RATIONALITY AND SPECIALIZATION

YURI TSCHINKEL

1. Introduction

This note, an extended version of my talk at the 9th Pan African Congress of Mathematicians, surveys several recent advances in birational algebraic geometry. The key problem in this area is to determine whether or not an algebraic variety X of dimension n over some field k is k-rational, i.e., k-birationally equivalent to the simplest projective variety, the projective space $\mathbb{P}^n$. One of the motivations, going back to the study of Pythagorean triples, i.e., solutions of the equation

$$x^2 + y^2 = z^2,$$

is to obtain a *parametrization* of solutions, which in the case at hand takes the form

$$x(t) = t^2 - 1, \quad y(t) = 2t, \quad z(t) = t^2 + 1. \tag{1.1}$$

Less known is the parametrization, with $\mathbb{Q}$-rational coefficients, of

$$x^3 + y^3 + z^3 = w^3 \tag{1.2}$$

worked out by N. Elkies:

$$\begin{array}{rcl} x(s,t) & = & -(s+r)t^2 + (s^2+2r^2)t - s^3 + rs^2 - 2r^2s - r^3 \\ y(s,t) & = & t^3 - (s+r)t^2 + (s^2+2r^2)t + rs^2 - 2r^2s + r^3 \\ z(s,t) & = & -t^3 + (s+r)t^2 - (s^2+2r^2)t + 2rs^2 - r^2s + 2r^3 \\ w(s,t) & = & (2r-s)t^2 + (s^2-r^2)t - s^3 + rs^2 - 2r^2s + 2r^3 \end{array}$$

Choosing $(s,t,r) \in \mathbb{Q}^3$ we obtain solutions of (1.2). However, even straightforward deformations of this equation, e.g.,

$$x^3 + y^3 + z^3 = 2w^3, \qquad 5x^3 + 9y^3 + 10z^3 = 12w^3, \tag{1.3}$$

lack such parametrizations. The first of these equations has infinitely many $\mathbb{Q}$-rational solutions, but the second has no nontrivial solutions at all, even though it has nontrivial solutions modulo p, for all primes

Date: July 4, 2018.

p (see, e.g., [Man86, Section VI] and [CG66])! The obstruction to parametrizing lies in the Galois cohomology group

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{Pic}(\bar{X})), \tag{1.4}$$

where $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$, and $\mathrm{Pic}(\bar{X})$ is the geometric Picard group of X, which is easily computable, knowing the Galois action on the 27 lines on the cubic surface.

The study of rationality revolves naturally around the following issues:

- Introduce *computable* obstructions to rationality and provide supporting examples;
- Investigate rationality for important classes of varieties, such as algebraic surfaces, or low-degree hypersurfaces;
- Study the behavior of rationality under basic geometric operations: fibrations, deformation and specialization.

There are several excellent surveys on these fascinating subjects, e.g., [MT86], [Bea16], [Voi16], [dF14], [Pir16], [Has06], [Pey16], [AB17], [CT18b], [CT18a]. Here we focus on the last of these issues, in particular, on specialization. Our presentation is inspired by our recent joint work [HKT16b], [HT16], [HPT16b], [HPT16a], [KT17a]; we hope that the point of view presented here might be of independent interest.

Acknowledgments: I am very grateful to Fedor Bogomolov, Ivan Cheltsov, Brendan Hassett, Andrew Kresch, and Alena Pirutka for discussions on related topics. The author was partially supported by NSF grant 1601912 and by the Laboratory of Mirror Symmetry NRU HSE, RF grant 14.641.31.0001.

2. Generalities

A smooth projective algebraic variety X over a field k of characteristic zero is called *rationally connected* if for any pair of points $x_0, x_\infty \in X(k)$ there exists a k-morphism

$$f : \mathbb{P}^1 \to X, \quad f(0) = x_0, \; f(\infty) = x_\infty,$$

i.e., any pair of points in $X(k)$ can be joined by an irreducible rational curve in X, defined over k. This property is a birational invariant. Moreover, it behaves well in families: limits of rationally connected varieties are rationally chain-connected, and if the limit has mild singularities there exist irreducible connecting rational curves.

An important class of geometrically (i.e., over an algebraic closure $\bar{k}$) rationally connected varieties is the class of (smooth) *Fano varieties*, i.e., those with ample anticanonical class $-K_X$. Examples of Fano varieties are:

- The projective line $\mathbb{P}^1$ and its nonsplit form, a conic without k-points.
- Del Pezzo surfaces: forms of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and blowups of $\mathbb{P}^2$ in up to 8 points in general position.
- Fano threefolds: cubics, quartics, ...

There are finitely many families of Fano varieties in every dimension. These constitute basic building blocks in the Minimal Model Program, and one expects that every rationally-connected variety admits, birationally, a Mori fiber space structure, i.e., a fibration with generic fiber a Fano variety. For example, in dimension 2, we are led to consider del Pezzo surfaces or conic bundles over $\mathbb{P}^1$.

An algebraic variety X over a field k is called

- rational: if X is birational to $\mathbb{P}^n$, over k, for some n;
- stably rational: if $X \times \mathbb{P}^n$ is rational, for some n;
- unirational: if X is dominated by $\mathbb{P}^n$, for some n.

We have logical implications:

Rational $\Rightarrow$ Stably rational $\Rightarrow$ Unirational $\Rightarrow$ Rationally connected.

While there are examples of stably rational but nonrational, unirational, but not stably rational varieties, very little is known about unirationality; finding examples of rationally connected but not unirational varieties over the complex numbers remains an elusive goal.

The above properties of X depend on the ground field k but are preserved under field extensions. Conics $C \subset \mathbb{P}^2$ are rational over k if and only if $C(k) \neq \emptyset$; projecting from a point $c \in C(k)$ establishes an isomorphism $C \simeq \mathbb{P}^1$, which leads to algebraic formulas as in (1.1). By a theorem of Lüroth, for curves, the above notions of rationality coincide. A major achievement of classical Italian geometry was the proof of the corresponding result for algebraic surfaces over $\mathbb{C}$, essentially, via classification. In particular, cubic surfaces in (1.3) are rational over $\mathbb{C}$; however, the first is unirational but not rational over $\mathbb{Q}$, as it has nonvanishing Galois cohomology group (5), and the second is not even unirational over $\mathbb{Q}$, as it lacks $\mathbb{Q}$-rational points (see, e.g., [CTKS87]).

After unsuccessful attempts by Fano and others to provide examples of unirational but nonrational complex threefolds, major developments occurred in 1971-72, with the introduction of:

- Birational rigidity (Iskovskikh–Manin [IM71]),
- Intermediate Jacobians (Clemens–Griffiths [CG72]),
- Brauer groups (Artin–Mumford [AM72]).

Each of these approaches triggered important advances in algebraic geometry: in-depth study of singularities and minimal models, abelian varieties, and unramified cohomology.

We recall the basic definitions: Let X be a rationally connected variety. It is *birationally rigid* if its Mori fiber space structure is unique, and *super-rigid* if its group of birational automorphisms $\mathrm{Bir}(X)$ coincides with the automorphisms $\mathrm{Aut}(X)$. Such varieties cannot be rational since the Cremona group $\mathrm{Bir}(\mathbb{P}^n)$ is enormous, for $n \geq 2$. The paper [IM71] established birational super-rigidity of smooth quartic threefolds.

If X is a complex projective rationally connected threefold, its *intermediate Jacobian* is the principally polarized abelian variety

$$\mathrm{IJ}(X) := \frac{H^{2,1}(X, \mathbb{C})^\vee}{H_3(X, \mathbb{Z})}.$$

For rational X, $\mathrm{IJ}(X)$ is isomorphic to a product of Jacobians of curves. The paper [CG72] proved that this fails for smooth cubic threefolds.

And finally, for smooth projective X, the nontriviality of the *Brauer group*

$$\mathrm{Br}(X) := H^2_{et}(X, \mathbb{G}_m)$$

is an obstruction to stable rationality, and not only rationality. Note that the second invariant does not distinguish between nonrational and stably rational varieties [BCTSSD85a]. No examples of birationally rigid but stably rational varieties are known; it seems plausible that smooth projective birationally rigid threefolds are not stably rational.

Each of these invariants is essentially preserved under deformations:

- No examples of smooth families of smooth projective varieties are known where some fibers are birationally rigid and some are not; for example, *every* smooth hypersurface of degree n in $\mathbb{P}^n$, with $n \geq 4$, is birationally rigid [dF13, dF16]. Small deformations of birationally rigid threefolds are also birationally rigid [dFH12].

- A small deformation of a principally polarized abelian variety which is not isomorphic to a product of Jacobians of curves remains so.
- For smooth varieties X over $\mathbb{C}$, the Brauer group has an interpretation as torsion in $H^3(X, \mathbb{Z})$, a topological invariant.

The Specialization methods discussed in the next section provided new powerful obstructions to stable rationality and allowed to construct examples of smooth families of projective varieties of dimension ≥ 3 with varying rationality behavior of the fibers.

An important class of threefolds whose rationality properties are accessible, in principle, via all three methods is the class of standard conic bundles over smooth projective rational surfaces, e.g.,

$$\pi : X \to \mathbb{P}^2,$$

with discriminant $D \subset \mathbb{P}^2$ (see Section 4.1 for more details). Then

- If $\deg(D) \geq 12$ then X is birationally rigid [Sar82].
- If D is general of degree ≥ 6 then the intermediate Jacobian $\mathrm{IJ}(X)$ is not a product of Jacobians of curves; there exist X of this type which are stably rational [BCTSSD85b].
- If $\deg(D) \leq 8$ then X is unirational [Mel14].
- If D has at least two smooth components D_1, D_2 of genus ≥ 1 such that the discriminant double cover is unramified over $D_1 \cap D_2$ then the Brauer group $\mathrm{Br}(X)$ is nontrivial, and X is not stably rational.

We will see below that conic and higher-dimensional quadric bundles play a special role in the study of rationality.

3. Specialization

The *Specialization idea*, in the context of the theory of intermediate Jacobians of complex threefolds, was introduced by Clemens in [Cle75]. It allows to reduce proofs of nonrationality of some threefolds to the case of conic bundles. It can be summarized as follows:

Theorem 1 (Specialization method I, [Cle75], [Bea77])**.** *Let*

$$\phi : \mathcal{X} \to B$$

be a flat family of projective threefolds with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) *Singularities: X has at most rational double points.*
(O) *Obstruction: the intermediate Jacobian of a desingularization $\tilde{X}$ of X is not a product of Jacobians of curves.*

Then there exists a Zariski open subset $B^\circ \subseteq B$ such that for all $b' \in B^\circ$ the fiber $\mathcal{X}_{b'}$ is not rational.

This idea was developed and applied by Beauville [Bea77] to prove nonrationality of several classes of smooth Fano threefolds, e.g., quartic and sextic double solids. Beauville found mild degenerations of Fano threefolds, satisfying the conditions (S) and (O) above, and admitting conic bundle structures over $\mathbb{P}^2$. Intermediate Jacobians of standard conic bundles over rational surfaces are much easier to analyze than those of general Fano threefolds. We review this theory in Sections 4.1 and 4.3.

A novel version of the specialization idea emerged in the work of Voisin [Voi15] in the form of *integral decomposition of the diagonal*:

$$\Delta_X = [x \times X] + Z \in \mathrm{CH}^n(X \times X), \quad \text{resp.} \quad H^{2n}(X \times X, \mathbb{Z}), \tag{3.1}$$

where Δ_X is the class of the diagonal (either in the Chow group or cohomology), $\dim(X) = n$, and Z is a cycle supported on $X \times D$, $D \subsetneq X$. The failure of an integral decomposition of the diagonal is an obstruction to stable rationality.

Theorem 2 (Specialization method II, [Voi15]). *Let*

$$\phi : \mathcal{X} \to B$$

be a flat family of projective varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) *Singularities: X has at most rational double points.*
(O) *Obstruction: the desingularization $\tilde{X}$ of X does not admit an integral decomposition of the diagonal, i.e., fails* (3.1).

Then a very general fiber of ϕ does not admit an integral decomposition of the diagonal, and in particular, is not stably rational.

Here very general refers to the complement of a countable union of Zariski closed subsets of B.

Theorem 3 connected the study of (stable) rationality with developments in the theory of algebraic cycles, in particular, the work of Bloch and Srinivas [BS83]. It also showed the way to new, more powerful and more direct, specialization results for (stable) rationality, relying on the following notions:

CH_0: Universal CH_0-triviality [CTP16];
K_0: the *Grothendieck ring* [NS17];
Burn: the *Burnside ring* [KT17a].

We now describe and compare these results. The common setup is as follows: Let $\mathfrak{o}$ be a discrete valuation ring, k its residue field and K its field of fractions. Let

$$\phi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o})$$

be a faithfully flat proper morphism, with generic fiber X_K and special fiber X_k. The specialization results state that if the generic fiber X_K is

- universally CH_0-trivial,
- stably rational, or
- rational,

then so is the special fiber, provided it has *mild singularities*. One essential difference is that in the first case, the characteristic $\mathrm{char}(k)$ of k is arbitrary, while in the other two cases $\mathrm{char}(k) = 0$. On the other hand, the last two cases bypass the theory of algebraic cycles. Moreover, they suffice to essentially settle the stable rationality problem in dimension 3. The definition of *mild* is also different in the three cases.

CH_0: Let $\mathrm{CH}_0(X_k)$ be the abelian group generated by zero-dimensional subvarieties $x \in X$ (e.g., points $x \in X(k)$), modulo k-rational equivalence. Assuming $X(k) \neq \emptyset$, there is a surjective degree homomorphism

$$\mathrm{CH}_0(X_k) \to \mathbb{Z}.$$

Here are examples when this an isomorphism:

- X is a unirational variety over $k = \mathbb{C}$,
- X is a Kummer surface over $k = \bar{\mathbb{F}}_p$ [BT05]; however, this property fails if we replace $\bar{\mathbb{F}}_p$ by an uncountable field.

A projective X/k is called *universally* CH_0*-trivial* if for *all* field extensions k'/k we have

$$\mathrm{CH}_0(X_{k'}) \xrightarrow{\sim} \mathbb{Z}.$$

In practice, it suffices to consider $k' := k(X)$.

This property is a strengthening of rational connectedness: for rationally connected varieties over k we have $\mathrm{CH}_0(X_k) \simeq \mathbb{Z}$, but this may fail after passage to field extensions of k. In particular, unirational varieties are *not* necessarily universally CH_0-trivial. Smooth projective X/k with $\mathrm{Br}(X) \neq \mathrm{Br}(k)$, or more generally, with nontrivial higher unramified cohomology, are not universally CH_0-trivial. On the other hand, smooth k-rational varieties are universally CH_0-trivial.

There is a relative version of this notion: A projective morphism

$$\beta : \tilde{X} \to X$$

of k-varieties is *universally* CH_0*-trivial* if for all field extensions k'/k one has

$$\beta_* : \mathrm{CH}_0(\tilde{X}_{k'}) \xrightarrow{\sim} \mathrm{CH}_0(X_{k'}).$$

For example, let

$$\beta : \mathrm{Bl}_Z(X) \to X,$$

be the blowup of a smooth variety X in a smooth subvariety Z. Then β is universally CH_0-trivial. The following theorem reflects the fact that, just as rational connectedness, universal CH_0-triviality behaves well under specializations.

Theorem 3 (Specialization method III, [CTP16])**.** *Let*

$$\phi : \mathcal{X} \to B$$

be a flat family of projective varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) *Singularities: X admits a desingularization*

$$\beta : \tilde{X} \to X$$

such that the morphism β is universally CH_0*-trivial;*

(O) *The group* $\mathrm{CH}_0(\tilde{X})$ *is not universally trivial.*

Then a very general fiber of ϕ is not universally CH_0*-trivial and in particular not stably rational.*

Again, one would like to find appropriate degenerations to conic bundles, specifically to those with nontrivial Brauer groups, to conclude failure of stable rationality of a very general member of the family.

K_0: Let $\mathrm{K}_0(Var_k)$ be the Grothendieck ring of varieties over k; it is generated by isomorphisms classes of schemes of finite type over k modulo scissor relations, with the ring structure given by producs of varieties. Let $\mathbb{L}$ be the class of the affine line. A result of Larsen and Lunts [LL03] states that

$$\mathrm{K}_0(Var_k)/\mathbb{L}$$

is isomorphic to the free abelian group spanned by stable rationality classes of algebraic varieties. Nicaise and Shinder [NS17] proved that Hrushovski and Kazhdan's motivic volume gives rise to a specialization homomorphism

$$\mathrm{K}_0(Var_K)/\mathbb{L} \to \mathrm{K}_0(Var_k)/\mathbb{L}$$

of abelian groups, where K is the fraction field of a discrete valuation ring $\mathfrak{o}$ and k is its residue field, both of characteristic zero. This implies specialization of stable rationality, when the special fiber has mild singularities, e.g., rational double points.

Burn(k): Let Burn(k) be the free abelian group spanned by birationality classes of algebraic varieties. The main result of [KT17a] states that there is a specialization homomorphism

$$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

defined by a formula similar to the motivic volume formula. This implies specialization of birational types, and in particular, of rationality and stable rationality, when the special fiber has mild singularities.

Both results rely on the Weak Factorization Theorem (WF) for birational maps between smooth proper varieties [AKMW02]. Indeed, the birational type of the exceptional divisor of a blowup of a smooth subvariety Z in a smooth variety X is simply the type of $Z \times \mathbb{A}^{\mathrm{codim}(Z)-1}$; (WF) says that such blowups are the only steps connecting birational varieties.

Remark 4. An important theme in higher-dimensional birational geometry is the study of G-varieties, i.e., varieties X with actions of algebraic groups G. For example, even the classification of birational involutions of $\mathbb{P}^3$ is not fully understood [Pro13]; this can be translated to a problem about regular involutions on a rational threefold X. Such problems amount to finding explicit G-equivariant birational maps – very similar in spirit and technique to finding rational parametrizations.

A G-equivariant version

$$\mathrm{Burn}^G(K)$$

of the Burnside group might help to distinguish nonequivalent actions via specializations. The proof in [KT17a] goes through, provided one *trivializes* linear actions, i.e., identifies total spaces of G-vector bundles with total spaces of vector bundles with trivial G-action.

4. Applications

Here we work over $\mathbb{C}$. We discuss applications of the Specialization methods to the problem of stable rationality of rationally-connected threefolds and fourfolds.

4.1. **Conic bundles.** A standard conic bundle structure on a smooth projective threefold X is a fibration

$$\pi : X \to S$$

over a smooth projective rational surface S, with generic fiber a conic. In particular, the discriminant curve $D \subset S$ is nodal. Let $\tilde{D} \to D$ be the discriminant double cover defined by π; note that X is determined, birationally, by the data

$$(S, \tilde{D} \to D).$$

The intermediate Jacobian is a principally polarized abelian variety, which is isomorphic to the Prym variety

$$\mathrm{IJ}(X) \simeq \mathrm{Prym}(\tilde{D} \to D).$$

This fails to be a product of Jacobians of curves in many cases, e.g., when $S = \mathbb{P}^2$ and D is irreducible of degree ≥ 6, which is not hyperelliptic or trigonal or quasi-trigonal (see, e.g., [Sho83], [Bea77]). An extensive discussion of conjectures and results concerning rationality and birational rigidity of conic bundles over rational surfaces can be found in [Pro17].

The Specialization methods presented in Section 3 allowed decisive progress on stable rationality. They shifted the focus to the construction of *families* of conic bundles with mild degenerations (see [KT17b, Proposition 3.1]).

Theorem 5. [HKT16b] *Let $\mathcal{L}$ be a linear system of effective divisors on a smooth projective surface S with smooth and irreducible general member. Let $\mathcal{M}$ be an irreducible component of the moduli space of pairs $(D, \tilde{D} \to D)$, where $D \in \mathcal{L}$ is nodal and reduced and $\tilde{D} \to D$ is an étale cover of degree 2. Assume that $\mathcal{M}$ contains a cover, which is nontrivial over every irreducible component of a reducible curve with smooth* *irreducible components. Then a conic bundle over S corresponding to a very general point of $\mathcal{M}$ is not stably rational.*

For example, very general conic bundles over $\mathbb{P}^2$ with discriminant D of degree ≥ 6 are not stably rational. We now focus on the case when the base of the conic bundle is a smooth del Pezzo surface S of degree d. When $\mathsf{d} \leq 7$, the surface S itself admits conic fibrations over $\mathbb{P}^1$, where a *conic* is a smooth rational curve in S of anticanonical degree two. Let $D \subset S$ be a smooth curve in $|-2K_S|$, which has genus $\mathsf{d}+1$. Consider a conic bundle

$$X \to S$$

with discriminant D. Note that for each conic fibration $S \to \mathbb{P}^1$ we obtain a fibration

$$\pi : X \to \mathbb{P}^1$$

with general fiber a degree 4 del Pezzo surface, admitting a conic bundle. We have:

(1) X is not birationally rigid.
(2) The intermediate Jacobian $\mathrm{IJ}(X) \simeq \mathrm{Prym}(\tilde{D} \to D)$ has dimension d.
(3) When $\mathsf{d} = 1, 2$, or 3 the intermediate Jacobian $\mathrm{IJ}(X) \simeq \mathrm{J}(C)$ where C is a curve of genus $1, 2$ or 3 respectively.
(4) The Brauer group of X is trivial.
(5) For very general D, X is not stably rational [HKT16a].

4.2. **Del Pezzo fibrations.** Here, we consider del Pezzo fibrations

$$\pi : X \to \mathbb{P}^1,$$

where X is a smooth projective threefold, the relative Picard rank equals 1, and the general fiber is a del Pezzo surface of degree d. These are rational when $\mathsf{d} \geq 5$, but need not be rational when $\mathsf{d} = 4$. Let

$$h(X) := \deg(c_1(\omega_\pi^3))$$

be the *height* (measuring the degree of coefficients of the defining equations of X, considered as a variety over $k[t]$). We focus on $\mathsf{d} = 4$, following [HT16] and [HKT16a]. In this case, $h(X)$ is an even positive integer. The moduli space of X with fixed height ≥ 12 is irreducible.

Theorem 6. *Assume that $\mathsf{d} = 4$ and that X is general: the discriminant is square-free and the monodromy of the family is maximal, i.e., the full Weyl group $W(\mathsf{D}_5)$. Then*

- *If $h(X) \leq 8$ or $h(X) = 12$ then X is rational,*

- *If $h(X) = 10$ then X is not rational*
- *If $h(X) \geq 14$ then a general X in the corresponding family is not stably rational.*

The case of del Pezzo fibrations of degrees $\mathsf{d} = 3, 2$, and 1 has been addressed in [KO17].

4.3. **Fano threefolds.** In this section, we study smooth Fano threefolds. Their basic invariants are

- $\rho = \rho(X) = \mathrm{rk}(\mathrm{Pic}(X))$, the Picard rank,
- $\mathsf{d} = \mathsf{d}(X) = -K_X^3$, the degree of the anticanonical class,
- $r = r(X)$, the index, the largest positive integer such that $-K_X = rH$, for some $H \in \mathrm{Pic}(X)$.

We focus on nonrational minimal Fano threefolds (not blowups of other Fano threefolds). There are 6 families with $\rho = r = 1$, corresponding to

$$\mathsf{d} = 2, 4, 6, 8, 10, 14. \tag{4.1}$$

and 3 families with $\rho = 1$, $r = 2$, corresponding to

$$\mathsf{d} = 8 \cdot 1, 8 \cdot 2, 8 \cdot 3. \tag{4.2}$$

Those with $\rho \geq 2$ are all conic bundles, whose nonrationality follows from the analysis of their intermediate Jacobians. However, the discriminant curves in these cases are *not* general in the respective linear series, and Theorem 5 does not apply. Nevertheless, we proved

Theorem 7. [HT16] *Let X be a general smooth nonrational Fano threefold. Assume that X is not birational to a cubic threefold. Then X is not stably rational.*

The key finding was that families with $\rho = r = 1$ specialize to nodal threefolds which are birational to conic bundles, and to which Theorem 5 does apply. After that, it suffices to invoke the Specialization theorem 3. The required conic bundle structure results from a representation as a degree 4 del Pezzo fibration over $\mathbb{P}^1$; blowing up a general section of this fibration we obtain a cubic surface over $k(\mathbb{P}^1)$ with a line, thus a conic bundle.

Example 8. [HT16] Let $X \subset \mathbb{P}^4$ be a smooth quartic, i.e., $\rho = r = 1$ and $\mathsf{d} = 4$. Consider an intersection of two forms in $\mathbb{P}^1 \times \mathbb{P}^4$ of bi-degree $(1, 2)$

$$sP^1 + tQ_1 = sP_2 + tQ_2.$$

Projection onto $\mathbb{P}^1$ exhibits this threefold as an intersection of two quadrics in $\mathbb{P}^4$ over $k(\mathbb{P}^1)$, i.e., a del Pezzo surface. Projection to $\mathbb{P}^4$ gives a *nodal* quartic threefold

$$\{P_1Q_2 - Q_1P_2 = 0\} \subset \mathbb{P}^4$$

with 16 nodes:

$$\{P_1 = P_2 = Q_1 = Q_2 = 0\}.$$

This is the desired degeneration of X.

In general, the required number n of nodes is given in the table:

d	2	4	6	8	10
n	$32+4$	16	8	4	2

The last entry in the list (4.1), with $\mathsf{d} = 14$ is birational to a cubic threefold. By [CG72], smooth cubic threefolds are all irrational; determining whether or not they are stably rational remains an open problem.

Remark 9. Let X be a cubic threefold and $P \subset \mathbb{P}^4$ a plane such that

$$X \cap P = \ell_1 \cup \ell_2 \cup \ell_3,$$

a triangle of lines. The fibration obtained by projecting from P

$$X \dashrightarrow \mathbb{P}^1$$

is a cubic surface fibration with three fixed lines. Blowing down one of them, we obtain a fibration

$$\pi : \mathcal{S} \to \mathbb{P}^1$$

where the generic fiber is a quartic del Pezzo surface with a pair of conic bundle fibrations, summing to the anticanonical divisor. These have a chance of being stably rational but *not* through the relative universal torsor construction associated with the fibration π as in [BCTSSD85b]. Indeed, let $K = k(\mathbb{P}^1)$ and consider the Galois module $\mathrm{Pic}(\bar{S})$, where S is the generic fiber of $\mathcal{S}$. Generically, $\mathrm{H}^1(G_K, \mathrm{Pic}(\bar{S})) = 0$, however, this fails over finite extensions of K. Thus, the Néron-Severi torus, i.e., the torus dual to $\mathrm{Pic}(\bar{S})$, is not stably-rational.

4.4. **Rationality in families.** Let $\pi : \mathcal{X} \to B$ be a family of projective varieties over a field k and

$$\mathrm{Rat}(\pi) := \{b \in B \mid \mathcal{X}_b \text{ is } k\text{-rational}\} \subseteq B$$

be the locus of k-rational fibers. We are interested in the properties of $\mathrm{Rat}(\pi)$. Over number fields, this has attracted considerable attention, already for conic bundles over $\mathbb{P}^1$ and, more generally, Brauer-Severi fibrations. For example, it can be shown that the locus of $\mathbb{Q}$-rational fibers in a nonsplit conic bundle is infinite but thin (see, e.g., [Ser90]). Another example, that of diagonal cubic surfaces, was mentioned in the introduction.

Over algebraically closed k the situation is more difficult. The rationality behavior does not change in smooth families of curves or surfaces. In dimension three, we have:

Theorem 10. [dFF13] *Let $\pi : \mathcal{X} \to B$ be a family of rationally connected theefolds over an algebraically closed field k. Then* $\mathrm{Rat}(\pi)$ *is a countable union of closed subsets of B.*

Using the stable rationality construction from [BCTSSD85a] and the Specialization methods applied to a conic bundle over a degree $\mathsf{d} = 5$ del Pezzo surface S, with degeneration over $D \in |-2K_S|$ as in Section 4.1, we have:

Theorem 11. [HKT18] *There exists a smooth family of complex projective threefolds with stably rational and not stably rational fibers.*

In dimension 4, we were able to show that the set of rational fibers can be dense.

Theorem 12. [HPT16a] *Let $\pi : \mathcal{X} \to B$ be the universal family of bi-degree (2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$, over $\mathbb{C}$. Then* $\mathrm{Rat}(\pi)$ *is dense with respect to Euclidean topology while a very general fiber of π is not stably rational.*

The proof of this theorem has two parts: exhibiting a dense set of rational fibers and proving failure of stable rationality for very general fibers. For the first part, we project a bi-degree (2,2) hypersurface $X \subset \mathbb{P}^2 \times \mathbb{P}^3$ onto the first factor, it is then presented as a quadric surface bundle over $\mathbb{P}^2$. Birationally, we can view X as a quadric surface over the function field $K = \mathbb{C}(x, y)$; the quadric is rational if it has a rational point over K, which implies rationality of the total space X. Intuitively, if the variation of a family of quadric surfaces over a field is sufficiently large, a dense set of members of this family will have rational points. This intuition is justified by considering the geometry of Noether-Lefschetz loci in the moduli space and invoking the integral Hodge conjecture.

The second part is proved via the Specialization theorem 3. The key is to produce *one* degeneration, satisfying properties (S) and (O), e.g.,

$$X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[s:t:u:v]}$$

is given by

$$yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

where

$$F(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2yz - 2xz.$$

Property (S) was checked by hand; and the computation of the obstruction to stable rationality, the Brauer group of X, is a special case of an algorithm in [Pir16].

The *same* degeneration works if one considers the family of complete intersections of three quadrics in $\mathbb{P}^7$, the very general such Fano fourfold fails stable rationality and the rational ones are dense in moduli [HPT], or the family of quartic double fourfolds [HPT16b]. Thus, the emphasis shifted towards finding suitable *reference varieties*, with computable obstructions and mild singularities.

More general results, extending the argument to higher-dimensional quadric bundles, have been obtained in [Sch18], [Sch17].

5. Rationality over nonclosed fields

In this section, X is a smooth projective geometrically rational variety over a field k of characteristic zero. In dimension 2, these admit minimal models over k, which are twisted forms of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, del Pezzo surfaces, or conic bundles over $\mathbb{P}^1$. Here *minimal* refers to the property that X cannot be simplified, over k; for del Pezzo surfaces this means that there is no Galois orbit of disjoint exceptional curves. Minimal del Pezzo surfaces of degree ≤ 4 are not rational.

The stable rationality problem for del Pezzo surfaces is widely open. A necessary condition is

Condition (H1).

$$\mathrm{H}^1(G_{k'}, \mathrm{Pic}(\bar{X})) = 0, \quad \text{for all} \quad k'/k.$$

In the case of del Pezzo surfaces, the Galois group acts through the symmetries of the geometric Picard lattice, preserving the intersection form. Such actions are computationally accessible. In particular, condition (H1) holds for del Pezzo surfaces of degree ≥ 5, and indeed, existence of k-points implies rationality in these cases. On the other hand, there are examples of minimal del Pezzo surfaces of degrees $1 \leq \mathsf{d} \leq 4$, or conic bundles with at least 4 degenerate fibers, failing (H1) and thus stable rationality over k.

A classification of Galois actions satisfying (H1) can be found in [TY18]; surprisingly, the list of such actions is rather small. One of the outputs of this classification is the following strengthening of a theorem of Segre: A minimal cubic surface is not stably rational. There is only one case, mentioned in Theorem 13, when the necessary condition (H1) is known to be also sufficient; in this case, the Galois group acts through the symmetric group $\mathfrak{S}_3$.

Theorem 13. *Let X be a smooth del Pezzo surface with $X(k) \neq \emptyset$.*

- $\mathsf{d} \geq 5$*: X is k-rational.*
- $\mathsf{d} = 4$*: If X admits a conic bundle over k, and satisfies Condition* (H1)*, then X is stably rational over k.*
- $\mathsf{d} = 4, 3, 2$*: X is k-unirational.*
- $\mathsf{d} = 1$*: If X admits a conic bundle over k then X is k-unirational* [KM17]*.*

One way of viewing Theorem 13 is to note that, geometrically, del Pezzo surfaces of degree ≥ 6 are toric, and that all 2-dimensional algebraic tori are rational over their field of definition. Consider, for example, a del Pezzo surface of degree 6. The 6 exceptional curves are intrinsically defined, over the ground field. Their complement is a principal homogeneous space for a 2-dimensional torus, if there is a k-point, the complement is a torus, and necessarily k-rational. The case of degree 5 is interesting: this surface is, geometrically, isomorphic to $\overline{\mathcal{M}}_{0,5}$, the moduli space of 5 points on $\mathbb{P}^1$, which is also a quotient of the Grassmannian $\mathrm{Gr}(2,5)$ by a 4-dimensional torus, and the Galois action factors through $\mathfrak{S}_5$. In degree ≤ 4, the Galois actions are more complicated, the surfaces more twisted, and stable rationality largely unexplored.

What can we say about smooth, geometrically rational Fano threefolds X with $X(k) \neq \emptyset$? There do exist nonrational tori in dimension 3 (classified in [Kun87]) and nonrational forms of $\overline{\mathcal{M}}_{0,6}$ [FR18]. Standard rationality (and unirationality) constructions rely on the existence of lines, and the variety of lines is typically of general type, thus need not have k-rational points. It would be interesting to explore an analog of Theorem 13 in dimension 3.

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Courant Institute, New York University, New York, NY 10012, USA

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA
E-mail address: `tschinkel@cims.nyu.edu`