Rational points of bounded height on hypersurfaces of dimension at most four

Let Z be a locally closed subset of $\mathbf{P}^n$ defined over $\mathbf{Q}$ and H: $\mathbf{P}^n(\mathbf{Q}) \rightarrow \mathbf{R}$ be the height function where $H(x)$ is the maximum of $|x_i|$ for a primitive integral n+1-tuple $(x_0, ..., x_n)$ representing a rational point x on $\mathbf{P}^n$. We shall write $N(Z,B)$ be the number of rational points of height at most B on Z. This is clearly an abuse of notation since $N(Z,B)$ depends both on the choice of the embedding $Z \subset \mathbf{P}^n$ and the choice of coordinates of $\mathbf{P}^n$.

We want to investigate the asymptotic behaviour of $N(Z,B)$ when $B \rightarrow \infty$. The following result is due to Pila [Pi$_1$] (cf. also [Pi$_2$]).

Theorem 1 *Let* $X \subset \mathbf{P}^n$ *an geometrically irreducible projective variety of degree d defined over* $\mathbf{Q}$. *Then ,*

$$N(X, B) = O_{d,n,\varepsilon}(B^{\dim X + 1/d + \varepsilon})$$

It is remarkable that the implied constant does not depend on X apart from its dependence of the degree and dimension of X. The importance of such uniform estimates is that one can use induction by dimension.

Heath-Brown [He] established uniform estimates for curves in $\mathbf{P}^2$ and $\mathbf{P}^3$ and for surfaces in $\mathbf{P}^3$ which are stronger than Pila´s. Some of these estimates were generalised to curves and surfaces of higher codimension in [Brb$_1$] . The techniques of Heath-Brown´s paper have also been applied to count rational points on smooth threefolds in $\mathbf{P}^4$ [Brw] and on general irreducible threefolds [Brb$_2$].

We shall in this paper give estimates for hypersurfaces of dimension at most four which are better than previous estimates. To obtain these, we combine techniques from [He] with new arguments from algebraic geometry. The most important of these is a Kodaira dimension argument which plays a central role in our estimates for smooth hypersurfaces. We obtain in particular (see theorem 11) improvements of a well-known bound of Hua from 1938 concerning the number of non-trivial positive solutions to the diagonal equation $x_0^d + x_1^d + x_2^d - x_3^d - x_4^d - x_5^d = 0$ when $d \geq 9$.

The following result is due to Heath-Brown [He] in the case where C is geometrically integral and n=2 or 3. The general case is due to Broberg [Brb$_1$] .

Theorem 2 *Let* $C \subset \mathbf{P}^n$ *be an irreducible curve of degree d over* $\mathbf{Q}$. *Then*

$$N(C, B) = O_{d,n,\varepsilon}(B^{2/d+\varepsilon})$$

For surfaces in $\mathbf{P}^3$ let us first quote the following result of Heath-Brown [He] .

Theorem 3 *Let* $X \subset \mathbf{P}^3$ *be a geometrically irreducible surface of degree d>1 over* $\mathbf{Q}$.*Then,*

$$N(X, B) = O_{d,\varepsilon}(B^{2+\varepsilon})$$

Browning [Brw] extended this result to surfaces in $\mathbf{P}^n$, n>3 by means of a a projection argument. It is easy to extend theorem 3 to the class of irreducible varieties since any rational point on an irreducible but not geometrically irreducible variety must be singular.

One cannot expect any improvement on theorem 3 since $N(L,B) \gg B^2$ for any line on X defined over **Q**. Let us therefore consider the complement X' of all lines on X . The following result is proved in [Sa].

Theorem 4 *Let* $X\subset\mathbf{P}^3$ *be an irreducible surface of degree d defined over* **Q**. *Then,*

$$N(X',B) = O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon}) \qquad \text{if } d\leq 8$$
$$N(X',B) = O_{d,\varepsilon}(B^{14/9+\varepsilon}) \qquad \text{if } d>8$$

We have in particular that :

$$N(X',B) = O_{d,\varepsilon}(B^{16/9+\varepsilon}) \qquad \text{when } d = 4$$
$$N(X',B) = O_{d,\varepsilon}(B^{76/45+\varepsilon}) \qquad \text{when } d \geq 5.$$

This should be compared with the estimates on p.558 in [He]

$$N(X',B) = O_{d,\varepsilon}(B^{52/27+\varepsilon}) \qquad \text{when } d = 3$$
$$N(X',B) = O_{d,\varepsilon}(B^{17/9+\varepsilon}) \qquad \text{when } d \geq 4$$

We can also generalise the estimate $N(X',B) = O_{d,\varepsilon}(B^{4/3+16/9d+\varepsilon})$ in (op.cit.) for smooth surfaces $X\subset\mathbf{P}^3$ to the class of irreducible surfaces $X\subset\mathbf{P}^3$ which contain no line of multiplicity d-2 or more.

For *smooth* **Q**-surfaces $X\subset\mathbf{P}^3$ of degree > 5, Heath-Brown (th. 11 in op.cit.) obtained a better bound . The following result [Sa] is a slight improvement of his result.

Theorem 5 *Let* $X\subset\mathbf{P}^3$ *be a smooth projective surface of degree d over* **Q.** *Let U be the open complement of all curves on X of degree at most d-2. Then,*

$$N(U, B) = O_{d,\varepsilon}(B^{f(d)+\varepsilon}) \quad \text{where}$$

$$f(d)= 3/\sqrt{d}+2/(d-1)- 1/(d-1)\sqrt{d} \qquad \text{if } d\leq 13$$
$$f(d)= 3/\sqrt{d} + 2/d - 1/2d\sqrt{d} \qquad \text{if } d\geq 14$$

Moreover,

$$N(X', B) = O_{d,\varepsilon}(B^{f(d)+\varepsilon}) \qquad \text{if } d<13$$

Remarks (a) When $d \geq 13$, then $N(X',B) = O_{d,\varepsilon}(B^{1+\varepsilon})$ by a result of Heath-Brown in (op.cit).

(b) From f(d) < $3/\sqrt{d} + 2/d$ we get from theorem 5 that $N(U, B) = O_{d,\varepsilon}(B^{3/\sqrt{d} + 2/d + \varepsilon})$. This should be compared with Heath-Brown´s estimate : $N(U, B) = O_{d,\varepsilon}(B^{3/\sqrt{d} + 2/(d-1) + \varepsilon})$.

(c) There are only finitely many curves on X of degree at most d-2 by a theorem of Colliot-Thélène [Co] . U is thus an open non-empty subset of X .

The following result is due to Broberg [Brb$_2$].

Theorem 6 *Let* $X \subset \mathbf{P}^4$ *be a geometrically irreducible projective hypersurface of degree d over* **Q.** *Then ,*

(a) $N(X,B) = O_\varepsilon(B^{(47+\sqrt{721})/168 + \varepsilon})$ *for d=3*

(b) $N(X,B) = O_\varepsilon(B^{(371+5\sqrt{721})/168 + \varepsilon})$ *for d=4*

(c) $N(X,B) = O_{d,\varepsilon}(B^{3+\varepsilon})$ *for* $d \geq 5$

We can improve this somewhat [Sa].

Theorem 7 *Let* $X \subset \mathbf{P}^4$ *be an irreducible projective hypersurface of degree d over* **Q.** *Then ,*

(a) $N(X,B) = O_\varepsilon(B^{55/18 + \varepsilon})$ *for d=3*

(b) $N(X,B) = O_{d,\varepsilon}(B^{3+\varepsilon})$ *for* $d \geq 4$

It was already known [He] that $N(X,B) = O_\varepsilon(B^{3+\varepsilon})$ for quadrics in $\mathbf{P}^4$. For smooth threefolds we prove in [Sa] the following result.

Theorem 8 *Let* $X \subset \mathbf{P}^4$ *be a smooth hypersurface of degree d>5 over* **Q**. *Then,*

$$N(X, B) = O_{d,\varepsilon}(B^{15 f(d)/16+5/4+\varepsilon} + B^{2+\varepsilon}) \quad \text{where}$$

$$f(d)= 3/\sqrt{d}+2/(d-1)- 1/(d-1)\sqrt{d} \quad \text{if } d \leq 13$$
$$f(d)= 3/\sqrt{d} + 2/d - 1/2d\sqrt{d} \quad \text{if } d \geq 14$$

By using the inequality $f(d)<3/\sqrt{d} + 2/d$ one obtains as a corollary the bound

$$N(X, B) = O_{d,\varepsilon}(B^{45/16\sqrt{d} +15/8d+5/4+\varepsilon} + B^{2+\varepsilon}) \qquad d>5$$

The following results for fourfolds are also proven in [Sa].

Theorem 9 *Let* $X \subset \mathbf{P}^5$ *be a smooth hypersurface of degree over* **Q** of degree d. *Then the following uniform estimates hold.*

$$N(X,B)= O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5(d-1) - 9/10(d-1)\sqrt{d} +12/5+\varepsilon}) \qquad \text{if} \quad 6< d \leq 13$$

$$N(X,B)= O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5d - 9/20d\sqrt{d} + 12/5+\varepsilon}) \qquad \text{if} \quad 14\le d \le 25$$
$$N(X,B)= O_{d,\varepsilon}(B^{3+\varepsilon}) \qquad \text{if} \quad d>25$$

Theorem 10 *Let $X\subset\mathbf{P}^5$ be a smooth hypersurface over* **Q** *given by an equation*

$$a_0x_0^d+\ldots+a_5x_5^d=0$$

where $a_0, \ldots , a_5 \in$ **Q .** *Let $V\subset X$ be the complement of the fifteen closed subsets given by the equations*

$$a_0x_0^d+ a_ix_i^d = a_jx_j^d+ a_kx_k^d =a_lx_l^d+ a_mx_m^d=0$$

where $\{i,j,k,l,m\}= \{1,2,3,4,5\}$. *Then,*

$$N(V,B)= O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5(d-1) - 9/10(d-1)\sqrt{d} + 12/5+\varepsilon}) \qquad \text{if} \quad 6< d \le 13$$
$$N(V,B)= O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5d - 9/20d\sqrt{d} + 12/5+\varepsilon}) \qquad \text{if} \quad 13< d \le 34$$
$$N(V,B)= O_{d,\varepsilon}(B^{131/45+\varepsilon}) \qquad \text{if} \quad d>34$$

Theorem 11 *Let $n_d(B)$ be the number of solutions in non-negative integers $x_i\le B$, $0\le i\le 5$ to the equation*

$$x_0^d+ x_1^d + x_2^d- x_3^d- x_4^d- x_5^d = 0$$

Then,

$$n_d(B) = O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5(d-1) - 9/10(d-1)\sqrt{d} + 12/5+\varepsilon}) \qquad \text{if} \quad 6< d \le 13$$
$$n_d(B) = 6B^3+ O_{d,\varepsilon}(B^{27/10\sqrt{d} + 9/5d - 9/20d\sqrt{d} + 12/5+\varepsilon}) \qquad \text{if} \quad 13< d \le 34$$

$$n_d(B) = 6B^3+ O_{d,\varepsilon}(B^{131/45+\varepsilon}) \qquad \text{if} \quad d>34$$

<u>Remark</u> Note that $27/10\sqrt{d} + 9/5d - 9/20d\sqrt{d} + 12/5<3$ when $d>25$ so that we get an asymptotic formula

$$n_d(B) = 6B^3+ O_{d,\varepsilon}(B^{3-\delta}) \qquad \text{for} \quad d>25$$

This improves upon [BrwHe] where the authors get an asymptotic formula for $d>32$.

Also , $$n_d(B) = O_{d,\varepsilon}(B^{7/2-1/80 +\varepsilon}) \qquad \text{for} \quad d\ge 9$$

which should be compared with Huas estimate (cf. [Hu] , [Da])

$$n_d(B) = O_{d,\varepsilon}(B^{7/2+\varepsilon})$$

from 1938. This was still the best known result until the paper of Heath-Brown [He] appeared He gives there an improvement on Huas estimate for $d\ge 24$.

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