BIRATIONAL TYPES

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1. Rationality

This note is a version of the talk of one of us (YT) at the inaugural Beijing International Congress of Basic Science.

Let X be a smooth projective algebraic variety over a field k. We assume that k has characteristic zero, since many of the constructions rely on *resolution of singularities* and *weak factorization*. A basic and well-studied problem in algebraic geometry is to understand its function field $k(X)$. This includes the classification of such fields up to isomorphism, i.e., classification of algebraic varieties up to birationality. Furthermore, one is interested in finite subgroups $G \subset \mathrm{Aut}(k(X)/k)$, as well as those preserving additional structures.

Of particular interest are function fields of projective spaces $\mathbb{P}^n$; indeed, the study of the *Cremona groups*

$$\mathrm{Cr}_n := \mathrm{Aut}(k(\mathbb{P}^n)/k), \quad n \geq 2,$$

is an extensive subject of its own. Recall that variety X is called *rational* if it is birational to $\mathbb{P}^n$ and *stably rational* if $X \times \mathbb{P}^m$ is rational, for some m. Both notions depend on the ground field k; allowing nonclosed fields is essential if one is interested in universal properties of these notions.

For instance, consider a smooth fibration

$$\pi : \mathcal{X} \to B,$$

over k, of relative dimension n. It is natural to hope for a relation between rationality properties of the generic fiber X over $k(B)$, and special fibers $\mathcal{X}_b$, for $b \in B$.

Informally, *rationality* means that X can be *parametrized* by algebraically independent rational coordinate functions $x_1(b), \dots, x_n(b)$. Thus, rationality of X over $k(B)$ yields rationality of all fibers X_b with b in some Zariski open subset $B^\circ \subset B$. Geometrically, one can think

Date: August 11, 2023.

about B° as the locus where a *generic* rationality construction can be applied; proving rationality in its complement, where, e.g., some of the coordinate functions x_i are not defined or fail to be independent, typically involves complicated *ad hoc* arguments. The following came as a surprise, not the least to us:

Theorem 1. [KT19a] *Rationality holds for all fibers of* π.

This result was inspired by a similar statement concerning stable rationality, proved in [NS19].

2. Burnside groups

The main conceptual idea which allowed to bypass all *ad hoc* considerations was the introduction of the *Burnside ring*

$$\mathrm{Burn}(k) = \oplus_{n\geq 0} \mathrm{Burn}_n(k).$$

As an abelian group

$$\mathrm{Burn}_n(k), \quad n \geq 0,$$

is freely generated by birational isomorphism classes of n-dimensional varieties over k, and the product on $\mathrm{Burn}(k)$ is induced by direct products of varieties. In the context of stable rationality, [NS19] considered

$$\mathrm{K}_0(\mathrm{Var}_k)/[\mathbb{L}],$$

the quotient of the Grothendieck ring of varieties over k by the ideal generated by the class of the affine line. This ring appeared prominently in motivic integration, as the universal ring for *motivic measures*; by [LL03], it is freely generated by stably birational isomorphism classes of algebraic varieties over k.

Similar rings can be considered in other contexts, e.g., for varieties equipped with actions of finite groups [KT22b], varieties with logarithmic volume forms [CLKT23], or orbifolds [KT19b]. These rings have an intricate internal structure, reflecting nontrivial stable birationalities. There is also an intriguing connection to cohomology of arithmetic groups and the theory of automorphic forms, uncovered in [KPT23]. And among first geometric applications, they serve as natural targets for invariants of (stable) birational automorphisms [LS22], [KT22a], [CLKT23, Section 7], yielding new structural information about the Cremona groups Cr_n.

3. Specialization

Specialization is ubiquitous in geometry and arithmetic. The main result of [KT19a] is:

Theorem 2. *Let $\mathfrak{o}$ be a discrete valuation ring, with residue field k and fraction field K. Then there is a well-defined (nontrivial) homomorphism of abelian groups*

$$\mathrm{Burn}_n(K) \to \mathrm{Burn}_n(k).$$

This was inspired by a corresponding specialization homomorphism in [NS19]:

$$\mathrm{K}_0(\mathrm{Var}_K)/[\mathbb{L}] \to \mathrm{K}_0(\mathrm{Var}_k)/[\mathbb{L}],$$

which in turn goes back to [DL01], [HK06]. Both homomorphisms are given by explicit formulas, incorporating information about function fields of *all* strata of the natural stratification of the special fiber.

With appropriate modifications, specialization holds in equivariant, logarithmic volume forms, and orbifold situations, respectively.

The explicit definition of the specialization homomorphisms yields, as an immediate corollary, specialization of (stable) rationality as in Theorem 1, and in other contexts. In fact, rationality holds even upon specialization to mildly singular fibers. Sometimes, the richer geometry of those singular fibers allows to extract obstructions to rationality which are invisible in the generic fiber, thus proving the failure of rationality of the generic fiber in a new, indirect, manner.

In another direction, Theorem 2 has been applied to deduce rationality of special fibers without the detailed analysis of the failure of general position arguments.

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