Notes on the arithmetic of Del Pezzo surfaces of degree 2

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Abstract. We study the arithmetic of certain Del Pezzo surfaces of degree 2.

1. Introduction

A general smooth Del Pezzo surface of degree 2 can be realized as a double cover of $\mathbb{P}^2$ ramified in a smooth curve of degree 4. In this note we consider sufaces S of the form

$$w^2 = Ax^4 + By^4 + Cz^4, \tag{1.1}$$

where $A, B, C \in \mathbb{Z}$. We compute their Galois-theoretic invariant $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ and the obstruction to the Hasse-principle. We match our theoretical computations with numerical data. For more background we refer to [**Man86**], [**CTKS87**].

2. Geometry

Consider the surface S given by the equation

$$w^2 = Ax^4 + By^4 + Cz^4$$

in the weighted projective space $\mathbb{P}(2, 1, 1, 1)$. Let a, b, c denote some chosen 4-th roots of A, B, C, respectively. The image in $\mathbb{P}^2$ of the 56 exceptional curves on S are the bitangents to the Fermat quartic curve. As is well-known, these are given by the following equations

$$\delta ax + by = 0, \quad \delta by + cz = 0, \quad \delta ax + cz = 0, \quad \text{where } \delta^4 = -1, \tag{2.1}$$

$$\alpha ax + \beta by + \gamma cz = 0 \ \ (\alpha^4 = \beta^4 = \gamma^4 = 1). \tag{2.2}$$

Multiplying the equation (2.2) by a scalar doesn't change the line it defines, so it is natural to index the line by an element $(\alpha, \beta, \gamma) \in \mu_4^3/\mu_4$. Each bitangent lifts to a pair of exceptional curves in S: for example, the preimage of the line given by $\delta ax + by = 0$ is the pair of curves with equations

$$w = \pm c^2 z^2 \ .$$

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These will be denoted by $L_{z,\delta,\pm}$. There are 24 exceptional curves lying over the lines in (2.1). The preimage of the line (2.2) is given by

$$w = \pm\sqrt{2}(\alpha\beta abxy + \beta\gamma bcyz + \alpha\gamma acxz) \ .$$

The ambiguity $\pm$ is resolved by scaling the tuple (α, β, γ), so that we can choose $+$ and consider $(\alpha, \beta, \gamma) \in \mu_4^3/\mu_2$. The 56 exceptional curves are denoted as follows:

$$\begin{array}{lccl}
L_{z,\delta,\pm} & : & \delta ax + by = 0, & w = \pm c^2z^2, \\
L_{x,\delta,\pm} & : & \delta by + cz = 0, & w = \pm a^2x^2, \\
L_{y,\delta,\pm} & : & \delta cz + ax = 0, & w = \pm b^2y^2, \\
L_{\alpha,\beta,\gamma} & : & \alpha ax + \beta by + \gamma cz = 0 & w = \sqrt{2}(\alpha\beta abxy + \beta\gamma bcyz + \alpha\gamma acxz) \ .
\end{array}$$

The intersections are as follows: each exceptional curve has self-intersection (-1). Each pair of curves lying above a bitangent to the Fermat quartic has intersection number 2. Among the two pairs of curves lying above two different bitangents there are exactly two pairs with intersection number 1 and two pairs with intersection number zero. We list below the pairs of lines with intersection number 1. Introduce the set T of tuples $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ such that (α, β, γ) is not proportional to $(\alpha', \beta', \gamma')$ and $\alpha/\alpha', \beta/\beta', \gamma/\gamma'$ are either pairwise distinct or $\mathbb{R}$-collinear.

$$\begin{array}{rcll}
L_{z,\delta,s} \cdot L_{x,\delta',\delta^2\delta'^2 s} & = & 1, & \\
L_{x,\delta,s} \cdot L_{y,\delta',\delta^2\delta'^2 s} & = & 1, & \\
L_{y,\delta,s} \cdot L_{z,\delta',\delta^2\delta'^2 s} & = & 1, & \\
L_{z,\delta,s} \cdot L_{z,\delta',s} & = & 1, & \delta \neq \delta', \\
L_{x,\delta,s} \cdot L_{x,\delta',s} & = & 1, & \delta \neq \delta', \\
L_{y,\delta,s} \cdot L_{y,\delta',s} & = & 1, & \delta \neq \delta', \\
L_{z,\delta,s} \cdot L_{\alpha,\beta,\gamma} & = & 1, & \text{when } \mathrm{Re}(\alpha^{-1}\beta\gamma^2\delta) > 0, \\
L_{\alpha,\beta,\gamma} \cdot L_{\alpha',\beta',\gamma'} & = & 1, & \text{when } (\alpha, \beta, \gamma, \alpha', \beta', \gamma') \in T \text{ and } \alpha\beta\gamma\alpha'\beta'\gamma' \in \{\pm 1\}.
\end{array}$$

Let $\zeta = e^{\pi i/4}$. The Picard group is a free abelian group with basis:

$$\begin{array}{llll}
v_1 = L_{x,\zeta,+} & v_2 = L_{x,\zeta^3,-} & v_3 = L_{y,\zeta,+} & v_4 = L_{y,\zeta^3,-} \\
v_5 = L_{z,\zeta,+} & v_6 = L_{z,\zeta^3,-} & v_7 = L_{i,i,i} & v_8 = L_{z,\zeta^7,-} + L_{z,\zeta^3,-} + L_{i,i,i} \ .
\end{array} \tag{2.3}$$

3. Galois group - generic case

Recall that $\zeta = e^{\pi i/4}$ and let G be the Galois group of the extension $K := \mathbb{Q}(\zeta, a, b, c)$ over $\mathbb{Q}$. The subextension $\mathbb{Q}(\zeta)/\mathbb{Q}$ corresponds to a normal subgroup H of index 4. The quotient group is Klein's four-group.

In the *generic* case, when the $|G| = 256$, we have the generators

$$\sigma, \tau, \iota_a, \iota_b, \iota_c$$

characterized by

$$\begin{array}{rclrclrclrcl}
\sigma(a) & = & a, & \sigma(b) & = & b, & \sigma(c) & = & c, & \sigma(\zeta) & = & \zeta^{-1}, \\
\tau(a) & = & a, & \tau(b) & = & b, & \tau(c) & = & c, & \tau(\zeta) & = & \zeta^3, \\
\iota_a(a) & = & ia, & \iota_a(b) & = & b, & \iota_a(c) & = & c, & \iota_a(\zeta) & = & \zeta, \\
\iota_b(a) & = & a, & \iota_b(b) & = & ib, & \iota_b(c) & = & c, & \iota_b(\zeta) & = & \zeta, \\
\iota_c(a) & = & a, & \iota_c(b) & = & b, & \iota_c(c) & = & ic, & \iota_c(\zeta) & = & \zeta \ .
\end{array}$$

The corresponding action of G on exceptional curves is given by

	σ	τ	ι_a	ι_b	ι_c
$L_{z,\delta,s}$	$L_{z,\sigma(\delta),s}$	$L_{z,\tau(\delta),s}$	$L_{z,i\delta,s}$	$L_{z,-i\delta,s}$	$L_{z,\delta,-s}$
$L_{x,\delta,s}$	$L_{x,\sigma(\delta),s}$	$L_{x,\tau(\delta),s}$	$L_{x,\delta,-s}$	$L_{x,i\delta,s}$	$L_{x,-i\delta,s}$
$L_{y,\delta,s}$	$L_{y,\sigma(\delta),s}$	$L_{y,\tau(\delta),s}$	$L_{y,-i\delta,s}$	$L_{y,\delta,-s}$	$L_{y,i\delta,s}$
$L_{\alpha,\beta,\gamma}$	$L_{\alpha^{-1},\beta^{-1},\gamma^{-1}}$	$L_{i\alpha^{-1},i\beta^{-1},i\gamma^{-1}}$	$L_{i\alpha,\beta,\gamma}$	$L_{\alpha,i\beta,\gamma}$	$L_{\alpha,\beta,i\gamma}$

In the basis (2.3), the action of the various generators on the Picard group $\mathrm{Pic}(S_K)$ is given by the following matrices:

$$\iota_a = \begin{pmatrix} -2 & -1 & -1 & -1 & -1 & -1 & -1 & -3 \\ -1 & -2 & -1 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -2 & -1 & -1 & -1 & -3 \\ -1 & -1 & 0 & -2 & -1 & -1 & 0 & -2 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & -2 \\ -1 & -1 & -1 & -1 & -2 & -1 & -1 & -3 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & -2 \\ 3 & 3 & 2 & 3 & 3 & 2 & 2 & 7 \end{pmatrix}, \iota_b = \begin{pmatrix} -1 & 0 & -1 & -1 & -1 & -1 & 0 & -2 \\ -2 & -1 & -1 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -2 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -2 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -1 & -1 & -2 & -1 & -3 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & -2 \\ -1 & 0 & -1 & -1 & 0 & -1 & -1 & -2 \\ 3 & 2 & 3 & 3 & 2 & 3 & 2 & 7 \end{pmatrix},$$

$$\iota_c = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 & -1 & -1 & -3 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 & -2 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & -2 \\ -1 & -1 & -2 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -1 & -2 & -1 & -1 & -3 \\ -1 & -1 & -1 & -1 & -1 & -2 & -1 & -3 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 & -2 \\ 2 & 3 & 3 & 2 & 3 & 3 & 2 & 7 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} -1 & 0 & -1 & -1 & -1 & -1 & 0 & -2 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 & -2 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & -2 \\ -1 & -1 & 0 & -1 & -1 & -1 & 0 & -2 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & -2 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 0 & 5 \end{pmatrix}, \tau = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 & -1 & -1 & -3 \\ -2 & -1 & -1 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -2 & -1 & -1 & -1 & -3 \\ -1 & -1 & -2 & -1 & -1 & -1 & -1 & -3 \\ -1 & -1 & -1 & -1 & -1 & -2 & -1 & -3 \\ -1 & -1 & -1 & -1 & -2 & -1 & -1 & -3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -2 & -3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 8 \end{pmatrix}$$

4. Review of group cohomology

Let G be a finite group and let M be a G-module. A standard free resolution of $\mathbb{Z}$ is given as follows:

$$\mathcal{C}^G_\bullet := \ldots \mathbb{Z}[G \times G \times G] \to \mathbb{Z}[G \times G] \to \mathbb{Z}[G] \ , \tag{4.1}$$

where the augmentation map $\mathbb{Z}[G] \to \mathbb{Z}$ is given by $g \mapsto 1$ (for all $g \in G$) and where each map in $\mathcal{C}^G_\bullet$ is of the form

$$(g_1, \ldots g_n) \mapsto \sum_{i=1}^{n} (-1)^{i+1} (g_1, \ldots, \widehat{g_i}, \ldots, g_n) \ .$$

The action of $g \in G$ on any of the terms in (4.1) is the diagonal left multiplication action. We may identify

$$\begin{array}{ccc} \mathbb{Z}[G \times G] & \simeq & \bigoplus_{g' \in G} \mathbb{Z}[G] \ , \\ (g, gg') & \mapsto & (0, \ldots, g, \ldots, 0) \ , \end{array} \tag{4.2}$$

where the unique non-zero entry g is in the g'-th position. We also identify

$$\begin{array}{ccc} \mathbb{Z}[G \times G \times G] & \simeq & \bigoplus_{(g',g'') \in G \times G} \mathbb{Z}[G] \ , \\ (g, gg', gg'') & \mapsto & (0, \ldots, g, \ldots, 0) \ , \end{array} \tag{4.3}$$

where the unique non-zero entry g is in the (g', g'')-th position.

After these identifications, the complex $\mathrm{Hom}(\mathcal{C}^G_\bullet, M)$ is identified with

$$\mathcal{C}^\bullet_{G,M} := M \xrightarrow{d^0} \bigoplus_{g' \in G} M \xrightarrow{d^1} \bigoplus_{(g',g'') \in G \times G} M \ldots \ . \tag{4.4}$$

Here the g'-th coordinate of the map d^0 is $m \mapsto m - g' \cdot m$ and the (g', g'')-th coordinate of d^1 is $(\dots, m_g, \dots) \mapsto m_{g'} - m_{g''} + g' \cdot m_{(g')^{-1}g''}$. Of course, $H^i(G, M)$ is identified with the i-th cohomology of (4.4). For instance, the kernel of d^0 is the module M^G of G-invariants of M.

Now let H be a subgroup of G. Since restriction is an exact functor, $\mathcal{C}^G_\bullet$ is a resolution of $\mathbb{Z}$ as an H-module. We choose a set $Q \subset G$ of coset representatives, so $G = \bigcup_{q \in Q} Hq$.

We have an isomorphism of H-modules

$$\begin{array}{ccc} \mathbb{Z}[G] & \simeq & \bigoplus_{q \in Q} \mathbb{Z}[H] \ , \\ hq & \mapsto & (0, \dots, h, \dots, 0) \ , \end{array} \tag{4.5}$$

where h appears in the q-th position $(h \in H, q \in Q)$. Also

$$\begin{array}{ccc} \mathbb{Z}[G \times G] & \simeq & \bigoplus_{(q,h',q') \in Q \times H \times Q} \mathbb{Z}[H] \ , \\ (hq, hh'q') & \mapsto & (0, \dots, h, \dots, 0) \ , \end{array} \tag{4.6}$$

where h appears in the (q, h', q') position. We can project the resolution $\mathcal{C}^G_\bullet$ to the standard resolution $\mathcal{C}^H_\bullet$. Under the identification (4.5) the map on the degree zero component is the sum of the $|Q|$ projection maps. Under the identifications (4.2) and (4.6) the map on the degree 1 component sends the element $(0, \dots, h, \dots, 0)$ from (4.6) to $(0, \dots, h, \dots, 0)$ with h in the h' position. Applying $\mathrm{Hom}_H(-, M)$ we get an inclusion of complexes $\mathcal{C}^\bullet_{H,M}$ into $\mathrm{Hom}_H(\mathcal{C}^G_\bullet, M)$, and via our identifications,

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\chi^0"] & \bigoplus_H M \arrow[r] \arrow[d, "\chi^1"] & \cdots \\ \bigoplus_Q M \arrow[r] & \bigoplus_{Q \times H \times Q} M \arrow[r] & \cdots \ . \end{tikzcd} \tag{4.7}$$

This allows us to take elements of $H^i(H, M)$, represented as cocycles via the standard resolution, and realize them as cocycles in the complex $\mathrm{Hom}_H(\mathcal{C}^G_\bullet, M)$.

5. Cohomology of group extensions

Assume that there is an exact sequence of groups

$$1 \to H \to G \to Q \to 1 \ .$$

Then Q acts on the cohomology $H^q(H, M)$ for all q and there is an associated standard spectral sequence

$$E_2^{p,q} = H^p(Q, H^q(H, M)) \Rightarrow H^{p+q}(G, M) \ . \tag{5.1}$$

This leads to a 5-term exact sequence

(5.2)
$$0 \to H^1(Q, M^H) \to H^1(G, M) \to H^1(H, M)^Q \xrightarrow{d_2^{0,1}} H^2(Q, M^H) \to H^2(G, M) \ .$$

For the purpose of computing $d_2^{0,1}$, we describe explicitly the spectral sequence at the E_0 level. There is an action of Q on the complex $\mathrm{Hom}_H(\mathcal{C}^G_\bullet, M)$, which is induced from the G action on this complex that combines the conjugation of G on itself with its action on M. This Q action has invariants $\mathrm{Hom}_G(\mathcal{C}^G_\bullet, M)$. Moreover, each term is acyclic as a Q-module. This leads to the E_0-term of the spectral sequence (5.1)

$$\begin{tikzcd}
\mathrm{Hom}_H(\mathbb{Z}[G^3],M) \arrow[r] & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G^3],M) \arrow[r] & \cdots \\
\mathrm{Hom}_H(\mathbb{Z}[G^2],M) \arrow[r] \arrow[u, "d_0^{0,1}"] & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G^2],M) \arrow[r] \arrow[u, "d_0^{1,1}"] & \cdots \arrow[u] \\
\mathrm{Hom}_H(\mathbb{Z}[G],M) \arrow[r] \arrow[u, "d_0^{0,0}"] & \bigoplus_Q \mathrm{Hom}_H(\mathbb{Z}[G],M) \arrow[r] \arrow[u, "d_0^{1,0}"] & \bigoplus_{Q^2} \mathrm{Hom}_H(\mathbb{Z}[G],M) \arrow[u]
\end{tikzcd}$$

In the special case where G is a *semi-direct* product of H and Q, we have identifications (4.5) and (4.6). Now $\tilde{q} \in Q$ acts on elements of the groups appearing in the bottom row of (4.7) as follows:

$$\tilde{q} \cdot (\ldots, m_q, \ldots) = (\ldots, \tilde{q} \cdot m_{\tilde{q}^{-1}q}, \ldots) \tag{5.3}$$

$$\tilde{q} \cdot (\ldots, m_{q,h',q'}, \ldots) = (\ldots, \tilde{q} \cdot m_{\tilde{q}^{-1}q,\tilde{q}^{-1}h'\tilde{q},\tilde{q}^{-1}q'}, \ldots) \tag{5.4}$$

6. Cohomology of abelian groups

For finite abelian groups G there are more efficient resolutions than the standard resolution. In each of the following representative cases we give an alternative resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules together with explicit maps from the standard resolution to the more efficient resolution. This allows us to compute the images of cocycles from the effecient resolution in the standard resolution.

Notation 6.1. Let G be a finite abelian group and $g \in G$ an element of order n. Put $N_g := 1 + g + \cdots + g^{n-1}$ and $\Delta_g := 1 - g$ in $\mathbb{Z}[G]$. For $g_1, \ldots, g_\nu \in G$ and $i_1, \ldots, i_\nu \in \mathbb{Z}$ the element in $\mathcal{C}_1^G$ which, under the identification (4.2) is the vector $(0, \ldots, 1, \ldots, 0)$ with 1 in the $(g_1^{i_1} g_2^{i_2} \cdots g_\nu^{i_\nu})$-th position, is denoted $\alpha_{i_1,\ldots,i_\nu}$. Similarly, given $i'_1, \ldots, i'_\nu \in \mathbb{Z}$ the element in $\mathcal{C}_2^G$ which, under the identification (4.3) is the vector $(0, \ldots, 1, \ldots, 0)$ with 1 in the $(g_1^{i_1} g_2^{i_2} \cdots g_\nu^{i_\nu}, g_1^{i'_1} g_2^{i'_2} \cdots g_\nu^{i'_\nu})$-th position is denoted $\alpha_{i_1,\ldots,i_\nu,i'_1,\ldots,i'_\nu}$.

Case 1: $G = \mathbb{Z}/n$.

$$\mathcal{C}_\bullet^{[n]} := \cdots \mathbb{Z}[G] \xrightarrow{N_g} \mathbb{Z}[G] \xrightarrow{\Delta_g} \mathbb{Z}[G] \ ,$$

with quasi-isomorphism

$$\sigma_\bullet^{[n]} : \mathcal{C}_\bullet^G \to \mathcal{C}_\bullet^{[n]}$$

given by

$$\begin{array}{rcl} \sigma_1^{[n]}(\alpha_i) & = & 1 + g + \cdots + g^{i-1} \ , \\ \sigma_2^{[n]}(\alpha_{i,i'}) & = & \left\{ \begin{array}{ll} 1 & \text{if } i > i' \\ 0 & \text{otherwise} \end{array} \right. \ , \end{array} \tag{6.1}$$

where g is a generator of G.

Case 2: $G = \mathbb{Z}/n \oplus \mathbb{Z}/m$.

$$\mathcal{C}_\bullet^{[n,m]} := \cdots \mathbb{Z}[G]^3 \xrightarrow{A^{[g,h]}} \mathbb{Z}[G]^2 \xrightarrow{(\Delta_g\ \Delta_h)} \mathbb{Z}[G] \ ,$$

where

$$A^{[g,h]} := \begin{pmatrix} N_g & \Delta_h & 0 \\ 0 & -\Delta_g & N_h \end{pmatrix}$$

with quasi-isomorphism

$$\sigma_\bullet^{[n,m]} \,:\, \mathcal{C}_\bullet^G \to \mathcal{C}_\bullet^{[n,m]}$$

given by

$$\sigma_1^{[n,m]}(\alpha_{i,j}) \quad = \quad (1+g+\cdots+g^{i-1}, g^i+g^ih+\cdots+g^ih^{j-1})\ , \tag{6.2}$$

where g (resp. h) is a generator of $\mathbb{Z}/n$ (resp. $\mathbb{Z}/m$).

Case 3: $G = \mathbb{Z}/n \oplus \mathbb{Z}/m \oplus \mathbb{Z}/\ell$.

$$\mathcal{C}_\bullet^{[n,m,\ell]} := \cdots \mathbb{Z}[G]^6 \xrightarrow{A^{[g,h,u]}} \mathbb{Z}[G]^3 \xrightarrow{(\Delta_g\ \Delta_h\ \Delta_u)} \mathbb{Z}[G]\ ,$$

where

$$A^{[g,h,u]} := \begin{pmatrix} N_g & \Delta_h & 0 & \Delta_u & 0 & 0 \\ 0 & -\Delta_g & N_h & 0 & \Delta_u & 0 \\ 0 & 0 & 0 & -\Delta_g & -\Delta_h & N_u \end{pmatrix}$$

with quasi-isomorphism

$$\sigma_\bullet^{[n,m,\ell]} \,:\, \mathcal{C}_\bullet^G \to \mathcal{C}_\bullet^{[n,m,\ell]}$$

given by

$$\begin{aligned} &\sigma_1^{[n,m,\ell]}(\alpha_{i,j,k}) = \\ &\qquad (1+g+\cdots+g^{i-1}, g^i(1+h+\cdots+h^{j-1}), g^ih^j(1+\cdots+g^ih^ju^k))\ , \end{aligned} \tag{6.3}$$

where g (resp. h, resp. u) is a generator of $\mathbb{Z}/n$ (resp. $\mathbb{Z}/m$, $\mathbb{Z}/\ell$).

In each case, given a G-module M we apply $\mathrm{Hom}_G(-,M)$ to every complex above. This provides a practical method for computing group cohomology of finite abelian groups. The dual maps are notated as above but with the super- and subscripts interchanged. For example, $A_{[g,h]} \,:\, M^2 \to M^3$ maps the element $(m,0)$ to $(m+g\cdot m+\cdots+g^{n-1}\cdot m, m-h\cdot m, 0)$.

7. Computation of $\mathrm{Br}(S)/\mathrm{Br}(\mathbb{Q})$ in the generic case

In this section we explain the computation of $H^1(G,M)$, where $M = \mathrm{Pic}(S_K)$, in the generic case. We have an exact sequence

$$1 \to H \to G \to Q \to 1$$

with $H = (\mathbb{Z}/4)^3$ and $Q = (\mathbb{Z}/2)^2$. In principle, $H^1(G,M)$ can be computed using the standard resolution (4.4). However, in this case the map d_1 would be given by a 524288×2046-matrix, which makes direct computations impractical. Exploiting the fact that G is an extension of one abelian group by another, we can use the spectral sequence technique, explained in Section 5, to simpify the computation significantly.

In the following, we will constantly refer to the diagram in Figure 1. First we compute $M^H = M^G = \mathbb{Z}$, spanned by the canonical class. In particular, $H^1(Q,M^H) = 0$. Thus $H^1(G,M)$ is equal to the kernel of the map

$$d_2^{0,1} \,:\, H^1(H,M)^Q \to H^2(Q,M^H)\ .$$

The group $H^1(H, M)$ is computed by the complex on the left side of the diagram. In this diagram the horizontal arrows labeled $\sigma^i_{[4,4,4]}$ and χ^i give quasi-isomorphisms of complexes. The linear algebra required to compute $\mathrm{Ker}(M^3 \to M^6)$ is quite modest and the cohomology group is identified as

$$H^1(H, M) = \mathbb{Z}/2 \ .$$

It remains to take a single cocycle representative of the non-zero element of $H^1(H, M)$ and follow it through the diagram to determine whether it lies in the kernel of $d_2^{0,1}$.

Remark 7.1. In this case the class is automatically Q-invariant since $\mathbb{Z}/2$ has only the identity automorphism. In general, as we point out below, there is a place in the diagram chase where this invariance is naturally tested (a certain element would otherwise fail to lift).

We start with a representative in M^3 for the nontrivial element $\lambda \in H^1(H, M)$, for instance

$$u = ((0, 0, 0, 0, -1, -1, -1, 1), (0, 0, 0, 0, -1, 1, 0, 0), (0, 0, 0, 0, -2, 0, -1, 1)).$$

Let v denote the image in $E_0^{1,1}$ of v by the composite of three horizontal maps in Figure 1. Now, we claim, v lies in the image of $d_0^{1,0}$ if and only if λ is Q-invariant (obviously true in this case). Indeed, a linear algebra solver can produce

$$v_0 = ((0, 0, 0, 0, -1, 1, 0, 0)^{*4}, (0, 0, 0, 0, -1, -1, -1, 1)^{*4}))$$

satisfying $d_0^{1,0}(v_0) = v$, where each vector with superscript $*4$ denotes the element in $\bigoplus_Q M$ with the vector repeated. Applying the cobounday map $E_0^{1,0} \to E_0^{2,0}$ to v_0 necessarily produces an element in the image of i_2, representing $d_2^{0,1}(\lambda)$ in $H^2(Q, M^H)$. In the present case, we get 0; in general, a linear solver can test whether or not it is a coboundary.

8. The non-generic case

We give examples when the Galois group is smaller than in the generic case.

Example 8.1. Consider the case $(A, B, C) = (-6, -3, 2)$. Here there are no local obstructions to the existence of rational points. The Galois group of the splitting field has order 32; it is an extension of the Klein four-group by $(\mathbb{Z}/4) \oplus (\mathbb{Z}/2)$. However, in this way it is not a split extension. On the other hand, we can use the *split* extension

$$1 \to H \to G \to \mathbb{Z}/2 \to 1$$

where $H = (\mathbb{Z}/4)^2$, generated by

$$\iota_a \iota_b \text{ and } \sigma \iota_a \iota_c \tau \ ,$$

and the image of $\mathbb{Z}/2$ in G is generated by σ. In this case, we compute $H^1(H, M) = 0$. By (5.2), $H^1(G, M)$ is isomorphic to $H^1(\mathbb{Z}/2, M^H)$. We find that M^H has rank 2, spanned by

$$(1, 1, 1, 1, 1, 1, 1, -3) \ ,$$
$$(1, 1, 1, 1, 1, 1, 0, -2) \ ,$$

hence M^H is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}'$, where $\mathbb{Z}'$ is free of rank 1 with non-trivial $\mathbb{Z}/2$-action. So, we have

$$H^1(G, M) = \mathbb{Z}/2.$$

FIGURE 1. E_0 spectral sequence

$$\begin{tikzcd}
M^6 \arrow[r, "\sigma^2_{[4,4,4]}"] & \bigoplus_{H\times H} M \arrow[r] & E_0^{0,2} \arrow[r] & E_0^{1,2} & \\
M^3 \arrow[u, "A_{[4,4,4]}"] \arrow[r, "\sigma^1_{[4,4,4]}"] & \bigoplus_H M \arrow[u] \arrow[r, "\chi^1"] & E_0^{0,1} = \bigoplus_{Q\times H\times Q} M \arrow[u, "d_0^{0,1}"] \arrow[r] & E_0^{1,1} = (\bigoplus_{Q\times H\times Q} M)^2 \arrow[u, "d_0^{1,1}"] \arrow[r] & E_0^{2,1} \\
M \arrow[u] \arrow[r, equal] & M \arrow[u] \arrow[r, "\chi^0"] & E_0^{0,0} = \bigoplus_Q M \arrow[u, "d_0^{0,0}"] \arrow[r] & E_0^{1,0} = (\bigoplus_Q M)^2 \arrow[u, "d_0^{1,0}"] \arrow[r] & E_0^{2,0} = (\bigoplus_Q M)^3 \arrow[u, "d_0^{2,0}"] \\
 & & M^H \arrow[u, hook, "i_0"] \arrow[r] & (M^H)^2 \arrow[u, hook, "i_1"] \arrow[r, "A_{[2,2]}"] & (M^H)^3 \arrow[u, hook, "i_2"]
\end{tikzcd}$$

As in the generic case, we have $M^G = \mathbb{Z}$, that is $\mathrm{Pic}(S)$ has rank 1.

EXAMPLE 8.2. The case $(A, B, C) = (1, 1, -2)$ is interesting because $\mathrm{Pic}(S)$ has rank 2. The Galois group G fits into an exact sequence

$$1 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/2 \to 1$$

with subgroup $H = \mathbb{Z}/4$ generated by $\iota_c\sigma\tau$ and image of quotient group generated by τ. As in example 8.1 we have $H^1(H, M) = 0$. Now M^H has rank 3, with generators

$$\begin{array}{l}(1,1,1,1,1,1,1,-3)\ ,\\ (0,0,0,0,1,-1,0,0)\ ,\\ (0,0,0,0,1,1,1,-1)\ ,\end{array}$$

and the action of τ fixes the first 2 vectors and negates the third. Hence

$$H^1(G, M) = \mathbb{Z}/2$$

and $\mathrm{Pic}(S)$ has rank 2.

EXAMPLE 8.3. The case $(A, B, C) = (1, 1, 1)$ yields $G = \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, the Klein four-group, and we directly compute

$$H^1(G, M) = (\mathbb{Z}/2)^3,$$

and $\mathrm{Pic}(S)$ has rank 1.

References

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