^3/G \approx \mathcal{S}_3/(G/N)$. This proves Theorem 3.6.

To prove Corollary 3.7 we consider two cases.

3.21. G/N **has a fixed point** $P \in \operatorname{Sing}(\mathcal{S}_3)$**.** Projection from this point is G/N-equivariant, and therefore $\mathcal{S}_3/(G/N)$ is birationally equivalent to $\mathbb{P}^3/(G/N)$, where G/N is a group of order ≤ 72. Corollary 3.7 in this case follows from the rationality of $\mathbb{P}^3/(G/N)$.

3.22. $\pi(G)$ **is a subgroup of some** $\mathfrak{S}_5 \subset \mathfrak{S}_6$**, a standard (non-transitive) permutation group.** This exactly means that one of the W_i (see (3.1)) is an eigenspace for G. Then $\mathbb{P}^3/G \approx \mathcal{S}_3/(G/N)$, where G/N is embedded into a transitive $\mathfrak{S}_5$. By Lemma 3.13 there is a G-invariant set Ω'' of 5 planes $\Pi_i = \Pi(\sigma_i)$ and $\Pi_i \cap \Pi_j \subset \operatorname{Sing}(\mathcal{S}_3)$ for $i \neq j$. We claim that there are exactly two planes $\Pi_i \in \Omega''$ passing through every singular point. Indeed, this follows by the fact that $\mathfrak{S}_5$ transitively acts on the set of Sylow 3-subgroups in $\mathfrak{S}_6$. Further, for every singular point $P \in \mathcal{S}_3$, there is exactly two small (possibly non-projective) resolutions $\mu_P\colon \bar{\mathcal{S}}_3 \to \mathcal{S}_3$ and $\mu'_P\colon \bar{\mathcal{S}}'_3 \to \mathcal{S}_3$. Recall that a birational contraction is said to be *small* if it does not contract any divisors. For one of them, say for μ_P, the proper transforms $\bar{\Pi}_i$ of planes $\Pi_i \in \Omega''$ does not meet each other over $\mu_P^{-1}(P)$. Thus there is exactly one small resolution $\mu\colon \bar{\mathcal{S}}_3 \to \mathcal{S}_3$ of all singular points such that the proper transforms $\bar{\Pi}_i$ of the planes $\Pi_i \in \Omega''$ are disjointed. This resolution must be $\mathfrak{S}_5$-equivariant (cf. [15, §5]). By our construction, $\bar{\Pi}_i \simeq \mathbb{P}^2$. It is easy to check that the $\bar{\Pi}_i$ satisfy the contractibility criterion, i.e., $\bar{\Pi}_i|_{\bar{\Pi}_i} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. So there is an $\mathfrak{S}_5$-equivariant contraction $\varphi\colon \bar{\mathcal{S}}_3 \to W$ of Moishezon complex manifolds. Since $\operatorname{Pic} \bar{\mathcal{S}}_3 \simeq \mathbb{Z}^{\oplus 6}$ (see [15]), we have $\operatorname{Pic} W \simeq \mathbb{Z}$. Obviously, the anticanonical divisor $-K_W$ is effective and divisible by four in $\operatorname{Pic} W$. By [41] the variety W is projective and $W \simeq \mathbb{P}^3$. Therefore, we have $\mathbb{P}^3/G \approx \mathcal{S}_3/(G/N) \approx \mathbb{P}^3/(G/N)$, where (G/N) is a subgroup of $\mathfrak{S}_5$. The latter reduces the question of rationality of $\mathbb{P}^3/G$ to a smaller group G of order ≤ 120. This will be discussed below.

In fact, it will be shown in §4 that there is another (different from $\mu\colon \bar{\mathcal{S}}_3 \to \mathcal{S}_3$) small projective G/N-equivariant resolution $\mu^+\colon \bar{\mathcal{S}}_3^+ \to \mathcal{S}_3$ and there is a G/N-equivariant $\mathbb{P}^1$-bundle structure on $\bar{\mathcal{S}}_3^+$ (see Proof of Proposition 4.7).

To finish the proof of Corollary 3.7 we note that, except for 20^o and 17^o, there are only two groups which do not satisfy conditions 3.21 or 3.22 above: $\pi(G) = \mathfrak{S}_6$ and $\pi(G) \subset \mathfrak{S}_6$ is a transitive $\mathfrak{S}_5$. The ring of invariants $\mathbb{C}[\mathcal{S}_3]^{\mathfrak{S}_6}$ is generated by symmetric functions s_2, s_4, s_5, s_6, where $s_k = \sum_{i=1}^{6} x_i^k$. Therefore, $\mathcal{S}_3/\mathfrak{S}_6 \simeq \mathbb{P}(2,4,5,6)$ is rational. Consider the case when $G/N \simeq \mathfrak{S}_5$ and $\pi(G) \subset \mathfrak{S}_6$ is a transitive $\mathfrak{S}_5$. Then $G/N = \mathfrak{S}_5$ fixes some of $x_1, \dots, x_6$ on $\mathcal{S}_3$. Assume that $\mathfrak{S}_5 \cdot x_6 = x_6$. The ring of invariants $\mathbb{C}[\mathcal{S}_3]^{\mathfrak{S}_6}$ is generated by x_6 and $s'_1, \dots, s'_5$, where $s'_k = \sum_{i=1}^{5} x_i^k$. Thus, $\mathbb{C}[\mathcal{S}_3]^{\mathfrak{S}_6} \simeq \mathbb{C}[s'_2, s'_4, s'_5, x_6]$ and $\mathcal{S}_3/\mathfrak{S}_5 \simeq \mathbb{P}(2,4,5,1)$ is rational. This proves Corollary 3.7. $\square$

Remark 3.23. Another approach to the treatment of case 3.22 is to note that there is a G/N-equivariant $\mathbb{P}^1$-bundle structure on $\mathcal{S}_3$. Indeed, consider the family of lines $\mathcal{L} = \mathcal{L}(\mathcal{S}_3)$ on $\mathcal{S}_3$. Let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be all covering irreducible components (i.e., components $\mathcal{L}_i$ such that there is a line from $\mathcal{L}_i$ passing through a general point of $\mathcal{S}_3$). Since there is at most 6 lines passing through a general point, $r \le 6$. On the other hand, there is a hyperplane section of the form $\Pi_1 + \Pi_2 + \Pi_3$, where Π_i are planes. A general line from a covering family meets exactly one of the planes Π_1, Π_2, Π_3. This shows that $r \ge 3$. Using the action of $\mathfrak{S}_6$ of $\mathcal{S}_3$ one can see that $r = 6$. Therefore, each family $\mathcal{L}_i$ gives us (birationally) a $\mathbb{P}^1$-bundle structure on $\mathcal{S}_3$. Now it remains to note that if G is such as in 3.22, then one of families $\mathcal{L}_i$ is G/N-invariant.

4. Groups of types (V)-(X)

Case (X). Recall that an element $g \in GL(n, \mathbb{C})$ of finite order is said to be (complex) *reflection* if exactly $n-1$ eigenvalues are equal to 1. For convenience of the reader we recall the following well-known theorem of Chevalley and Shephard-Todd:

Theorem 4.1 (see [51], [7], see also [52]). *Let $V = \mathbb{C}^n$ and let $G \subset GL(V)$ be a finite subgroup. The following are equivalent:*

(i) *G is generated by reflections,*
(ii) *$V/G \simeq \mathbb{C}^n$.*

Outline of Proof. Let $W := V/G$ and $f: V \to W$ be the quotient morphism.

(ii) $\Longrightarrow$ (i). Assume the converse. Let $G_0 \subset G$ be the maximal subgroup generated by reflections. Then the morphism $V/G_0 \to W$ is étale over $W \setminus Z$, where Z is of codimension of least two. On the other hand, $\pi_1(V \setminus Z) = \{1\}$, a contradiction.

(i) $\Longrightarrow$ (ii). Put $R := \mathbb{C}[x_1, \ldots, x_n]$. First we claim that R is a free R^G-module. Let $I \subset R$ be the ideal generated by homogeneous invariants of positive degree.

Lemma 4.2. *Assume that for some homogeneous elements $y_i \in R$ and $z_i \in R^G$ the following relation holds*

$$z_1 y_1 + \cdots + z_m y_m = 0. \tag{4.3}$$

If $z_1 \notin R^G z_2 + \cdots + R^G z_m$, then $y_1 \in I$.

Proof. We can the take y_i so that $\deg y_1$ is minimal. If $\deg y_1 = 0$, then applying the map $\frac{1}{|G|}\sum_{g \in G} g$ to (4.3) we obtain $z_1 \in R^G z_2 + \cdots + R^G z_m$. Assume that $\deg y_1 > 0$. Let $s \in G$ be a reflection and let $\{l_s = 0\}$ be its fixed hyperplane. For any homogeneous $f \in R$, the polynomial $s \cdot f - f$ is divisible by l_s. Define a map

$$\Delta_s: R_d \to R_{d-1}, \qquad \Delta_s(f) = (s \cdot f - f)/l_s.$$

It is easy to check that

$$\Delta_s(f_1 f_2) = (\Delta_s f_1) f_2, \quad \forall f_1 \in R, \quad \forall f_2 \in R^G.$$

This gives us

$$z_1 \Delta_s(y_1) + \cdots + z_m \Delta_s(y_m) = 0.$$

Since $\deg \Delta_s(y_i) < \deg y_i$, we may assume that $\Delta_s(y_1) \in I$. Therefore, $s \cdot y_1 - y_1 \in I$ for all reflections $s \in G$. This implies that $g \cdot y_1 - y_1 \in I$ for all $g \in G$, so $y_1 \in I$. □

Take homogeneous elements $y_i \in R$ so that the images $\bar{y}_i$ in R/I form a basis over $\mathbb{C}$. It is clear that the y_i generate R as an R^G-module. By the above lemma these y_i are linearly independent over R^G. Indeed, if $z_1 y_1 + \cdots + z_m y_m = 0$ for some $z_i \in R^G$, then $z_1 = z_2 u_2 + \cdots + z_m u_m$, $u_i \in R^G$. So,

$$z_2(y_2 + y_1 u_2) + \cdots + z_m(y_m + y_1 u_m) = 0$$

and we can apply the induction by m. Since R is integral over R^G, the basis y_i is finite. This proves our claim.

Further, let J be the the ideal of R^G generated by homogeneous elements of positive degree. Since R^G is a Noetherian algebra, J is finitely generated. Take a minimal system of generators $f_1, \ldots, f_r$. It is clear that $f_1, \ldots, f_r$

generate R^G as $\mathbb{C}$-algebra. On the other hand, one can check that they are algebraically independent. This proves theorem. $\square$

Note that the statement of Theorem 4.1 fails if the characteristic of the base field divide $|G|$. However the field of invariants is rational in this case under the additional assumption that G is irreducible [26].

Corollary 4.4. *Let $G \subset GL(n, \mathbb{C})$ be a finite group generated by reflections. Then $\mathbb{C}[x_1, \dots, x_n]^G \simeq \mathbb{C}[f_1, \dots, f_n]$, where $f_1, \dots, f_n$ are homogeneous polynomials.*

Numbers $d_i := \deg f_i$ are called the *degrees* of G. They are uniquely determined by G. Thus for any group generated by reflections we have $\mathbb{C}^n/G \simeq \mathbb{C}^n$ and $\mathbb{P}^{n-1}/G$ is a weighted projective space $\mathbb{P}(d_1, \dots, d_n)$.

The list of all complex reflection groups can be found in [51], [10]. According to this list there is a finite group No. 32 of order $25920 \cdot 6$ generated by complex reflections of order 3. The degrees of this group are 12, 18, 24, 30 and the intersection with $SL(4, \mathbb{C})$ is exactly our first group. Therefore, the quotient $\mathbb{P}^3/G_{25920} \simeq \mathbb{P}(12, 18, 24, 30) \simeq \mathbb{P}(2, 3, 4, 5)$ is rational.

The group $SL(2, \mathbb{F}_5)$. Let $\delta \colon SL(2, \mathbb{F}_5) \hookrightarrow SL(2, \mathbb{C})$ be a faithful representation whose image is the icosahedron group $\mathbb{I}$. For short, we identify $SL(2, \mathbb{F}_5)$ with $\mathbb{I}$. Then $S^3\delta \colon \mathbb{I} \to SL(4, \mathbb{C})$ is an irreducible faithful representation as in (VIII). This gives as the action of $\mathfrak{A}_5 = \mathbb{I}/\{\pm E\}$ on $\mathbb{P}^3$ which leaves a rational cubic curve C invariant. Since $\mathbb{I}$ has a faithful two-dimensional representation, $\mathbb{P}^3/\mathbb{I}$ is stably rational. We prove more:

Theorem 4.5 ([29], [30])**.** *$\mathbb{P}^3/\mathbb{I}$ is rational.*

Proof. Let $\sigma \colon X \to \mathbb{P}^3$ be the blowup of C. Then X is a Fano threefold and there is a K-negative extremal contraction [39] different from σ. This contraction is given by the birational transform of the linear system of quadrics passing through C and the fibers are birational transforms of 2-secant lines of C. This defines a $\mathbb{P}^1$-bundle structure $\varphi \colon X \to \mathbb{P}^2$. We have the following $\mathbb{I}$-equivariant diagram:

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\varphi"] & \\ \mathbb{P}^3 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

Thus $\mathbb{P}^3/\mathbb{I} \approx X/\mathbb{I}$ and there is a rational curve fibration $f \colon X/\mathbb{I} \to \mathbb{P}^2/\mathbb{I}$. The action $\mathbb{I}$ on $\mathbb{P}^2$ is induced by an irreducible representation $\beta \colon \mathbb{I}/\{\pm E\} = \mathfrak{A}_5 \to SL(3, \mathbb{C})$. Note that the group $\beta(\mathfrak{A}_5) \cdot \{\pm E\}$ is generated by reflections and has degrees $2, 6, 10$ (see [6, Ch 17, §266]). Therefore, $\mathbb{P}^2/\mathfrak{A}_5$ is the weighted projective plane $\mathbb{P}(2, 6, 10) \simeq \mathbb{P}(1, 3, 5)$. Let $\Delta \subset \mathbb{P}^2/\mathfrak{A}_5$ be the minimal curve such that F is smooth over $(\mathbb{P}^2/\mathfrak{A}_5) \setminus \Delta$. It is easy to see that Δ is the image of the reflection lines in $\mathbb{P}^2$. Since there are exactly 15 such lines (corresponding to order 2 elements in $\mathfrak{A}_5$), the degree of Δ on $\mathbb{P}(1, 3, 5)$

is equal to 15 and Δ is irreducible. Let $\mathbb{P}(1,3,5) = \operatorname{Proj}\mathbb{C}[x_0, y_0, z_0]$, where $\deg x_0 = 1$, $\deg y_0 = 3$, $\deg z_0 = 5$. Using only the equality $\deg \Delta = 15$ one can see that Δ is given by the following equation

$$c_1 z_0^3 + c_2 y_0^5 + c_3 x_0 y_0^3 z_0 + c_4 x_0^2 y_0 z_0^2 + c_5 x_0^3 y_0^4 + c_6 x_0^4 y_0^2 z_0 + c_7 x_0^5 z_0^2 + c_8 x_0^6 y_0^3 + \\ + c_9 x_0^7 y_0 z_0 + c_{10} x_0^9 y_0^2 + c_{11} x_0^{10} z_0 + c_{12} x_0^{12} y_0 + c_{13} x_0^{15} = 0,$$

where the c_i are some constants. Consider the open set $U = \mathbb{P}(1,3,5) \cap \{x_0 \neq 0\}$. Then in coordinates $y = y_0/x_0^3$, $z = z_0/x_0^5$ on $U \simeq \mathbb{A}^2$ the curve Δ is defined by

$$c_2 y^5 + c_5 y^4 + (c_3 z + c_8)\, y^3 + (c_6 z + c_{10})\, y^2 + \left(c_4 z^2 + c_9 z + c_{12}\right) y + \\ + c_1 z^3 + c_7 z^2 + c_{11} z + c_{13} = 0.$$

Put $S_0 := U \setminus \Delta$ and $V := f^{-1}(S_0)$. Then $f|_V \colon V \to S_0$ is a smooth morphism whose geometric fibers are isomorphic to $\mathbb{P}^1$. Thus, $V \to S_0$ is a Severi-Brauer scheme.

Lemma 4.6. *Let S be a smooth projective rational surface and let $D \subset S$ be a reduced curve. Let $S_0 := S \setminus D$ and let V/S_0 be a Severi-Brauer scheme. Assume that there is an irreducible component $D_1 \subset D$ which is a smooth rational curve and such that D_1 meets the closure $\overline{D - D_1}$ at a single point. Then the Severi-Brauer scheme V/S_0 can be extended to $S \setminus \overline{D - D_1}$.*

Proof. According to general theory, there is 1-1 correspondence between isomorphism classes of Severi-Brauer S_0-schemes of relative dimension $n-1$ and isomorphism classes of Azumaya $\mathcal{O}_{S_0}$-algebras of rank n^2. Let A be the corresponding Azumaya algebra over S_0. Denote by $[A]$ its class in the Brauer group of the function field $\operatorname{Br}\mathbb{C}(S)$. Taking into account that S is rational we consider the Artin-Mumford exact sequence [1]

$$0 \longrightarrow \operatorname{Br}\mathbb{C}(S) \xrightarrow{\ a\ } \bigoplus_{C \subset S} H^1(\mathbb{C}(C), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\ r\ } \\ \longrightarrow \bigoplus_{P \in S} \boldsymbol{\mu}^{-1} \xrightarrow{\ s\ } \boldsymbol{\mu}^{-1} \longrightarrow 0,$$

where $\boldsymbol{\mu}^{-1} := \cup_n \operatorname{Hom}(\boldsymbol{\mu}_n, \mathbb{Q}/\mathbb{Z})$, the first sum runs through all irreducible curves $C \subset S$ while the second runs through all closed points $P \in S$ (for details we refer to [1], [56]). Note that the map r “measures” ramification and the map s is just the sum. Now let $P := D_1 \cap \overline{D - D_1}$. Since there are no cyclic coverings of $D_1 \simeq \mathbb{P}^1$ ramified only at P, the $H^1(\mathbb{C}(D_1), \mathbb{Q}/\mathbb{Z})$-component of $a([A])$ is zero, i.e., the algebra A is *unramified* over D_1. In this situation, A can be extended over D_1, see, e.g., [56, Prop. 6.2]. $\square$

Now we consider the natural embedding $U = \mathbb{A}^2 \hookrightarrow \mathbb{P}^2 = \bar{U}$. Let (x, y, z) be homogeneous coordinates on $\mathbb{P}^2$ so that $\mathbb{P}^2_{x,y,z} \cap \{z \neq 0\} = \mathbb{A}^2_{x,y} = U$. Let $\bar{\Delta} \subset \mathbb{P}^2$ be the closure of $\Delta \cap U$. Then $\bar{\Delta}$ intersects the infinite line $N := \{x = 0\}$ at a single point $P := (0,0,1)$ which is cuspidal. By the above lemma the Severi-Barauer scheme V can be extended to N. Let L_t be the pencil of lines on $\mathbb{P}^2$ through P. Then a general member of L_t meets $\bar{\Delta}$ at P and three more points. According to [45] there is a *standard conic bundle* $g\colon Y \to \widetilde{\mathbb{P}}^2$ and a commutative diagram:

$$\begin{tikzcd} X/\mathbb{I} \arrow[d, dashed] & Y \arrow[l, dashed] \arrow[d, "g"] \\ \mathbb{P}^2 & \widetilde{\mathbb{P}}^2 \arrow[l, "\psi"'] \end{tikzcd}$$

Here $\widetilde{\mathbb{P}}^2$ is a smooth surface, ψ is a birational morphism, and $Y \dashrightarrow X/\mathbb{I}$ is a birational map. By the above the discriminant curve $\widetilde{\Delta}$ of g is contained in $\psi^{-1}(\bar{\Delta})$. Moreover, again by Lemma 4.6 the Severi-Barauer scheme V can be extended to all exceptional divisors over P (because $P \in \bar{\Delta}$ is a cuspidal point). Thus we may assume that $\widetilde{L}_t$ is a base point free pencil such that $\widetilde{L}_t \cdot \widetilde{\Delta} = 3$. In this situation, Y is rational (see [23]), so are both $X/\mathbb{I}$ and $\mathbb{P}^3/\mathbb{I}$. $\square$

Groups of type (V). Since the ring of invariants of the standard representation of $\mathfrak{S}_5$ on $\mathbb{C}^4$ is generated by symmetric polynomials $s_2, \dots, s_5$, the quotient $\mathbb{P}^3/\mathfrak{S}_5 \simeq \mathbb{P}(2,3,4,5)$ is rational. Note also that $\mathfrak{A}_5$ has a faithful three-dimensional representation, so $\mathbb{P}^3/\mathfrak{A}_5$ is stably rational (more precisely, $\mathbb{P}^3/\mathfrak{A}_5 \times \mathbb{P}^3$ is rational). T. Maeda [34] (see also [27]) proved the rationality of $\mathbb{A}^5/\mathfrak{A}_5 \approx \mathbb{P}^3/\mathfrak{A}_5 \times \mathbb{P}^2$ (over an arbitrary field). The rationality of $\mathbb{P}^3/\mathfrak{A}_5$ over $\mathbb{C}$ was proved in [27] by an algebraic method. Here we propose an alternative, geometric approach.

Proposition 4.7. *There is the following $\mathfrak{S}_5$-equivariant diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi_0"] \arrow[dd, "\sigma"'] & & X^+ \arrow[dl, "\varphi_0^+"'] \arrow[dd, "\varphi"] \\ & \mathcal{S}_3 & \\ \mathbb{P}^3 \arrow[rr, dashed] & & W \end{tikzcd}$$

where χ is a flop, φ_0 and φ_0^+ are small contractions to the Segre cubic $\mathcal{S}_3$, W is a smooth del Pezzo surface of degree 5 and φ is a $\mathbb{P}^1$-bundle.

As in the proof of Theorem 4.5 the rationality of $\mathbb{P}^3/\mathfrak{A}_5$ can be proved by a detailed analysis of the discriminant curve of the rational curve fibration $X^+/\mathfrak{A}_5 \to W/\mathfrak{A}_5$.

Proof. Let $G = \mathfrak{S}_5$. Consider the standard representation $G = \mathfrak{S}_5 \hookrightarrow GL(4,\mathbb{C})$ and the corresponding linear action on $\mathbb{P}^3$. Then G permutes five points $P_1, \dots, P_5 \in \mathbb{P}^3$. Let $\sigma\colon X \to \mathbb{P}^3$ be the blowup of $P_1, \dots, P_5$ and let $S_i = \sigma^{-1}(P_i)$ be the exceptional divisors. Let H be the class of hyperplane section on $\mathbb{P}^3$ and let $H^* := \sigma^* H$. Since $\sum P_i$ is an intersection of quadric (scheme-theoretically), the linear system $|2H^* - \sum S_i| = |-\frac{1}{2}K_X|$ is base point free. In particular, $-K_X$ is nef and big, i.e., X is a *weak Fano threefold*. It is easy to check that $\dim|-\frac{1}{2}K_X| = 4$ and the morphism $\varphi_0\colon X \to X_0 \subset \mathbb{P}^4$ given by $|-\frac{1}{2}K_X|$ is birational onto its image, a three-dimensional cubic. Moreover, φ_0 small and contracts proper transforms $L_{i,j}$ of lines passing through P_i and P_j. Thus X_0 has ten singular points $\varphi_0(L_{i,j})$. Therefore, X_0 is the Segre cubic $\mathcal{S}_3$ and our construction is inverse to the construction in §3, Proof of Corollary 3.7.

Since $(\operatorname{Pic} X)^G$ is of rank two, there is G-equivariant flop $\chi\colon X \dashrightarrow X^+$. Here X^+ is a small resolution of $X_0 = \mathcal{S}_3$ obtained from X by "changing all the signs" [15]. There is a unique K-negative G-extremal ray on X^+ [39]. Let $\varphi\colon X^+ \to W$ be its contraction.

Assume that $\rho(X^+/W) > 1$. Then φ passes through a (non-G-equivariant) extremal contraction $\varphi_1\colon X^+ \to W_1$. If φ_1 is birational, then taking into account that $-K_{X^+}$ is divisible by 2 and the classification [39] we get that W_1 is smooth and φ_1 is the blowup of a point. Let S_1 be the corresponding exceptional divisor and let $S_1, \dots, S_r$ be the G-orbit. By the extremal property the S_i are disjointed and give us extremal rays on X^+/W. Since $\rho(X^+/W) \le 5$, $r \le 5$. On the other hand, the image of S_i on $X_0 = \mathcal{S}_3$ is a plane. By Lemma 3.13 we have $r = 5$ and the S_i are proper transforms of σ-exceptional divisors. Hence, $\chi = \mathrm{id}$ and $W = \mathbb{P}^3$. Clearly, this is impossible. Thus, φ_1 is not birational. We claim that W is a surface. Indeed, W cannot be a point because $-K_{X^+}$ is not ample. Assume that W is a curve and let F be a general fiber. Since $K_F = K_{X^+}|_F$ is divisible by 2, $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$. On the other hand the Mori cone $\overline{NE}(X^+/W)$ has at least 5 (non-birational) extremal rays. Each of them gives a contraction to a surface over W. This is impossible because the fibers of these contractions are contained in the fibers of φ. Thus W is a surface. By construction the linear system $|-\frac{1}{2}K_{X^+}|$ is the pull-back of the system of hyperplane sections of $\mathcal{S}_3$. In particular, $|-\frac{1}{2}K_{X^+}|$ is base point free and a general element $D \in |-\frac{1}{2}K_{X^+}|$ is a smooth cubic surface. Since the restriction $\varphi|_D\colon D \to W$ is birational, the surface also is smooth and $-K_W$ is ample, i.e., W is a del Pezzo surface. The group G faithfully acts on W. Clearly, this is not possible if $\rho(W) \le 5$. Therefore, $\rho(W) = 5$ and $K_W^2 = 5$. Finally, $-K_{X^+}$ is divisible by 2, so the fibration φ has no degenerate fibers. This proves the statement. □

5. Monomial groups

An action of a group Γ on a field $\mathbb{C}(x_1, \dots, x_n)$ is said to be *monomial* (with respect to $x_1, \dots, x_n$) if for every $g \in \Gamma$ one has

$$g(x_i) = \lambda_i(g) x_1^{m_{i,1}} \cdots x_n^{m_{i,n}}, \quad \lambda_i(g) \in \mathbb{C}^*, \quad m_{i,j} \in \mathbb{Z}. \tag{5.1}$$

Rationality of the fields of invariants of such actions were studied in the series of works [19], [20], [22], [21].

Any monomial action (5.1) defines a representation

$$\pi\colon \Gamma \to GL(n, \mathbb{Z}), \qquad g \to (m_{i,j}).$$

Now let $V := \mathbb{C}^{n+1}$ and let $G \subset GL(V)$ be an imprimitive group of type (1^{n+1}). Then G permutes the one-dimensional subspaces V_i from Definition 1.3 and contains a normal Abelian subgroup A which acts diagonally in the corresponding coordinates $x_1, \dots, x_{n+1}$. Let $\Gamma := G/A$. By the above, there is a natural embedding $\Gamma \hookrightarrow \mathfrak{S}_n$. Since we are assuming that the representation $G \hookrightarrow GL(V)$ is irreducible, the group $\Gamma \subset \mathfrak{S}_{n+1}$ is transitive. Put

$$y_1 = \frac{x_1}{x_{n+1}}, \dots, y_n = \frac{x_n}{x_{n+1}} \in \mathbb{C}(\mathbb{P}(V)).$$

The action of A on $y_1, \dots, y_n$ is diagonal in these coordinates. If $f = \sum_I a_I y_I \in \mathbb{C}[y_1, \dots, y_n]$ is an A-invariant, then so are all the monomials $a_I y_I$. Hence the ring $\mathbb{C}[y_1, \dots, y_n]^A$ and its fraction field $\mathbb{C}(y_1, \dots, y_n)^A$ are generated by invariant monomials. Let $z_1, \cdots, z_n$ be a basis of the free $\mathbb{Z}$-module (A-invariant monomials in $y_i)^*/\mathbb{C}^*$. Then the field of invariants $\mathbb{C}(\mathbb{P}(V))^A$ is generated by these $z_i = z_i(y_1, \dots, y_n)$ and they are algebraically independent. Further, the action of G on $\mathbb{C}(\mathbb{P}(V))^A$ is monomial with respect to $z_1, \dots, z_n$. Thus the rationality question of $\mathbb{P}^n/G$ is reduced to the rationality question of the invariant field $\mathbb{C}(z_1, \dots, z_n)^\Gamma$ of a monomial (but in general non-linear) action.

We have two representations π_0, $\pi\colon \Gamma \to GL(n, \mathbb{Z})$, corresponding monomial actions of Γ on $\mathbb{C}(y_1, \dots, y_n)$ and $\mathbb{C}(z_1, \dots, z_n)$, respectively. Moreover, π is a restriction of π_0 to an invariant sublattice of finite index. In particular, $\pi \otimes \mathbb{Q} \simeq \pi_0 \otimes \mathbb{Q}$.

In some cases the rationality of the field $\mathbb{C}(z_1, \dots, z_n)^\Gamma$ can be proved by purely algebraic methods. For example, $\mathbb{C}(z_1, \dots, z_n)^\Gamma$ is rational in the following cases:

- (i) Γ is cyclic of order m, where the class number of mth cyclotomic field is 1 [19], [20],
- (ii) $\dim V \le 4$ and G is meta-Abelian [20] (in fact, the author proved the rationality of V/G),
- (iii) $n \le 3$ and the action of Γ on $\mathbb{C}(z_1, \dots, z_n)$ is purely monomial with one exception (i.e., all constants $\lambda_i(g)$ in (5.1) are equal to 1) [22].

¿From now on we consider the four-dimensional case, i.e., the case $n = 4$.

Theorem 5.2. *Let $G \subset GL(4, \mathbb{C})$ be an imprimitive group of type (1^4). Then $\mathbb{P}^3/G$ is rational.*

There are the following possibilities for $\Gamma \subset \mathfrak{S}_4$: 1) Γ is a cyclic group of order 4, 2) Klein group $\mathfrak{V}_4$, 3) Γ is a dihedral group $\mathfrak{D}_4$ of order 8, 4) $\Gamma = \mathfrak{A}_4$, and 5) $\Gamma = \mathfrak{S}_4$. By the above it is sufficient to prove the rationality of the field $\mathbb{C}(z_1, z_2, z_3)^\Gamma$. The theorem was proved in the unpublished manuscript of I. Kolpakov-Miroshnichenko and the author (1986) by case by case consideration of the action of Γ on $\mathbb{C}(z_1, z_2, z_3)$. We consider here only the case

$\Gamma = \mathfrak{A}_4$. Case $\Gamma = \mathfrak{S}_4$ can be treated in a similar way. Cases $\Gamma = \mathbb{Z}_4$, $\mathfrak{V}_4$, and $\mathfrak{D}_4$ are easier. Moreover, in the these three cases G is also imprimitive of type (2^2) and then the rationality can be proved also by another method, see §6.

The group $\mathfrak{A}_4$ is generated by two elements: $\delta = (1,2)(3,4)$ and $\theta = (1,2,3)$. One has:

$$\pi_0(\theta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \pi_0(\delta) = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Using the classification of finite subgroups in $GL(3,\mathbb{Z})$ [54] we get the following possibilities:

$$\Gamma_9^{12}: \quad \pi(\theta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \pi(\delta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma_{10}^{12}: \quad \pi(\theta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \pi(\delta) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\Gamma_{11}^{12}: \quad \pi(\theta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \pi(\delta) = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Case Γ_9^{12}. By (5.1) the action of $\Gamma = \mathfrak{A}_4$ on $\mathbb{C}(z_1, z_2, z_3)$ has the following form

$$(a_1 z_2, a_2 z_3, a_3 z_1) \xleftarrow{\theta} (z_1, z_2, z_3) \xrightarrow{\delta} (b/z_1, c z_2, b'/z_3), \quad a_i, b, b', c \in \mathbb{C}^*.$$

After coordinate change of the form $z_i \to \lambda_i z_i$, $\lambda_i \in \mathbb{C}^*$ we may assume that $a_1 = a_2 = a_3 = b' = 1$. Since $\delta^2 = 1$, $c = \pm 1$. From other relations between θ and δ we get $b = c = \pm 1$. Therefore,

$$(z_2, z_3, z_1) \xleftarrow{\theta} (z_1, z_2, z_3) \xrightarrow{\delta} (b/z_1, b z_2, 1/z_3), \qquad b = \pm 1.$$

Assume that $b = c = 1$. Then after some coordinate change

$$z_1' = (z_1 + 1)/(z_1 - 1), \qquad z_2' = (z_2 + 1)/(z_2 - 1), \qquad z_3' = (z_3 + 1)/(z_3 - 1)$$

the action will be linear:

$$(z_2', z_3', z_1') \xleftarrow{\theta} (z_1', z_2', z_3') \xrightarrow{\delta} (-z_1', z_2', -z_3')$$

By proposition 1.2 the field $\mathbb{C}(z_1', z_2', z_3')^\Gamma$ is rational.

Now assume that $b = c = -1$. Regard z_1, z_2, z_3 as non-homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We get an action of $\Gamma = \mathfrak{A}_4$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by regular automorphisms. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$

$$(z_1, z_2, z_3) \longrightarrow (z_1 z_2 z_3,\ z_1,\ z_2,\ z_3,\ z_2 z_3,\ z_1 z_3,\ z_1 z_2,\ 1)$$

and let $t_1, \dots, t_8$ be the corresponding coordinates in $\mathbb{P}^7$. This induces the following representation of the tetrahedron group $\mathbb{T}$ into $GL(8, \mathbb{C})$:

$$(t_1,\ t_3,\ t_4,\ t_2,\ t_6,\ t_7,\ t_5,\ t_8) \xleftarrow{\widetilde{\theta}} (t_1, \dots, t_8) \xrightarrow{\widetilde{\delta}} (t_3,\ -t_4,\ -t_1,\ t_2,\ -t_7,\ -t_8,\ t_5,\ t_6).$$

It is easy to check that this representation is the direct sum of four two-dimensional faithful representations. Therefore, there are four Γ-invariant lines L_i in $\mathbb{P}^7$. Clearly $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ contains no $\mathfrak{A}_4$-invariant lines. On the other hand, the intersection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \cap L_i$ consists of at most two points (because the image of the Segre embedding is an intersection of quadrics). This immediately implies that all the L_i are disjointed from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $V \simeq \mathbb{P}^5$ be the linear span of $L_2,\ L_3,\ L_4$ and let $\mathcal{H}$ be the pencil of hyperplane sections of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ passing through V.

We claim that the general member of $H \in \mathcal{H}$ is smooth. Let $B = V \cap \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the base locus of $\mathcal{H}$. By Bertini's theorem $\operatorname{Sing}(H) \subset B$. If H is not normal, then by the adjunction formula H is singular along a line. On the other hand, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ does not contain any $\mathfrak{A}_4$-invariant lines. This immediately implies that H is normal. Since $-K_H$ is ample, H is either a cone over an elliptic curve or a del Pezzo surface with Du Val singularities. In both cases the number of singular points is at most two. On the other hand, $\operatorname{Sing}(H)$ coincides with the set of points where the dimension of Zariski tangent space $T_{P,B}$ jumps. Therefore, $\operatorname{Sing}(S)$ is $\mathfrak{A}_4$-invariant. This immediately implies the existence of $\mathfrak{A}_4$-invariant point $P \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, a contradiction.

Thus, a general member $H \in \mathcal{H}$ is a smooth del Pezzo surface of degree 6. Now consider the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ from V to L_1. By blowing up B we obtain a $\mathfrak{A}_4$-equivariant fibration $f\colon Y \to \mathbb{P}^1$ whose general fiber is a smooth del Pezzo surface of degree 6. This induces a fibration $Y/\mathfrak{A}_4 \to \mathbb{P}^1/\mathfrak{A}_4$ with the same type of general fiber. Such a fibration is rational over $\mathbb{C}(\mathbb{P}^1/\mathfrak{A}_4)$ (see [36, Ch. 4]), so $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/\mathfrak{A}_4 \approx Y/\mathfrak{A}_4$ is also rational.

Case Γ^{12}_{10}. Regard $z_1,\ z_2,\ z_3$ as non-homogeneous coordinates on $\mathbb{P}^3$. We get a linear action of $\Gamma = \mathfrak{A}_4$ on $\mathbb{P}^3$. This action is either reducible or imprimitive. In the first case, $\mathbb{C}(\mathbb{P}(V))^G$ is rational. In the second one, we can repeat inductive procedure to reduce the problem to the smaller order of Γ.

Case Γ^{12}_{11}. Then after coordinate change the action of $\Gamma = \mathfrak{A}_4$ on $\mathbb{C}(z_1, z_2, z_3)$ has the following form

$$(z_2, z_3, z_1) \xleftarrow{\theta} (z_1, z_2, z_3) \xrightarrow{\delta} (1/(z_1 z_2 z_3), z_3, z_2).$$

i.e., it is purely monomial. The field of invariants is rational by [22].

6. Imprimitive case (2^2)

Theorem 6.1. *Let $G \subset GL(4, \mathbb{C})$ be an imprimitive group of type (2^2). Then $\mathbb{P}^3/G$ is rational.*

Outline of the proof. Let $G \subset SL(V)$, $\dim V = 4$ be an imprimitive group of type (2^2). Then there is a decomposition $V = V_1 \oplus V_2$, $\dim V_i = 2$ and a subgroup $N \subset G$ of index 2 such that $N \cdot V_i = V_i$. We may assume that N is not Abelian (otherwise G is reducible).

The decomposition $V = V_1 \oplus V_2$ defines two skew lines L_1, L_2 in $\mathbb{P}^3 = \mathbb{P}(V)$. Let H be the class of hyperplane in $\mathbb{P}^3$. The image of the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given by the linear system $2H - L_1 - L_2$ is the two-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$ and fibers of this map are lines meeting both L_1 and L_2. We have the following G-equivariant diagram:

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^3 \arrow[dl, "\sigma"'] \arrow[dr, "\varphi"] & \\ \mathbb{P}^3 \arrow[rr, dashed] & & \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

where $\sigma\colon \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ is the blow up of $L_1 \cup L_2$ and φ is a $\mathbb{P}^1$-bundle. Let S_i be the corresponding exceptional divisors. There is a rational curve fibration $f\colon \widetilde{\mathbb{P}}^3/G \to S = (\mathbb{P}^1 \times \mathbb{P}^1)/G$. The rationality of $\mathbb{P}^3/G$ follows from detailed analysis of the action of G on $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. Proof of Theorem 4.5).

Consider for example the case when restrictions $N \to GL(V_i)$ are injective. Let $\pi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1 \times \mathbb{P}^1)/G$ be the quotient map and let $\Delta \subset S$ be the discriminant curve. Then, $B := \pi^{-1}(\Delta)$ coincides with the ramification divisor, the union of one-dimensional components of the locus of points with non-trivial stabilizer. Take a point $P = (x, y) \in B$. If $g \in \mathrm{St}(P)$, then $g(x) = x$, $g(y) = y$. If $g \in N$ or $g^2 \neq \lambda E$, there are only a finite number of such points. This implies that the ramification index over each component of Δ is equal to 2. Consider the pencil C_t of $(1,0)$-curves of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $D_t := \pi(C_t)$. Then D_t is a base point free pencil. Hence,

$$K_S \cdot D_t = 2p_a(D_t) - 2 - D_t^2 = -2.$$

Further, by the Hurwitz formula,

$$K_{\mathbb{P}^1 \times \mathbb{P}^1} = \pi^* \left(K_S + \frac{1}{2}\Delta \right).$$

This yields

$$\begin{aligned} 0 > \frac{2}{\deg \pi} K_{\mathbb{P}^1 \times \mathbb{P}^1} \cdot \pi^* D_t = \frac{2}{\deg \pi} \pi^* \left(K_S + \frac{1}{2}\Delta \right) \cdot \pi^* D_t = \\ = 2 \left(K_S + \frac{1}{2}\Delta \right) \cdot D_t = \Delta \cdot D_t - 4. \end{aligned}$$

Therefore, $\Delta \cdot D_t \leq 3$. Finally, as in the proof of Theorem 4.5 we deduce the rationality of $\widetilde{\mathbb{P}}^3/G$ using [45] and [23]. $\square$

7. Final remarks and open questions

As a consequence of the above results we have the following

Theorem 7.1. *Let $G \subset GL(4, \mathbb{C})$ a finite subgroup. Assume that G is solvable. Then $\mathbb{P}^3/G$ is rational.*

Remaining cases. The rationality question for $\mathbb{P}^3/G$ and $\mathbb{C}^4/G$ is still open for the following groups (up to scalar multiplication). For convenience of the reader we give a short description of group action.

Type (I). The rationality of $\mathbb{P}^3/G$ is unknown only for $G = \Psi(\mathbb{O}, \mathbb{I})$. Note that $\mathbb{P}^3/\Psi(\mathbb{O}, \mathbb{I})$ is stably rational, see Theorem 2.3.

Types (VI), (VII) and (VIII). Unsolved cases are $\widetilde{\mathfrak{S}}_5$, $\widetilde{\mathfrak{A}}_6$, $\widetilde{\mathfrak{S}}_6$, and $\widetilde{\mathfrak{A}}_7$. The corresponding actions on $\mathbb{P}^3$ are given by projective representations of $\mathfrak{S}_5$, $\mathfrak{A}_6$, $\mathfrak{S}_6$, and $\mathfrak{A}_7$ into $PGL(4, \mathbb{C})$. Recall that the Schur multiplier $H^2(\mathfrak{S}_n, \mathbb{C}^*)$ of the symmetric group is isomorphic to $\boldsymbol{\mu}_2$ for $n \geq 4$. Therefore, any projective representation of $\mathfrak{S}_n$, $n \geq 4$ is induced by a linear representation of a central extension $\tilde{\mathfrak{S}}_n$ by $\boldsymbol{\mu}_2$.

Following Schur [46], [47] we give an explicit matrix representation $\tilde{\mathfrak{S}}_6 \to GL(4, \mathbb{C})$. Consider the following matrices

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \quad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Next take the following Kronecker products

$$M_1 := C \otimes A, \; M_2 := C \otimes B, \; M_3 := A \otimes E, \; M_4 := B \otimes E, \; M_5 := \sqrt{-1} C^{\otimes 2}.$$

One can easily check the relations

$$M_j^2 = -E_4, \quad M_j M_k = -M_k M_j, \qquad 1 \leq j \neq k \leq 5, \tag{7.2}$$

where E_4 is the identity 4×4 matrix. Now put

$$T_k := \frac{1}{\sqrt{2k}} \left(-\sqrt{k-1} M_{k-1} + \sqrt{k+1} M_k \right), \quad k = 1, \dots, 5. \tag{7.3}$$

¿From (7.2) we have

$$\begin{aligned} T_k^2 &= -E_4, \qquad (T_k T_{k+1})^3 = -E_4, \\ T_j T_k &= -T_k T_j \quad \text{for} \quad k > j+1. \end{aligned} \tag{7.4}$$

Generators T_j and relations (7.4) determine an abstract group $\tilde{\mathfrak{S}}_6$ that is the central extension of $\mathfrak{S}_6$ by $\boldsymbol{\mu}_2$. Here T_j corresponds to the transposition interchanging j and $j+1$. In fact, the constructed representation $\tilde{\mathfrak{S}}_6 \to GL(4, \mathbb{C})$ is obtained from the standard action of $\mathfrak{S}_6$ on the Clifford algebra $A(\mathbb{C}^5)$ (cf., e.g., [8]). Taking compositions with embeddings into $\mathfrak{S}_6$ we get also projective representations $\mathfrak{S}_5 \hookrightarrow PGL(4, \mathbb{C})$ and $\mathfrak{A}_6 \hookrightarrow PGL(4, \mathbb{C})$.

For $\tilde{\mathfrak{A}}_7$, similarly put

$$M_1 := C^{\otimes 2} \otimes A,\ M_3 := C \otimes A \otimes E,\ M_5 := A \otimes E^{\otimes 2},$$
$$M_2 := C^{\otimes 2} \otimes B,\ M_4 := C \otimes B \otimes E,\ M_6 := B \otimes E^{\otimes 2},\ M_7 := \sqrt{-1}C^{\otimes 3}$$

and define T_k by formula (7.3). As above, we get an irreducible linear representation $\tilde{S}_7 \to GL(8,\mathbb{C})$. The restriction to $\tilde{\mathfrak{A}}_7$ splits as a direct sum of two 4-dimensional faithful representations. The explicit generators of $\mathfrak{A}_7 \subset PGL(4,\mathbb{C})$ can be taking as follows

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^4 & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix} \qquad W = \frac{1}{\sqrt{-7}} \begin{pmatrix} p^2 & 1 & 1 & 1 \\ 1 & -p & -q & -p \\ 1 & -p & -p & -q \\ 1 & -q & -p & -p \end{pmatrix}$$

where β is a primitive 7-th root of unity, $p := \beta+\beta^2+\beta^4$, and $q := \beta^3+\beta^5+\beta^6$, see [4]. Then the isomorphism between the subgroup in $PGL(4,\mathbb{C})$ generated by S, W and $\mathfrak{A}_7$ is given by

$$S \to (1,2,3,4,5,6,7), \qquad W \to (2,3,5)(4,6,7).$$

Remark 7.5. The group $\mathfrak{A}_6$ has also a central extension $\hat{\mathfrak{A}}_6$ by $\boldsymbol{\mu}_3$ which admits a 3-dimensional representation $\delta\colon \hat{\mathfrak{A}}_6 \to SL(3,\mathbb{C})$. The group $\delta(\hat{\mathfrak{A}}_6)$ is projectively equivalent to the complex reflection group of order $360 \cdot 6$ [51], [10]. Possibly this can be used to prove the stable rationality of $\mathbb{P}^3/\tilde{\mathfrak{A}}_6$.

Type (IX), $G \simeq SL(2,\mathbb{F}_7)$. The representation $G \hookrightarrow SL(4,\mathbb{C})$ can be described as follows. Fix a character $\chi\colon \mathbb{F}_7^* \to \mathbb{C}^*$ and consider the following $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on $\mathbb{F}_7^2 \setminus \{0\}$

$$V_\chi := \left\{ f\colon \left(\mathbb{F}_7^2 \setminus \{0\}\right) \to \mathbb{C} \mid f(\lambda x) = \chi(\lambda) f(x),\ \forall \lambda \in \mathbb{F}_7^* \right\}.$$

It is easy to see that $\dim_{\mathbb{C}} V_\chi = 8$ and $SL(2,\mathbb{F}_7)$ naturally acts on V_χ. Thus we have a representation $\rho_\chi\colon SL(2,\mathbb{F}_7) \to SL(V_\chi) = SL(8,\mathbb{C})$. If $\chi^2 = 1$, $\chi \neq 1$, then ρ_χ splits as a direct sum of two 4-dimensional faithful representations. This gives as a subgroup in $SL(4,\mathbb{C})$ isomorphic to $SL(2,\mathbb{F}_7)$. Explicit matrices can be found, e.g., in [4]. The ring of invariants is completely described, see [35]. Note that $SL(2,\mathbb{F}_7)$ has a 3-dimensional non-faithful representation $\delta\colon SL(2,\mathbb{F}_7) \to SL(3,\mathbb{C})$. The group $G' := G \times \{\pm E\}$ is generated by complex reflections and the degrees are 4, 6 and 14 (see [59, §139], [10]).

Type (XI). The rationality of $\mathbb{P}^3/G$ is unknown only for two groups of order $64 \cdot 360$ and $64 \cdot 60$, see Corollaries 3.7 and 3.8. Using Theorem 3.6 one can easily get equations for birational models of these quotients in terms of discriminants.

p-groups. It is known that the answer to Noether's problem is negative in higher dimensions: there are examples of p-groups such that $\Bbbk(\{x_g\}_{g\in G})^G$ is not rational (and even not stably rational) [44], [50]. Moreover, for any prime p there is a group G of order p^6 such that $\Bbbk(\{x_g\}_{g\in G})^G$ is not stably rational [5]. On the other hand, it is known that the field of invariant of any linear action of a p-group of order $\leq p^4$ on $\Bbbk(x_1,\ldots,x_N)$ is rational whenever $\Bbbk$ contains a primitive p^e-th root of unity, where p^e is the exponent of G [9], see also [3]. The following question is open:

> is it true that for any group G of order p^5 and any linear action of G on $\mathbb{C}^n$ the quotient $\mathbb{C}^n/G$ is stably rational?

It seems that the answer is positive (the author checked this for $p=2$). Note that in characteristic $p>0$ any linear action of p-group is rational [32].

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Department of Algebra, Faculty of Mathematics (and Mechanics), Moscow State University, Moscow 117234, Russia

E-mail address: prokhoro@mech.math.msu.su