Jumps in Mordell-Weil rank and Arithmetic Surjectivity.

In our work establishing a "converse" to the theorem of Graber-Harris-Starr (see [G-H-S],, G-H-M-S 1], [G-H-M-S 2]) the four of us came up with some questions and examples that we didn't put in our published articles. So that they don't get lost, we collect them here.

Recall that a (smooth, proper) morphism $f : X \to B$ over a number field K is said to be **arithmetically surjectivity** if for all number field extensions L/K, the mapping on L-rational points, $f(L) : X(L) \to B(L)$, is surjective.

We begin with a qualitative question.

Question 1. Let $f : X \to B$ be a pencil of curves of genus one over a number field K. Are the following two conditions equivalent?

(1) The morphism f is arithmetically surjective.

(2) The morphism f admits a section over K.

Here is a convenient way of restating this question.

Consider B an open subscheme of $\mathbf{P}^1$ over K and $X \to B$ a family of (smooth, proper) curves of genus 1 defined over K. Letting $E \to B$ be the jacobian (i.e., Pic^o) of the family, we have that $X \to B$ is a torsor over $E \to B$ and is represented therefore by some class $h \in H^1(B, E)$. This cohomology group is torsion, so let n denote the order of the element h. Invoking the "Kummer sequence" of group schemes over $B_{/\mathbf{Q}}$,

$$0 \to E[n] \to E \to E \to 0,$$

where the mapping $E \to E$ is multiplication by n, we see that there is an element $\tilde{h} \in H^1(B, E[n])$ which maps to $h \in H^1(B, E)$. Choose such an $\tilde{h}$. The family $X \to B$ is determined up to isomorphism by the element $\tilde{h} \in H^1(B, E[n])$ and to reflect this fact we may refer to the family as $X_{\tilde{h}} \to B$.

Question 2. Fix an integer $n > 1$, and $E \to B$, an abelian scheme of dimension one over B an open subscheme of $\mathbf{P}^1$ defined over a number field K. Is it the case that if a class $\tilde{h} \in H^1(B, E[n])$ is *not* in the image of $H^0(B, E) = E(B)$ under the connecting homomorphism of the long exact sequence on cohomology induced by the Kummer sequence, then does there exist a number field extension L/K and an L-valued point of B, $\beta = \mathrm{Spec}\ L \to B$, such that if

$$\tilde{h}(\beta) \in H^1(\beta, E[n]) = H^1(\mathrm{Spec}\ L, E[n])$$

is the restriction of the class $\tilde{h} \in H^1(B, E[n])$ to β, $\tilde{h}(\beta) \in H^1(\beta, E[n])$ is *not* in the image of $H^0(\beta, E) = E(L)$? (Or equivalently, $h(\beta) \neq 0$?)

An affirmative answer to **Question 2** for all $n > 1$ and all $E \to B$ is the same as an affirmative answer to **Question 1.**

As for quantitative questions, we might fix the number field K and ask for asymptotics regarding the number of fibers X_b in the family $X \to B$ that have K-rational points, where $b \in B(K)$ runs over the K-rational points of height bounded by some real number C:

Question 3. Let $\tilde{h}$ and h be as in **Question 2.** Is there a positive number e such that for all $\tilde{h} \in H^1(B, E[n])$ *not* in the image of $H^0(B, E) = E(B)$

$$\text{card}\{b \in B(K) \mid \text{Height}(b) < C \text{ and } h(b) \neq 0\} \;>>\; C^e \;?$$

Here are two comments about this question and its connection to jump points. (By a **jump point** $b = \text{Spec } \mathbf{Q} \to B$ (for $E_{/B}$) let us mean a $\mathbf{Q}$-rational point of B such that the induced mapping on Mordell-Weil groups $E(B) \to E(b)$ is not surjective.)

1. Quadratic twist families.

Let

$$E_1 : \quad y^2 \;=\; x^3 + ax + b$$

be an elliptic curve over the number field K. Let $B := \mathbf{P}^1 \;-\; \{0, \infty\}$. By the **quadratic twist family** $E \to B$ attached to E_1 over K we mean the pencil of elliptic curves

$$E_t : \quad ty^2 \;=\; x^3 + ax + b$$

for t ranging through B.

Proposition. Let $K = \mathbf{Q}$ and $n = 2$. Let E_1 be any elliptic curve over $\mathbf{Q}$ with none of its points of order two rational over $\mathbf{Q}$. Let $E \to B$ be its associated quadratic twist family. Then **Question 3** has an affirmative answer in such a case, and we can take $e = 1 - \epsilon$ for any positive ϵ.

Proof. We will be dealing with cohomology of the Kummer sequence

$$0 \to E[2] \to E \to E \to 0$$

over B and the restriction of that cohomology to rational points $b \in B$. We are in a particularly nice situation because the group scheme $E[2]$ over B is isomorphic to the pullback via $B \to \text{Spec}(\mathbf{Q})$ of the group scheme $E_1[2]$ over $\text{Spec}(\mathbf{Q})$.

Let

$$\bar{B} := B \otimes_{\text{Spec}\mathbf{Q}} \text{Spec}\bar{\mathbf{Q}}.$$

Lemma. $E(\bar{B}) = E[2](\bar{B})$.

Proof. This is very easy and I assume that it is somewhere in the literature?? Let a bar mean making the base change from Spec($\mathbf{Q}$) to Spec($\bar{\mathbf{Q}}$).

Consider $\bar{B}' \to \bar{B}$ the double cover ramified at 0 and ∞ (i.e., extract a square root of the parameter t) and note that the pullback $\bar{E}' = \bar{E} \otimes_{\bar{B}} \bar{B}'$ is a product, $\bar{E}' = \bar{E}_1 \times \bar{B}'$. A section $\sigma : \bar{B} \to \bar{E}$ gives us a section $\sigma' : \bar{B}' \to \bar{E}'$ such that $\sigma' \cdot i = j \cdot \sigma'$ where $i : \bar{B}' \to \bar{B}'$ is the involution of $\bar{B}'$ as double cover of $\bar{B}$ and $j : \bar{E}' = \bar{E}_1 \times \bar{B}' \to \bar{E}_1 \times \bar{B}'$ is given by $j(e_1, b') = (-e_1, i(b'))$ for $e_1 \in \bar{E}_1$ and $b' \in \bar{B}'$. Since there are no nonconstant morphisms from $\bar{B}'$ to $\bar{E}_1$, σ' is a constant section, i.e., $\sigma'(b') = (e_1, b')$ for a fixed $e_1 \in \bar{E}_1$ and all $b' \in \bar{B}'$. Therefore e_1 is in $\bar{E}_1[2]$ proving the lemma.

Let $G := \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. By the Hochschild-Serre Spectral we have an exact sequence

$$0 \to H^1(G, H^0(\bar{B}, E[2])) \to H^1(B, E[2]) \to H^1(\bar{B}, E[2])^G.$$

Since the group scheme $E[2]_{/B}$ comes by pullback from $E_1[2]_{/\mathbf{Q}}$ we see that

$$H^1(G, H^0(\bar{B}, E[2])) = H^1(G, E_1[2]),$$

and

$$H^1(\bar{B}, E[2])^G = \mathrm{Hom}(H_1(\bar{B}, \mathbf{Z}/2\mathbf{Z}), E[2])^G = \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, E[2])^G = E_1[2]^G = 0.$$

Therefore we get that the natural mapping

$$H^1(G, E_1[2]) \to H^1(B, E[2])$$

is an isomorphism, which tells us that for any $\mathbf{Q}$-rational point of B, i.e., for any morphism $b = \mathrm{Spec}\ \mathbf{Q} \to B$, the induced morphism on cohomology,

$$H^1(B, E[2]) \to H^1(b, E[2])$$

is an isomorphism.

¿From the above discussion and our hypotheses we have that $E(B) = 0$. It follows that for any nontrivial $\tilde{h} \in H^1(B, E[2])$ the specialization of its image $h \in H^1(B, E)$ to a $\mathbf{Q}$-rational point b of B, $h(b) \in H^1(b, E)$, is nontrivial if the Mordell-Weil group $E(b)$ has rank zero. To conclude the proof of our proposition, then, it suffices to show that for any positive ϵ,

$$\mathrm{card}\{b \in B(\mathbf{Q}) \mid \mathrm{Height}(b) < C \text{ and } E_1(b) \text{ finite}\} \;>>\; C^{1-\epsilon} \text{ ?}$$

For this, we use the fact that E_1 is modular. Let f be the newform of weight two which is attached to E_1. Ono and Skinner have estimates on the number of nonzero Fourier coefficients of the modular form F of weight 3/2 associated to such a newform f via the Shimura lift. By classsical results of Waldspurger, Ono and Skinner then get that for any positive ϵ

$$\text{card}\{b \in B(\mathbf{Q}) \mid \text{Height}(b) < C \text{ and analytic rank of } E_1(b) \text{ is zero}\} \;>>\; C^{1-\epsilon},$$

and they conclude the same estimate for arithmetic rank using the results of Kolyvagin. I am thankful to Brian Conrey for pointing out the Ono-Skinner reference.

2. The Cassels-Schinzel Example. Since jump points seem to be relevant to arithmetic surjectivity questions, it may be useful to recall a classic example due to Cassels-Schinzel of a pencil of elliptic curves over $\mathbf{Q}$ where *every* $\mathbf{Q}$-rational point of the parameter space B is a jump point. Their example is a certain pencil E_t of twists of the elliptic curve $X_0(32)$ over $\mathbf{Q}$ where the Mordell-Weil group of its sections (over $\mathbf{Q}$) is equal to the Klein 4-group (consisting of four sections in the 2-torsion subgroup $E_t[2]$) and yet for each $t_o \in \mathbf{Q}$ the elliptic curve E_{t_o} has odd, hence nonzero, (analytic) rank (the game in finding such examples is to "work the equations" to guarantee that odd parity happens for all rational t_o's). Of course, the "rank" computed is the analytic rank, but by a recent result of Nekovar, it is the same parity as the Selmer rank. To pass from this "parity of the Selmer rank" to the parity of Mordell-Weil rank we must invoke the Shafarevich-Tate conjecture (at least when the Mordell-Weil rank is > 1 for then we can't invoke Heegner points and Kolyvagin's theorem; this Mordell-Weil rank will even be > 1 surprisingly often: Elkies has extensive tables of this). Here is the explicit example, then, due to Cassels-Schinzel (Bull London Math Soc 14(1982)345-348).

$$E_t : \quad y^2 = x(x^2 - (7 + 7t^4)^2).$$

It would be interesting to get asymptotic results (which are either like or unlike the estimates given in the proposition above) for torsors over such an example. It would also be interesting to get asymptotic results over $\mathbf{Q}$ for pencils of plane cubics possessing no sections.