Equivariant birational types

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Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic zero. We assume that X is rational, i.e., birational to $\mathbb{P}^n$, and that it carries a regular and generically free action of a finite group G. A classical problem is to decide whether or not this action is *linearizable*, i.e., whether or not X is G-equivariantly birational to $\mathbb{P}^n$, with a (projectively) linear action of G. There is an extensive literature on this problem, already for $n = 2$, going back to Bertini, Castelnuovo, Kantor, and many others, and culminating in the work of Dolgachev–Iskovskikh [1].

Among the known equivariant birational invariants is:

(1) Existence of fixed points upon restriction to abelian subgroups of G.

A more sophisticated invariant was introduced in [9], for *abelian* G:

(2) Let $\mathfrak{p} \in X$ be a point fixed by G. Let $\{a_1, \dots, a_n\}$ be its weights, i.e., characters of G in the tangent space to X at $\mathfrak{p}$, and

$$\det(a_1, \dots, a_n) = a_1 \wedge \cdots \wedge a_n \in \wedge^n(G^\vee)$$

the *determinant*. Let $Y \to X$ be a G-equivariant blowup. Then Y contains a G-fixed point $\mathfrak{q}$ (in the preimage of $\mathfrak{p}$) with weights $\{b_1, \dots, b_n\}$, such that

$$\det(b_1, \dots, b_n) = \pm \det(a_1, \dots, a_n),$$

i.e., this is an *equivariant birational invariant.*

Inspired by applications of ideas from motivic integration to the study of rationality properties of algebraic varieties [8, 4], and by keen interest in equivariant birational geometry, the following generalization of (2) was introduced in [3]:

Let G be *abelian*, and $X^G = \sqcup_\alpha F_\alpha$ the decomposition of the G-fixed locus into irreducible components. Recording the G-eigenvalues

$$[b_{1,\alpha}, \dots, b_{n,\alpha}], \quad b_{j,\alpha} \in G^\vee = \mathrm{Hom}(G, k^\times),$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$, we put, formally,

$$\beta(X) := \sum_\alpha [b_{1,\alpha}, \dots, b_{n,\alpha}].$$

Let $\mathcal{S}_n(G)$ be the free abelian group generated by *unordered* tupels $[b_1, \dots, b_n]$, with $b_i \in G^\vee$, such that $\sum_i \mathbb{Z} b_i = G^\vee$. Consider the quotient

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G),$$

by the *blow-up* relations

$$\beta(Y) - \beta(X) = 0,$$

for every G-equivariant blowup $Y \to X$. It turns out, that *all* such relations can be encoded in a compact form:

(B) for all $b_1, b_2, b_3, \dots, b_n \in G^\vee$ we have

$$
\begin{aligned}
[b_1, b_2, b_3, \dots b_n] &= [b_1 - b_2, b_2, b_3, \dots, b_n] + [b_1, b_2 - b_1, b_3, \dots, b_n], \qquad && b_1 \neq b_2,\\
&= [b_1, 0, b_3, \dots, b_n], && b_1 = b_2.
\end{aligned}
$$

Equivariant Weak Factorization yields:

(3) The class $\beta(X) \in \mathcal{B}_n(G)$ is a G-equivariant birational invariant [3].

The groups $\mathcal{B}_n(G)$ exhibit a rather intricate internal structure, they are equal to cohomology of certain congruence subgroups, carry Hecke operators etc., see [3]. First geometric applications of this new invariant can be found in [2].

The next development, in [5], addressed three issues:

- extension to *nonabelian* groups,
- considerations of *all* possible, and not just maximal, stabilizers, and
- inclusion of the function-field information of strata, with induced actions.

The geometric input data for the definitions in [5] are:

- $\mathrm{Bir}_d(k)$ – birationality classes of varieties of dimension d over k,
- $\mathrm{Alg}_N(K_0)$ – isomorphism classes of Galois algebras K over $K_0 \in \mathrm{Bir}_d(k)$ for a finite group N, subject to a certain **Assumption 1.**

Let $\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$ be the $\mathbb{Z}$-module, generated by symbols

$$(H, N \curvearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup, with character group $H^\vee$, and $N := N_G(H)/H$,
- $K \in \mathrm{Alg}_N(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \le n$,
- $\beta = (b_1, \dots, b_{n-d})$, a sequence, up to order, of *nonzero* elements of $H^\vee$, that generate $H^\vee$.

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, N \curvearrowright K, \beta) = (H', N' \curvearrowright K, \beta')$, when $H' = gHg^{-1}, N' = N_G(H')/H'$, and β and β' are related by conjugation by $g \in G$.

(B1): $(H, N \curvearrowright K, \beta) = 0$, when $b_1 + b_2 = 0$.

(B2): $(H, N \curvearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2,\\ (H, N \curvearrowright K, \beta_1) + (H, N \curvearrowright K, \beta_2), & \text{otherwise,}\end{cases}$$

with

$$\beta_1 := (b_1 - b_2, b_2, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}),$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i,\\ (\overline{H}, \overline{N} \curvearrowright \overline{K}, \bar{\beta}), & \text{otherwise,}\end{cases}$$

with

$$\overline{H}^\vee := H^\vee/\langle b_1 - b_2\rangle, \quad \bar\beta := (\bar b_2, \bar b_3, \dots, \bar b_{n-d}), \quad \bar b_i \in \overline{H}^\vee,$$

and a specified action on a new algebra $\overline{K}$. The Burnside groups $\mathrm{Burn}_n(G)$ also have an intricate internal structure: they admit interesting filtrations, forgetful homomorphisms, restriction, induction, comparison homomorphisms, see [6].

The class of a G-variety X is computed on a *standard model* (X, D):

- X is smooth projective, D a normal crossings divisor,
- G acts freely on $U := X \setminus D$,
- $\forall g \in G$ and irreducible components D, either $g(D) = D$ or $g(D) \cap D = \varnothing$.

Passing to a standard model X, define the class:

$$[X \righttoleftarrow G] := \sum_H \sum_F (H, N \curvearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) *abelian* subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H. The symbols record the generic eigenvalues of H in the normal bundle along F, as well as the $N = N_G(H)/H$-action on the function field of F, respectively the orbit of F. Note that, on a standard model, all stabilizers are abelian, and all symbols satisfy **Assumption 1**.

(4) The class $[X \righttoleftarrow G] \in \mathrm{Burn}_n(G)$ is a well-defined G-equivariant birational invariant [5, Theorem 5.1].

Using this invariant, we found new examples of finite groups G admitting intransitive, nonbirational actions on $\mathbb{P}^2$, addressing a problem raised in [1, Section 9], and $\mathbb{P}^3$ [7].

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