APPLICATION OF MULTIPLIER IDEALS AND ARC SPACES TO BIRATIONAL RIGIDITY

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Abstract. This paper is devoted to an informal discussion of the methods (old and new) that are involved in the proof of the birational superrigidity of smooth complex hypersurfaces of degree N in $\mathbb{P}^N$ for $N \geq 4$.

Introduction

The problem of proving or disproving rationality for complex manifolds has a long history, tracing back at least to the nineteenth century. The beginning of the 1970's was marked by three important results on nonrationality: Clemens and Griffith's theorem on smooth cubic threefolds [CG], Iskovskikh and Manin's theorem on smooth quartic threefolds in $\mathbb{P}^4$ [IM], and Artin and Mumford's theorem on conic bundles [AM]. The method introduced in the paper of Iskovskikh and Manin has been extended by several people to prove nonrationality for many other varieties as well, and in fact a even stronger condition, called *birational rigidity*, which is deeply motivated by the minimal model program.

Following the publication of [IM] and of the analogous theorem of Pukhlikov on quintic fourfolds [Pu1], it was suggested that these results should generalize to all the higher dimensions. The explicit conjecture, namely, that for any $N \geq 6$, every smooth hypersurface of degree N in $\mathbb{P}^N$ is birationally rigid (in the strongest possible sense), and hence nonrational, was then made by Pukhlikov in the late 1990's [Pu2]. Partial results were established in [Pu2, Ch, dFEM], and the complete answer (in the affirmative) to this conjecture has been recently given in [dF].

The general argument to prove birational rigidity for Fano varieties essentially relays on quantifying singularities of birational maps. The so-called Noether–Fano inequality provides the first step, leading to a condition on canonical thresholds; from here, one needs to deduce a bound on multiplicities. For the quartic threefold, Iskovskikh and Manin did this by analyzing the graph associated to a resolution of indeterminacies. An alternative way to treat this case was later found by Corti [Co], who proposed to use the connectedness principle of Shokurov and Kollár (hence local vanishings) to pass from canonical thresholds to log-canonical thresholds. However, to obtain the full result, neither of these methods appear suitable when the dimension is large: a new approach is needed.

Using a result on log-discrepancies via arcs from [ELM], we prove in [dF] an adjunction formula for restriction of multiplier ideals that, under suitable conditions, generalizes the formula for hyperplane sections derived from the connectedness principle. This formula allows us to generalize Corti's approach, and thence to prove birational rigidity for smooth hypersurfaces of degree N in $\mathbb{P}^N$, for all $N \geq 4$.

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The purpose of the present article is to overview these techniques, explaining how multiplier ideals sheaves and arc spaces come into the picture. We will keep an informal tone. The interested reader will be able to find more details in the paper [dF], to which we also refer for a more comprehensive list of references.

1. Indeterminacies and thresholds

A projective Fano manifold X with $\mathrm{Pic}(X) \cong \mathbb{Z}$ is said to be *birationally superrigid* if there are no birational maps from X to any Mori fiber space other than isomorphisms. This condition clearly implies that X is nonrational, and also that $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

We now restrict ourselves to the case of Fano hypersurfaces. Using linear projections, one can modify any smooth Fano hypersurface of index ≥ 2 into a Mori fiber space over a positive base. The reminder of the paper is devoted to an outline of the proof of the following result, in which Fano hypersurfaces of index 1 are considered.

Theorem 1.1 ([dF]). *For $N \geq 4$, every smooth complex hypersurface $X \subset \mathbb{P}^N$ of degree N is birationally superrigid.*

In order to prove that X is birationally superrigid, the idea is to assume on the contrary the existence of a birational map

$$\phi \colon X \dashrightarrow \widetilde{X},$$

other than an isomorphism, between X and some Mori fiber space $\widetilde{X}$. We fix an embedding $\widetilde{X} \subseteq \mathbb{P}^m$ of the target of ϕ. By Lefschetz hyperplane theorem, ϕ is defined by the restriction of $m+1$ homogeneous forms of some degree $r \geq 1$, which generate an ideal $I \subset H^0(\mathcal{O}_X(r))$. We recall that the locus of indeterminacy of ϕ has codimension ≥ 2 in X. Therefore, two general elements of I define a complete intersection subscheme $B \subset X$ of codimension 2 and degree Nr^2.

We consider a resolution of indeterminacies of ϕ, given by a diagram

$$\begin{tikzcd} & X' \arrow[dl, "f"'] \arrow[dr] & \\ X \arrow[rr, dashed, "\phi"] & & \widetilde{X}. \end{tikzcd}$$

We can assume that f is a *log-resolution* of the pair (X, B), so that the union of the exceptional locus of f and the schematic inverse image of B is a divisor with simple normal crossings.

Every prime divisor E on X' naturally determines a discrete valuation val_E of $\mathbb{C}(X)$ that associates to any element $h \in \mathbb{C}(X)^*$ the order of vanishing (or minus the order of polarity) of $h \circ f$ at the generic point of E. We can then define the *canonical threshold* and the *log-canonical threshold* of the pair (X, B) to be, respectively,

$$\mathrm{can}\,(X,B) := \min_E \frac{\mathrm{ord}_E(K_{X'/X})}{\mathrm{val}_E(B)} \quad \text{and} \quad \mathrm{lc}\,(X,B) := \min_E \frac{\mathrm{ord}_E(K_{X'/X})+1}{\mathrm{val}_E(B)},$$

where the minima are taken over all prime divisors $E \subset X'$ with $f(E) \subseteq B$. Another important invariant of singularities associated to the pair (X, cB) (here $c > 0$ is like a weight given to B) is the *multiplier ideal sheaf*, which is defined by

$$\mathcal{J}\,(X, cB) := f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor cf^{-1}(B) \rfloor) \subseteq \mathcal{O}_X.$$

It is not difficult to see that $\mathcal{J}\,(X, cB) \subsetneq \mathcal{O}_X$ if and only if $c \geq \mathrm{lc}\,(X, B)$.

Coming back to our problem, the following fact represents the first step towards a contradiction.

Proposition 1.2 (Noether–Fano Inequality [IM]). *With the above notation, the canonical threshold of the pair (X, B) satisfies*

$$\mathrm{can}\,(X, B) < 1/r.$$

The inequality in the proposition is equivalent to the existence of a prime divisor E on X' such that

$$\tfrac{1}{r} \cdot \mathrm{val}_E(B) > \mathrm{ord}_E(K_{X'/X}).$$

An argument of Pukhlikov, based on general linear projections and residual cycles, easily shows that $f(E)$ cannot be positive dimensional. The difficult case to exclude is precisely the one in which the center of E is equal to a point $P \in X$.

At least in principle, the strategy of the proof is to show that, given this situation, the multiplicity $e_P(B)$ of B at P is greater than Nr^2, in order to get a contradiction, via Bezout, with the degree of $[B]$. The idea would be to cut down X near P until B becomes zero-dimensional, in order to apply the following fact.

Theorem 1.3 ([dFEM]). *Let $(R, \mathfrak{m})$ be an n-dimensional regular local ring essentially of finite type over $\mathbb{C}$, let $\mathfrak{a} \subset R$ be an $\mathfrak{m}$-primary ideal, and fix $c > 0$. Denoting by $\mathcal{J}(R, \mathfrak{a}^c)$ the multiplier ideal of $(R, \mathfrak{a}^c)$, assume that*

$$\mathfrak{m}^k \not\subseteq \mathcal{J}(R, \mathfrak{a}^c)$$

for some integer $k \geq 0$. Then the Samuel multiplicity of $\mathfrak{a}$ satisfies

$$e(\mathfrak{a}) \geq (k+1)(n/c)^n.$$

When $N = 4$, this was the approach followed by Corti in his alternative proof of Iskovskikh and Manin's theorem. More precisely, one takes a general hyperplane section $Y \subset X$ passing through P, so that $B|_Y$ is zero dimensional. By the connectedness principle, the log-canonical threshold of the restricted pair $(Y, B|_Y)$ satisfies

$$\mathrm{lc}\,(Y, B|_Y) < 1/r.$$

This is equivalent to saying that $\mathcal{J}\big(Y, (\frac{1}{r} - \epsilon)B|_Y\big)$ is non-trivial near P for $\epsilon > 0$ sufficiently small. Then, by the case $(n, k) = (2, 0)$ of Theorem 1.3 (which already appears as a theorem in [Co]), one obtains $e_P(B|_Y) > 4r^2$, and hence $e_P(B) > 4r^2$. But this is impossible in view of the fact that $\deg[B] = 4r^2$.

Unfortunately this approach does not quite work if $N \geq 5$. Indeed, in this case one would need to take $N - 3$ general hyperplane sections through P in order to get a surface $Y \subset X$, so that $B|_Y$ is zero dimensional. Let $f\colon X' \to X$ be a log-resolution of (X, B) such that, if $Y' \subset X'$ is the proper transform of Y, then Y' intersects transversally the exceptional locus of f and f restricts to a log-resolution $Y' \to Y$ of $(Y, B|_Y)$. Then, letting $E \subset X'$ be the exceptional divisor as described above, the main question is: *does Y' meets E?* If it does, and F is one of the components of $E|_{Y'}$, then the usual adjunction formula gives

$$\tfrac{1}{r} \cdot \mathrm{val}_F(B|_F) - (N-4) \cdot \mathrm{val}_F(P) > \mathrm{ord}_F(K_{Y'/Y}) + 1.$$

This implies that

$$\mathfrak{m}_{P,Y}^{N-4} \not\subseteq \mathcal{J}\big(Y, (\tfrac{1}{r} - \epsilon)B|_Y\big) \quad \text{for small} \quad \epsilon > 0,$$

and therefore we obtain $e_P(B) > 4(N-3)r^2$ from Theorem 1.3, which is impossible for $N \geq 4$ since $\deg[B] = Nr^2$. However, in general Y' will not meet E at all, and the connectedness principle does not generalize to the case when more than one hyperplane section is taken. To better understand the problem, one can consider the following simple example, which will also give us some directions on where to look for a solution.

Example 1.4. Let $D = \{y^2 = x^3\} \subset \mathbb{C}^2$ and $c = 5/6$. Then $\mathcal{J}\left(\mathbb{C}^2, cD\right) = (x,y)$, and if $L \subset \mathbb{C}^2$ is a general line through the origin, then $\mathcal{J}\left(\mathbb{C}^2, cB|_L\right) = (y)$. However, if we take the principal tangent $T = \{y = 0\}$ of D, then we obtain $\mathcal{J}\left(T, cD|_T\right) = (x^2)$.

2. Going beyond the log-canonical threshold

The previous example seems to suggest that we should take the hyperplane sections with suitable tangency conditions. Roughly speaking, this operation *pushes Y' closer to E*, at least in the first order of approximation, and therefore it is more likely to produce a deep enough multiplier ideal on the restriction. This can be made precise if we investigate the whole situation using arcs.

It was Nash to first relate invariants of singularities encoded in resolutions with the geometry of *arc spaces*, namely, those spaces parameterizing formal arcs

$$\gamma\colon \operatorname{Spec}\mathbb{C}[[t]] \to V$$

on any given variety V. The "log version" of this viewpoint was first considered by Mustaţă, and then further studied in [ELM]. In particular, one can use the diagram

$$\begin{tikzcd}[column sep=small] & X'_\infty \arrow[dl] \arrow[dr, "f_\infty"] & \\ X' \arrow[dr, "f"'] & & X_\infty \arrow[dl] \\ & X, & \end{tikzcd}$$

where X_∞ is the arc space of X and X'_∞ is that of X', to pass back and forth from one theory (resolutions) to the other (arc spaces). The advantage of working with arc spaces is that we can better understand, so to speak, where Y' stands with respect to the exceptional divisors of f, by looking at Y_∞ inside X_∞.

Assuming for simplicity that E is a smooth divisor on X', the correspondence is given by taking the set $W' \subset X'_\infty$ of arcs with order of contact 1 along E, and then defining

$$W := \overline{f_\infty(W')} \subset X_\infty.$$

It can be proven that W is an irreducible closed set defined by finitely many equations in X_∞. In particular, even if X_∞ is infinite dimensional, we can still define the codimension of W inside it. Moreover, associated to this set W, there is a valuation val_W on $\mathbb{C}(X)$ that is defined on regular functions by taking the minimum of order of vanishing along the elements of W, and the following result gives the correspondence.

Theorem 2.1 ([ELM])**.** *With the above notation,* $\mathrm{val}_W = \mathrm{val}_E$ *and* $\mathrm{codim}(W, X_\infty) = \mathrm{ord}_E(K_{X'/X}) + 1$.

Using this result, we can translate conditions on multiplier ideals in terms of the geometry of arc spaces. The advantage is that now we can investigate quite explicitly how Y_∞ intersects W. Then the idea is to pick a (suitable) irreducible component of

$W \cap Y_\infty$, in order to produce a divisor F over Y satisfying some interesting condition in the computation of the multiplier ideal sheaf of the restricted pair. To do so, we need the following converse of Theorem 2.1.

Theorem 2.2 ([ELM])**.** *If $C \subset Y_\infty$ is an irreducible closed set defined by finitely many equations, then there is a proper birational morphism $f\colon Y' \to Y$ (with Y' smooth), a divisor $F \subset Y'$, and an integer $q \geq 1$, such that* $\mathrm{val}_C = q \cdot \mathrm{val}_F$ *and* $\mathrm{codim}(C, X_\infty) \geq \mathrm{ord}_F(K_{Y'/Y}) + 1$.

The tangency condition we plan to impose to Y is necessary for Y_∞ to meet W in an appropriate way. However, this first order of approximation does not suffice in general. To resolve this difficulty, we reduce ourselves to work with a linear space $\mathbb{C}^n$; the subvariety Y will now be a linear subspace of $\mathbb{C}^n$. This reduction gives us an extra toll, namely, the natural $\mathbb{C}^*$-action of $\mathbb{C}^n$; we intend to use this action in order, now, to *bring E closer to Y'*. Technically speaking, we deform W into a subset of X_∞ that satisfies better intersection properties with Y_∞.

It is precisely the combination of a suitable tangency condition on the restriction and such limiting process (corresponding to a flat degeneration of the ideal of the subscheme to an appropriate homogeneous ideal) that leads us to the desired adjunction formula.

Passing to some details, let $Z \subset \mathbb{C}^n$ be a proper closed subscheme, and suppose that, for some $c > 0$, the origin O of $\mathbb{C}^n$ is a center of non log-terminality for the pair $(\mathbb{C}^n, cZ)$. This means that there is a prime divisor E over $\mathbb{C}^n$ with center O and such that

$$c \cdot \mathrm{val}_E(Z) \geq \mathrm{ord}_E(K_{X'/X}) + 1.$$

Let $Z_0 \subset \mathbb{C}^n$ be the subscheme cut out by the top-degree initial homogeneous forms of the polynomials in the ideal of Z, and let $Y \subset \mathbb{C}^n$ be linear subspace of codimension $k < n$ that is not contained in Z_0. Then we have the following result.

Theorem 2.3 ([dF])**.** *With the above notation, assume that Y is tangent to the direction determined by a sufficiently general point of the center of E in $\mathrm{Bl}_O\mathbb{C}^n$. Then*

$$\mathfrak{m}_{O,Y}^k \not\subseteq \mathcal{J}(Y, cZ_0|_Y).$$

Although we cannot apply this theorem directly in our previous setting, a suitable use of generic linear projections, combined with a formula on log-canonical thresholds from [dFEM] and Nadel's vanishing theorem, enables us to use this result to complete the proof of Theorem 1.1.

We close this paper with some comments on Theorem 2.3. In the case the divisor E induces a homogeneous valuation, the formula in the theorem still holds with $Z_0|_Y$ replaced by $Z|_Y$. However, in general, taking the degeneration to Z_0 seems to be a necessary step for the adjunction to work properly, at least if we require that Y is a linear subspace of $\mathbb{C}^n$. It comes natural then to pose the following question.

Question 2.4. With the notation and assumptions as in Theorem 2.3, and for any fixed $1 \leq k < n$, can one always find a smooth subvariety $V \subset \mathbb{C}^n$ of codimension k, not contained in Z and passing through O, such that $\mathfrak{m}_{O,V}^k \not\subseteq \mathcal{J}(V, cZ|_V)$?

Remark 2.5. Since we are not considering anymore degenerations to homogeneous ideals, the question makes sense on any smooth ambient variety, not just $\mathbb{C}^n$.

We would expect the answer to this question to be affirmative, and that one would need in general to allow sufficiently large degrees to the equations defining V, depending on the divisor E.

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