Geometry over nonclosed fields

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Abstract. I discuss some arithmetic aspects of higher-dimensional algebraic geometry. I focus on varieties with many rational points and on connections with classification theory and the minimal model program.

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1. Introduction

Let k be a field, X an algebraic variety over k and $X(k)$ the set of k-rational points on X. Broadly speaking, arithmetic geometry is concerned with the set $X(k)$, but usually with additional structure, coming from the Zariski topology on X or the Galois group of k. Questions of arithmetic nature arise even in classical algebraic geometry over $\mathbb{C}$ – a rational section of a fibration $X \to B$ is a rational point of the generic fiber, considered as a variety over the function field $\mathbb{C}(B)$ of the base. Of particular importance are "small" ground fields k, such as finite fields $\mathbb{F}_p$ or the rational numbers $\mathbb{Q}$, as they are closest to the theory of diophantine equations, with its wealth of challenging problems.

The geometric approach to diophantine equations is inspired by the analogy "numbers – functions" going back at least to Riemann, Dedekind, Kronecker, Weber, Hensel, Weil, and many others. More concretely, one expects close relationships between global geometric properties of X, considered as an algebraic variety over "large" fields such as $\mathbb{C}$, and arithmetic properties. Sometimes one may need to stabilize the arithmetic situation by passing to finite or even infinite extensions of k, or by excluding exceptional subsets.

This area of arithmetic and geometry is dominated by the following themes:

- Existence of rational points. In dimension one this includes the work of Mazur and Merel on torsion points on elliptic curves, conjectures of Birch and Swinnerton-Dyer, Wiles' work on Fermat's conjecture, the theorems of Gross-Zagier, Kolyvagin and their generalizations. In higher dimensions, the attention is on the Brauer-Manin obstruction to the Hasse principle and weak approximation, its uniqueness and effective computation (see [54]).

- Density of rational points in various topologies, e.g., Zariski density or density in analytic topologies (see [51], [39]).
- Heights. This covers questions of bounding heights of points on higher genus curves (effective Mordell), finding lower bounds for heights of rational and algebraic points, Manin's conjecture about asymptotics of rational points of bounded height and its refinements (see [13], [55]).

The field is evolving rapidly, opening new directions of research and raising many concrete and interesting questions.

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2. Classification schemes

In this section we work over $\mathbb{C}$. Smooth projective algebraic varieties X can be classified by their topological and algebraic invariants: fundamental group $\pi_1(X)$, Brauer group $\mathrm{Br}(X)$, or a property of the canonical line bundle K_X:

$$\kappa(X) := \limsup_{n\to\infty} \log(h^0(X, nK_X))/\log(n) \in \{-\infty, 0, 1, \ldots, \dim(X)\},$$

the Kodaira dimension. Alternatively, there are several geometric notions reflecting how close X is to a projective space $\mathbb{P}^n$: rationality, unirationality, uniruledness or rational connectedness, resp. chain-connectedness. Sometimes, X is realized by particularly simple equations, e.g., as a complete intersection in projective or weighted projective space. In this case, one could also classify by the degree $\deg(X)$ in this embedding. Sometimes, there is a distinguished polarization: (multiples of) the canonical, resp. the anticanonical line bundle; in the first case, $\kappa(X) = \dim(X)$, and one speaks of varieties of *general type*, and in the second, of *Fano* varieties. For historical reasons, Fano surfaces are called Del Pezzo surfaces. All other varieties are called varieties of *intermediate type*; an important subclass are varieties with trivial canonical class, e.g., abelian varieties and Calabi-Yau varieties.

The only Fano variety in dimension 1 is $\mathbb{P}^1$. Curves of general type are curves of genus ≥ 2. Smooth Del Pezzo surfaces divide into following types: $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, and blowups of $\mathbb{P}^2$ in ≤ 8 points in general position. In each dimension, there are only finitely many families of Fano varieties [49], [19].

The notions above are intertwined in many ways. For example, a smooth hypersurface $X \subset \mathbb{P}^n$ is Fano if $\deg(X) < n+1$, and of general type if $\deg(X) > n+1$, so that we generally think of Fano varieties as being varieties of *small* degree, and geometrically not "too far" from $\mathbb{P}^n$. A Fano variety is rationally connected.

If X is rationally connected, then it has trivial $\pi_1(X)$. If X is smooth and rational then $\mathrm{Br}(X)$ is trivial. A rationally connected X is rational, provided $\dim(X) \leq 2$, but not necessarily if $\dim(X) = 3$, etc. There are conjectured criteria linking algebraic and geometric properties, e.g., X is uniruled if and only if $\kappa(X) = -\infty$. A fundamental result is the following:

Theorem 2.1 (Graber-Harris-Starr [36]). *Let $\pi : X \to B$ be a morphism from a smooth projective variety to a smooth projective curve. Assume that the generic fiber is a rationally connected variety. Then π has a section.*

For applications, one needs to better understand the class of varieties of intermediate type. We know that every X admits an *MRC quotient*, a rational map $\pi_r : X \to Y = R(X)$ with rationally connected fibers and base not uniruled [49], [19], [36]. If Y is not a point, this is followed by another fibration $Y \to Z$, with $\dim(Z) = \kappa(Y)$ and generic fiber of Kodaira dimension zero. However, Z is not necessarily of general type, and the construction may have to be iterated.

An alternative classification scheme has been proposed by Campana. A variety X is *special* if there are no (birational) dominant maps $\pi : X \to Y$, where Y is a smooth variety of $\dim(Y) \geq 1$ and of *orbi*-general type, with *orbi*-canonical class computed taking into account *multiplicities* of singular fibers of π. The class of special varieties includes rationally connected varieties and varieties of Kodaira dimension zero. Every X admits (birationally) a fibration $\pi_c : X \to Y = C(X)$ onto its *core* (or *le cœur*), with *special* generic fiber and base of orbi-general type [20]. For examples of simply connected X which don't admit birational maps onto varieties of general type of dim ≥ 1, but have a nontrivial core $C(X)$, see [10].

3. Potential density

One says that rational points on an algebraic variety X over a field k are *potentially dense* if there exists a finite extension k'/k such that $X(k')$ is Zariski dense in X. This birational property holds for projective spaces and abelian varieties and thus for varieties dominated by these, e.g., unirational varieties or Kummer varieties. It is preserved under étale covers of proper varieties. An important problem is to find necessary and sufficient geometric conditions for potential density of rational points. In this direction, we have the following generalization of Mordell's conjecture to higher dimensions.

Conjecture 3.1 (Bombieri-Lang). *Let X be a variety of general type over a number field k. Then $X(k)$ is contained in a proper subvariety, i.e., rational points on X are not potentially dense.*

The outstanding result is Faltings' proof of this conjecture for subvarieties of abelian varieties [32], and in particular, curves of genus ≥ 2. Conjecture 3.1 has surprising consequences for *uniform* bounds for the number of rational points on curves of higher genus: it implies that there is a constant $c(k, \mathsf{g})$ such that for every curve C of genus $\mathsf{g} \geq 2$ over k one has $\#C(k) \leq c(k, \mathsf{g})$ [22].

In the opposite case of Fano varieties it is expected that their arithmetic properties are not too far from those of the projective space:

Conjecture 3.2 ([38]). *Let X be a Fano variety over a number field k. Then rational points on X are potentially dense.*

Conjecture 3.2 holds in dimension ≤ 2, for smooth quartic hypersurfaces of dimension ≥ 3 and for smooth hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$ [38], [8]. There remains only one family of smooth Fano threefolds for which this question is open:

Problem 3.3. Let $X \to \mathbb{P}^3$ be a double cover ramified in a smooth surface S of degree 6. Show that rational points on X are potentially dense.

For singular S this has been resolved in [25].

The above conjectures are subsumed in

Conjecture 3.4 (Campana [20]). *Let X be a smooth projective variety over a number field k. Rational points on X are potentially dense if and only if X is special.*

An interesting subclass of special varieties to consider is the class of Calabi-Yau varieties, e.g., K3 surfaces and their higher-dimensional analogs – holomorphic symplectic varieties. Potential density holds for Enriques surfaces, elliptic K3 surfaces and K3 surfaces with infinite automorphism groups [9]; symmetric products of arbitrary K3 surfaces have been treated in [40]. Potential density holds *a posteriori* for varieties dominated by these.

Problem 3.5. Let X be a K3 surface over a number field, with geometric Picard number one. Show that rational points on X are potentially dense.

It would also be worthwhile to find nontrivial examples of three-dimensional Calabi-Yau varieties over number fields with dense sets of rational points.

The proofs of potential density rely on either automorphisms or on fibration structures, with fibers abelian varieties. It would be interesting to involve endomorphisms and to study orbits of the corresponding dynamical systems, in the spirit of [21]. Examples of quite nontrivial endomorphisms on certain holomorphic symplectic fourfolds are given in [67].

4. Points of bounded height

Assuming that rational points are Zariski dense one may seek some quantitative understanding of their distribution. A natural approach to this proceeds via the theory of heights. Let X be a projective algebraic variety over a number field k and $\mathcal{L} = (L, \|\cdot\|)$ an adelically metrized very ample line bundle on X (i.e., L is equipped with a family of v-adic norms, for each place v of k, which at almost all places are induced from a fixed global integral model, see [55]). Let

$$H_{\mathcal{L}} : X(k) \to \mathbb{R}_{>0}$$

be the associated height function, $X^\circ \subset X$ a subvariety of X and

$$\mathcal{N}(X^\circ, \mathcal{L}, B) := \#\{x \in X^\circ(k) \mid H_{\mathcal{L}}(x) \le B\} < \infty, \tag{1}$$

the counting function. The goal is to investigate its asymptotic behavior as $B \to \infty$, in terms of the pair $(X^\circ, \mathcal{L})$.

For example, let $X \subset \mathbb{P}^n$ be a hypersurface given by $f(\mathbf{x}) = 0$, where f is a homogeneous form in the variables $\mathbf{x} = (x_0, ..., x_n)$ of degree d with coefficients in $\mathbb{Z}$. It is Fano for $n \ge d$. A counting function over $\mathbb{Q}$ is defined as

$$\mathcal{N}(X, \mathcal{O}(1), B) := \#\{\mathbf{x} \in (\mathbb{Z}^{n+1}_{\mathrm{prim}} \setminus 0)/\pm, f(\mathbf{x}) = 0, \max_j |x_j| \le B\},$$

where $\mathbb{Z}^{n+1}_{\mathrm{prim}}$ is the set of primitive vectors. The counting problem is to understand the asymptotic of this function as $B \to \infty$. The classical circle method solves this problem when the hypersurface X is smooth, the number of variables $n + 1 \gg 2^d$ and if there exist solutions to $f(\mathbf{x}) = 0$ in all completions of $\mathbb{Q}$. Strong uniform bounds for points on hypersurfaces were recently established in [18], [46], [60]; these papers use in a crucial way the case of curves considered in [58], [14].

Problem 4.1. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \le n$, over $\mathbb{Q}$. Show that there exists a Zariski open subset $X^\circ \subset X$ such that

$$\mathcal{N}(X^\circ, \mathcal{O}(1), B) = O(B^{n+1-d+\epsilon}),$$

for all $\epsilon > 0$. This is open already for $d = n = 3$.

In general, it is also difficult to produce rational points, even numerically. Interesting examples of varieties with *a priori* dense sets of rational points arise from group actions. More precisely, let G/k be a linear algebraic group, e.g., the Heisenberg group, the additive group $\mathbb{G}_a^d$ or the algebraic torus $\mathbb{G}_m^d$. Let

$$\rho : \mathsf{G} \to \mathrm{PGL}_{n+1} \tag{2}$$

be a representation over k. Fixing a point $\mathbf{x} \in \mathbb{P}^n(k)$, we can consider the *flow* $\rho(\mathsf{G}) \cdot \mathbf{x}$ and count

$$\#\{\gamma \in \mathsf{G}(k) \,|\, H_{\mathcal{O}(1)}(\rho(\gamma) \cdot x) \le B\},$$

respectively, k-points in G/H, when the stabilizer H is nontrivial.

The asymptotic will depend on the group, the representation, the initial point $\mathbf{x}$, and the choice of the height. The difficulty is that already for $\mathsf{G} = \mathbb{G}_a^2$ there is no reasonable classification of possible representations ρ. This necessitates a change of language, from representation-theoretic to algebro-geometric: Let X be the Zariski closure of $\rho(\mathsf{G}) \cdot \mathbf{x} \subset \mathbb{P}^n$. Then X is

- an equivariant compactification of $X^\circ := \mathsf{G}/\mathsf{H}$,
- with a G-linearized very ample line bundle L,
- which is equipped with an adelic metrization $\mathcal{L} = (L, \|\cdot\|)$.

Then the counting problem is as in (1). Using equivariant resolution of singularities we can now reduce the counting problem to the case when X is smooth, and the boundary $X \setminus X^\circ$ is a divisor with normal crossings.

Over number fields, in available examples arising from group actions, such as flag varieties [33], toric varieties [4], horospherical varieties [66], equivariant compactification of $\mathbb{G}_a^n$ [23], De Concini-Procesi equivariant compactifications of semi-simple group of adjoint type [64], [34], bi-equivariant compactifications of unipotent groups [63], the *algebraic-geometric* picture is as follows:

- the Picard group Pic(X) is a torsion free $\mathbb{Z}$-module,
- the (closed) cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic(X)}\otimes_{\mathbb{R}}$ is finitely generated,
- $-K_X$ is contained in the interior of $\Lambda_{\mathrm{eff}}(X)$,

The arithmetic picture, reflected in asymptotic formulas for the counting function (1), is described in terms of the geometric invariants as follows:

$$\mathcal{N}(X^\circ(k), \mathcal{L}, B) = c(\mathcal{L}) B^{a(L)} \log(B)^{b(L)-1}(1 + o(1)), \ B \to \infty, \tag{3}$$

where

- $a(L) := \inf\{a \,|\, aL + K_X \in \Lambda_{\mathrm{eff}}(X)\}$,
- $b(L)$ is the maximal codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing the class $a(L)L + K_X$.

In particular, $a(-K_X) = 1$ and $b(-K_X)$ is the rank of the Picard group of X, over k. The constant $c(\mathcal{L})$ depends on the choice of an adelic metrization, which determines the height. It was defined for $L = -K_X$ in [52] and in general in [5]:

$$c(-\mathcal{K}_X) := \alpha(X)\beta(X)\tau(\mathcal{K}_X) \tag{4}$$

where

- $\alpha(X)$ is the volume of certain polytope (intersection of the dual cone to $\Lambda_{\mathrm{eff}}(X)$ with the affine hyperplane $\langle -K_X, \cdot\rangle = 1$),
- $\beta(X) = |\mathrm{Br}(X)/\mathrm{Br}(k)|$ is the order of the nontrivial part of the Brauer group,
- $\tau(-\mathcal{K}_X)$ is a Tamagawa type number.

For general polarizations, $c(\mathcal{L}) = \sum'_{y \in Y(k)} c(\mathcal{L}_y)$, where $X \to Y$ is a certain fibration arising in Fujita's version of the minimal model program; the summation is over a possibly infinite subset of the set of rational points on the base and $c(\mathcal{L}_y)$ are constants similar to (4).

Some of the above results have been extended to function fields $\mathbb{F}_q(B)$, where B is a curve over a finite field $\mathbb{F}_q$ [57], [16]. Over these fields, the constant $c(-\mathcal{K}_X)$ differs by an additional integral factor, $\neq 1$ already for some toric varieties [16].

Tamagawa numbers occurring here are natural generalizations of those in the theory of algebraic groups. An adelic metrization of K_X gives rise to v-adic measures ω_v on v-adic analytic manifolds $X(k_v)$. A regularized global measure $\omega_{\mathcal{K}}$ on the adeles $X(\mathbb{A}_k)$ is given by

$$\omega_{\mathcal{K}} := \mathrm{L}^*_{\mathsf{S}}(1, \mathrm{Pic}(X))|\mathrm{disc}(k)|^{-\dim(X)/2} \prod_v \lambda_v^{-1}\omega_v, \tag{5}$$

where S is a finite set of valuations of k, including the archimedean ones, $\mathrm{disc}(k)$ is the discriminant of k, the regularizing factors

$$\lambda_v := \begin{cases} \mathrm{L}_v(1, \mathrm{Pic}(X)) & v \notin \mathsf{S}, \\ 1 & v \in \mathsf{S} \end{cases}$$

and

$$\mathrm{L}^*_{\mathsf{S}}(1, \mathrm{Pic}(X)) := \lim_{s\to 1}(s-1)^r \mathrm{L}_{\mathsf{S}}(s, \mathrm{Pic}(X))$$

is the leading coefficient at the pole of the partial Artin L-function associated to the Galois representation on the module $\mathrm{Pic}(\bar{X})$. Then

$$\tau(-\mathcal{K}_X) := \int_{\overline{X(k)}\subset X(\mathbb{A}_k)} \omega_{\mathcal{K}}, \tag{6}$$

an integral over the closure of rational points in the adeles, in the direct product topology.

The proofs of these asymptotics use a combination of techniques from arithmetic geometry and analysis. A useful tool is the *height zeta function* defined by the series

$$\mathcal{Z}(X^\circ, \mathcal{L}, s) = \sum_{x\in X^\circ(k)} H_{\mathcal{L}}(x)^{-s}.$$

Tauberian theorems relate the asymptotic behavior of the counting function to analytic properties of the height zeta function. Analytic properties of $\mathcal{Z}$ are investigated via harmonic analysis on adelic groups or ergodic theory.

Problem 4.2. Prove Formula (3) for general equivariant compactifications of unipotent and solvable groups.

Problem 4.3. Prove Formula (3) for general spherical varieties over number fields.

An alternative approach to asymptotics uses universal torsors. The essence of the method consists in the lifting of the counting problem for rational points on X to a counting problem for integral points on an auxiliary variety of higher dimension. This method has been used to reprove asymptotic results on toric varieties [60]. It has also been successfully applied to nonhomogeneous varieties, such as $\bar{\mathcal{M}}_{0,5}$ [17] and certain singular cubic surfaces [47], [26].

In the simplest case of a projective space, this can be explained by the familiar

$$(\mathbb{Z}^{n+1}_{\mathrm{prim}} \setminus 0)/\pm \xrightarrow{\mathbb{G}_m} \mathbb{P}^n(\mathbb{Q}),$$

certain integral points on the torsor $\mathbb{A}^{n+1} \setminus 0$, modulo units, are in bijection with rational points on the projective space. In general, the study of universal torsors leads to interesting algebraic problems, involving ideas from geometric invariant theory and toric geometry [2], [59], [42].

Problem 4.4. Compute equations of universal torsors for singular Del Pezzo surfaces in degrees $1, \ldots, 4$.

The initial conjecture of Batyrev-Manin was that Formula (3) should hold for *all* Fano varieties, after enlarging the ground field, and restricting to appropriate Zariski open subsets X°. It is necessary to allow finite field extensions, since a Fano variety may not have any rational points over the ground field (e.g., a nonsplit conic). The restriction to Zariski open subsets is also necessary since the variety X may contain *accumulating* subvarieties, and the asymptotic of rational points on them can violate the conjecture (e.g, lines on a cubic surface). The Batyrev-Manin conjecture had to be adjusted, already in dimension 3, after the realization that certain fibrations may lead to differing $b(L)$ [3], [5]. The results mentioned above are convincing evidence, that the Batyrev-Manin conjecture and its refinement by Peyre should hold for Del Pezzo surfaces and for equivariant compactifications of all linear algebraic groups and their homogeneous spaces. A first step would be

Problem 4.5. Prove Formula (3) for singular Del Pezzo surfaces whose universal torsor is a hypersurface.

5. Integral points

Let k be a number field, X a smooth projective algebraic variety over k and $D \subset X$ a divisor with strict normal crossings. In this log-geometric setup, it is the ampleness of $(K_X + D)$, resp. $-(K_X + D)$, which characterizes the opposites in the classification picture. We need a theory of log-uniruledness, log-rational connectivity, and classification results and notions comparable to those in Section 2. e.g., log-special varieties etc. Like for rational points, one is interested in the distribution of integral points in Zariski topology and with respect to heights.

Choose models $\mathcal{X}, \mathcal{D}$ of X, D over the ring of integers $\mathfrak{o}_k$ of k. A rational point $x \in X(k)$ gives rise to a section $\pi_x : \mathrm{Spec}(\mathfrak{o}_k) \to \mathcal{X}$ of the structure morphism. Choose a finite set of places S of $\mathfrak{o}_k$. A $(\mathcal{D}, \mathsf{S})$-integral point is a section π_x such that for all $v \notin \mathsf{S}$ one has

$$\pi_{x,v} \cap \mathcal{D}_v = \emptyset,$$

i.e., π_x avoids $\mathcal{D}$ over $\mathrm{Spec}(\mathfrak{o}_k) \setminus \mathsf{S}$. We say that D-integral points on X are potentially dense, if there exists a finite extension k'/k, a finite set of places S' of $\mathfrak{o}_{k'}$, models $\mathcal{X}', \mathcal{D}'$ of X, D over S' such that $(\mathcal{D}', \mathsf{S}')$-integral points in $X(k')$ are Zariski dense.

Conjecture 5.1 (Vojta). *Assume that $K_X + D$ is ample. Then $(\mathcal{D}, \mathsf{S})$-integral points are contained in a proper subvariety, i.e., integral points are not potentially dense.*

In analogy with Conjecture 3.2 we can formulate

Conjecture 5.2. *Assume that* $-(K_X + D)$ *is ample. Then* D*-integral points on* X *are potentially dense.*

An instance of what is expected in intermediate cases is the following "puncturing" conjecture, which could serve as a guiding principle:

Conjecture 5.3 ([41]). *Let* (X, Z) *be a pair consisting of a smooth projective variety* X *and a smooth irreducible subvariety* $Z \subset X$ *of codimension* ≥ 2*, defined over a number field* k*. Let* $\tilde{X} := \mathrm{Bl}_Z(X)$ *be the blowup of* X *with center in* Z*, and* D *the exceptional divisor. Assume that rational points on* X *are potentially dense. Then* Z*-integral points on* $\tilde{X}$ *are potentially dense.*

Problem 5.4. Let X be an abelian variety of dimension ≥ 2 and Z a point. Prove potential density of D-integral points on the blowup $\tilde{X}$, where D is the exceptional divisor.

This holds if X is a product of at least two abelian varieties [41]. For numerical evidence in dimension 2, see [48].

Problem 5.5. Let X be a surface and $D \subset X$ a reduced effective Weil divisor such that the pair (X, D) has log-terminal singularities and $(K_X + D)$ is trivial. Show that D-integral points on X are potentially dense.

A special case, for $D = \emptyset$ and X smooth, is stated in Problem 3.5. Another open case arises for X a smooth Del Pezzo surface, and D a singular anticanonical curve.

When $X = \mathbb{P}^2$ this problem been treated in [65] and [6]. Cubic surfaces X and smooth D have been considered in [7], this has been extended to smooth Del Pezzo surfaces X and smooth D in [41].

Assume now that $-(K_X + D)$ is ample and that $(\mathcal{D}, \mathsf{S})$-integral points are Zariski dense. Like in the case of rational points, group actions give a rich supply of quasi-projective projective varieties with many integral points. For example, replacing the projective representation ρ in (2) by a representation of an algebraic group G into GL_n and fixing $\mathbb{Z}$-structures leads to the study of points in $\mathsf{G}(\mathbb{Z})$, or integral points on the corresponding homogeneous spaces [27], [29], [30], [15].

For a Zariski open $X^\circ \subset X$, let

$$\mathcal{N}(X^\circ, -(\mathcal{K}_X + \mathcal{D}), B) := \#\{x \mid H_{-(\mathcal{K}_X+\mathcal{D})}(x) \leq B\}$$

be the counting function on the set of $(\mathcal{D}, \mathsf{S})$-integral points in $X^\circ(k)$.

Our goal is a geometric interpretation of asymptotic formulas, similar to the one in Section 4 (see [24] for more details). Available asymptotics, proved via ergodic theory or harmonic analysis, are of the shape

$$\mathcal{N}(X^\circ, -(\mathcal{K}_X + \mathcal{D}), B) \sim c_{\mathsf{S}} B \log(B)^{b_S - 1}, \tag{7}$$

where

- $b_{\mathsf{S}} = \mathrm{rk}\ \mathrm{Pic}(X \setminus D) + \sum_{v \in \mathsf{S} \cup \mathsf{S}_\infty} r_v$;
- $r_v = \dim \mathrm{Cl}(\mathcal{D}_v)$, the dimension of the Clemens polytope associated to the set of boundary components of the reduction of $\mathcal{D}$ modulo v (vertices correspond to irreducible components, faces of codimension one to pairs of intersecting components, etc.);
- c_{S} is a constant similar to the one in (3), it involves a Tamagawa volume of the adeles outside S of $X \setminus D$ as in (6), with respect to the restriction of the Tamagawa measure from (5), and for each $v \in \mathsf{S} \cup \mathsf{S}_\infty$ and $\sigma \in \mathrm{Cl}_{\max}(\mathcal{D}_\nu)$, the Tamagawa volume of the closed subvariety $Z_{\sigma,v} \subset X(k_v)$, the intersection of corresponding components from σ. Here $\mathrm{Cl}_{\max}(\mathcal{D}_\nu)$ is the set of faces of the Clemens polytope of maximal dimension, and the Tamagawa measure on $Z_{\sigma,v}$ is obtained by adjunction.

Problem 5.6. Prove (7) for $X = \mathbb{P}^n$, $D \subset X$ a smooth hypersurface of low degree, defined by a form $f \in \mathbb{Z}[x_0, \ldots, x_n]$, and S a finite set of primes of good reduction for D.

6. Arithmetic over function fields of curves

Here we work over $k = \mathbb{C}(B)$, where B is a smooth projective curve. Let X be a smooth projective variety over k and $\pi : \mathcal{X} \to B$ a model of X over B. Points in $X(k)$ correspond to sections of π, i.e., to certain curves in $\mathcal{X}$. A reformulation of Theorem 2.1 is

Theorem 6.1 ([36]). *Let X be a rationally connected variety over $k = \mathbb{C}(B)$. Then $X(k) \neq \emptyset$.*

One corollary of Theorem 2.1 is the proof of potential density, Conjecture 3.2, for rationally connected varieties over $k = \mathbb{C}(B)$. More precisely, after choosing finitely many smooth fibers and a point in each of these fibers one can find a section passing through these points [50]. This can be strengthened: after choosing finite *jets* (reductions of local analytic sections) in finitely many smooth fibers one can find a section which reduces to these jets [43]. It would be important to extend this property, called *weak approximation*, to singular fibers. Careful analysis of desingularizations of compound du Val singularities should yield a solution to

Problem 6.2. Prove weak approxiation for cubic surfaces over function fields of curves.

Another step beyond what can be currently proved over number fields, are examples of K3 surfaces, and more generally, Calabi-Yau varieties, of geometric Picard number one with Zariski dense sets of rational points in [45].

Problem 6.3. Prove potential density of rational points for all K3 surfaces over $k = \mathbb{C}(B)$.

There are analogous questions for log-Fano varieties. For example,

Problem 6.4. Prove potential density of integral points on log-Fano varieties (Conjecture 5.2) over function fields of curves.

One of the main advantages of the field $k = \mathbb{C}(B)$ is that curves often deform. This opens the door for a systematic application of deformation theory, the theory of moduli spaces of stable curves and maps etc. The proofs of properties like potential density or weak approximation proceed by finding and smoothing special chains of rational curves (*combs* and *combs with broken teeth*).

An alternative approach to potential density, based on endomorphisms from [67], leads to examples of holomorphic symplectic fourfolds X over $\mathbb{C}(x)$ with geometric Picard number one and dense rational points [1].

7. Geometry over finite fields

In some aspects, finite fields are similar to function fields of curves. For example, an analog of Theorem 6.1 is:

Theorem 7.1 ([31]). *Let X be a smooth projective rationally connected variety over a finite field $k = \mathbb{F}_q$. Then $X(k) \neq \emptyset$.*

On the other hand, the classification schemes as outlined in Section 2 have to be adjusted. In particular, the relation between the ampleness of the canonical, resp. anticanonical, line bundle and rational connectedness is much less clear. For example, in positive characteristic, there exist unirational, and thus rationally connected, varieties of general type. But over finite fields $k = \mathbb{F}_q$, and their closures $\bar{k}$, there are also examples of surfaces X of general type which are not uniruled, have a nontrivial Brauer group, nontrivial fundamental group, and still have the property that on some dense Zariski open subset $X^\circ \subset X$, every finite set of $\bar{k}$-points lies a geometrically irreducible rational curve in X, defined over k. Moreover, every Kummer K3 surface over a finite field has this rather strong rational connectedness property [11], [12].

Problem 7.2. Let X be an elliptic K3 surface over a sufficiently large finite field. Show that X is rationally connected (in the above sense).

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