EQUIVARIANT BIRATIONAL TYPES

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Throughout, k is a field of characteristic zero, since many of the constructions rely on *resolution of singularities* and *weak factorization*.

Birational geometry is concerned with function fields $k(X)$ of algebraic varieties X over k. It is a central and classical topic in higher-dimensional geometry. Apart from its intrinsic beauty, it has important applications, e.g., to diophantine geometry, algebraic dynamical systems, and even to string theory, under the headline of "wall-crossings". In turn, ideas from neighboring areas have given new and powerful tools that allowed to settle long-standing problems.

For example, consider actions of finite groups G on projective spaces $\mathbb{P}^n$ over an algebraically closed field k, for $n \leq 3$. The classification of such actions, up to conjugation in the projective linear group $\mathsf{PGL}_{n+1}(k)$, has been completed at least 120 years ago. But we are still lacking a full classification, up to conjugation in the Cremona group

$$\mathrm{Cr}_n = \mathrm{Aut}\,\mathrm{Bir}(\mathbb{P}^n),$$

the group of *birational* automorphisms of $\mathbb{P}^n$!

There is an extensive literature on this problem, within the framework of *birational rigidity*, which in turn relies on tools of the *Minimal Model Program*. Inspired by ideas of *motivic integration*, the papers [KT19], [KPT23], [KT22a], [KT21] introduced invariants of very different character. The main conceptual innovation was the definition of several types of *Burnside rings*, capturing information about (equivariant) birational types. This allowed to prove long-anticipated results about *specialization* of (equivariant) birationalities. Furthermore, this formalism yielded many new examples of nonbirational actions which were not distinguishable with previous methods.

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In detail, let

$$\mathrm{Burn}(k) = \oplus_{n\geq 0} \mathrm{Burn}_n(k),$$

where

$$\mathrm{Burn}_n(k), \quad n \geq 0,$$

is freely generated by birational isomorphism classes of n-dimensional varieties over k, and the product on $\mathrm{Burn}(k)$ is induced by direct products of varieties. These rings have an intricate internal structure, reflecting nontrivial stable birationalities over k. Similar rings can be considered in the context of varieties equipped with actions of finite groups [KT22a], varieties with logarithmic volume forms [CLKT23], or orbifolds [KT21].

Concretely, for G abelian, consider the $\mathbb{Q}$-vector space

$$\mathcal{B}_n(G) := \mathbb{Q}[a_1, \dots, a_n]/\sim,$$

an abelian group on symbols, consisting of sequences of characters of G, up to permutations, generating $G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$, and subject to relations

$$[a_1, a_2, a_3, \dots, a_n] = [a_1 - a_2, a_2, a_3, \dots, a_n] + [a_1, a_2 - a_1, a_3, \dots, a_n].$$

Given a regular action of G on X, we have a stratification of the fixed locus $X^G = \sqcup X_\alpha$, recording the sequence of characters of G appearing in the normal bundle to some $x_\alpha \in X_\alpha$, we have an assignment

$$[X \circlearrowleft G] := \sum_\alpha [a_{1,\alpha}, \dots, a_{n,\alpha}].$$

By [KPT23], the class

$$[X \circlearrowleft G] \in \mathcal{B}_n(G)$$

is a well-defined birational invariant of the action. The more complicated

$$\mathrm{Burn}_n(G),$$

defined in [KT22a], work for nonabelian G as well, and take into account *all* strata with nontrivial stabilizers, as well as birational types of residual actions on the strata themselves.

Numerous geometric applications can be found in, e.g., [HKT21] and [KT22b]. New results concerning birational classification of linear actions on $\mathbb{P}^2$ and $\mathbb{P}^3$ are in [TYZ23].

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