PERVERSE SHEAVES OF CATEGORIES AND NON-RATIONALITY

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Contents

1. Introduction

In this paper we take a new look at the classical notions of rationality and stable rationality from the perspective of sheaves of categories.

Our approach is based on three recent developments:

(1) The new striking approach to stable rationality introduced by Voisin and developed later by Colliot-Thélène and Pirutka, Totaro, Hassett, Kresch and Tschinkel.

(2) Recent breakthroughs made by Haiden, Katzarkov, Kontsevich, Pandit [HKKP], who introduced an additional, to the Harder-Narasimhan, filtration on the semistable but not polystable objects.
(3) The theory of categorical linear systems and sheaves of categories developed by Katzarkov and Liu, [KLb]. The main outcome of this paper was a proposal of a new perverse category of sheaves analog of unramified cohomology.

An important part of our approach is the analogy between the theory of Higgs bundles and the theory of perverse sheaves of categories (PSC) initiated in [KLa], [KLb]. In the same way as the moduli spaces of Higgs bundles record the homotopy type of projective and quasi-projective varieties, sheaves of categories should record the information of the rationality of projective and quasi-projective varieties. It was demonstrated in [KNPS15] and [KNPS13] that there is a correspondence between harmonic maps to buildings and their singularities with stable networks and limiting stability conditions for degenerated categories, degenerated sheaves of categories. In this paper we take this correspondence further. We describe this correspondence in the table below.

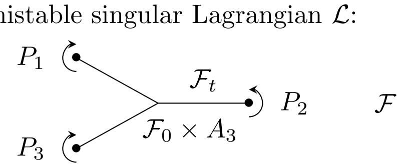
Table 1. Higgs bundles $\leftrightarrow$ Perverse sheaves of Categories

$\mathrm{Func}(\Pi_1^{\leqslant}(X,s), \mathrm{Vect})$ / | groupoid category of vector spaces	$\mathrm{Func}(\Pi^{\leqslant\infty}(X,s), \mathrm{dg\ Cat})$ / / 2 category dg category
Higgs bundles	Perverse sheaves of categories
Complex var. Hodge structures	Classical LG models

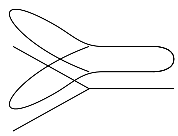
In this paper we describe a technology for finding such "good" flat families of perverse sheaves of categories. This is done by deforming LG models as sheaves of categories. The main geometric outcomes of our work are:

Classical	Categorical
$W = P$ equality for tropical varieties	"$W = P$" for perverse sheaves of categories
Voisin theory of deformations	Good flat deformations of PSC
Canonical deformations and compactification of moduli spaces	HN and additional filtrations of perverse sheaves of categories

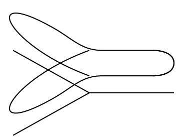
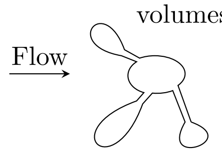
We will briefly discuss our procedure. We start with a perverse sheaf of categories (we will say more precisely what we mean by perverse sheaf of categories in the next section) $\mathcal{F}$ over $\mathbb{P}^1$ ($\mathbb{P}^2$, etc.). We then use a graph Γ (cell complex) in $\mathbb{P}^1$ ($\mathbb{P}^2$, etc.) to construct a semistable singular Lagrangian $\mathcal{L}$:



Here $\mathcal{F}_t$ are the fiber categories and P_i are categories equipped with spherical functors to $\mathcal{F}_t$. A global section in $\mathcal{F}$ defines a semistable object in the category of global sections of this PSC, which is analogous to the Lagrangian $\mathcal{L}$ shown in the following diagram.



Here, the fiber category is the category $A_1 \oplus A_1$ and the categories $P_1, \ldots, P_3$ are the category A_1 with the diagonal embedding into $A_1 \oplus A_1$ being the associated spherical functor. Observe that this object in the category of global sections can depend on the initial category or its degeneration. For most of the paper $\mathcal{L}$ will be a generator. We proceed with a correspondence:

PSC and degenerations	Deformations of categories
generators – graphs with sections 	Semistable objects
volumes of necks  ↔	Filtration on semistable generators

The filtration above is a refinement of the Harder-Narasimhan filtration. It will be defined in section 5.

We formulate the main conjecture of the paper.

Conjecture 1.1 (The main conjecture).
The weights of semistable generators are birational invariants.

We will confirm this conjecture on some examples. We indicate that our technique contains Voisin's technique which uses CH_0-groups of degenerations. On the A side, these weights produce symplectic invariants.

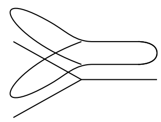
We briefly summarize our technique. We start with a perverse sheaf of categories (PSC) or its deformation. This produces a representation:

$$\rho : \pi_1(\mathbb{P}^1/\text{pts}) \to \mathrm{Aut}(\mathcal{F}_t).$$

Observe that this gives us more possibilities than in the classical case, where only cohomology groups are acted upon,

$$\rho : \pi_1(\mathbb{P}^1/\text{pts}) \to \mathrm{GL}(\oplus \mathrm{H}^*).$$

Our semistable objects (e.g. $\mathcal{L}$) correspond to global sections such as



$$\longleftrightarrow \qquad \rho : \pi_1(\mathbb{P}^1/\mathrm{pts}) \to \mathrm{Aut}(\mathcal{F}_t).$$

Observe that our filtration contains the filtrations

(1) Coming from degenerations of cohomologies. (Clemens' approach)
(2) Degenerations and nontrivial Brauer groups. (Voisin's approach)

We propose that our categorical method generalizes the methods of both Clemens and Voisin.

Clemens		
	$\subset$	
		Filtrations
	$\subset$	
Voisin		

In a very general sense our filtration is a generalization of classical Hodge theory. There should be an analogy between nilpotent representations of $\mathbb{Z}$ (which correspond to degenerations over the punctured disc with nilpotent monodromy) and their associated weight filtrations and the filtrations on an Artinian category $\mathcal{A}$ obtained from a central charge $Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R}$,

$$\{\rho : \pi_1(Z) \to Nil\} \longleftrightarrow \left\{\begin{array}{c}\text{an Artinian category with}\\ Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R}\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Artinian category of}\\ \text{nilpotent representations}\end{array}\right\} \longleftrightarrow \{\text{any Artinian category}\}$$

Based on this observation and based on many examples explained in this paper, we propose a correspondence:

Classical	Categorical
Unramified cohomologies	Hybrid models with filtrations

The paper is organized as follows:

In section 2, we introduce briefly the theory of perverse sheaves of categories and their deformations. In section 3, we give examples of deformations of categories. In section 4, we show how our approach relates to Voisin's approach. In section 5, we introduce hybrid models and explain how the examples given in this paper support our main conjecture. A more detailed treatment will appear in another paper.

This paper outlines a new approach. More details, examples and calculations will appear elsewhere.

Acknowledgments:

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2. Perverse sheaves of categories

2.1. Definitions. In this section we develop the theory of sheaves of categories and their deformations. First, we will explain what we mean when we talk about perverse sheaves of categories. Let M be a manifold with stratification S. Let K be a singular Lagrangian subspace of M so that each $K_i = K \cap S_i$ is a deformation retract of S_i. Furthermore, assume that the functor $\mathcal{R}$ which assigns to each perverse sheaf $\mathcal{F}$ on (M, S) a constructible sheaf on K with singularities in $S_K := K \cap S$ the sheaf $\mathbb{H}_K^{\dim M}(M, \mathcal{F})$ is faithful. Then a perverse sheaf of categories on M with singularities in K will be a constructible sheaf of categories on (M, S) is a constructible sheaf of categories on (K, S_K) which satisfies some appropriate conditions.

Such conditions are not known in general, and depend upon the singularities of K, but the general idea is that one should find appropriate translations of conditions which define the image of $\mathcal{R}$ into the language of pretriangulated dg categories. The most basic form of this condition is found in work of Kapranov-Schectman [KS16]. If one takes the stratified space $(\mathbb{C}, 0)$, then an appropriate skeleton K is the non-negative real line. The restriction functor expresses each perverse sheaf on $(\mathbb{C}, 0)$ as a constructible sheaf on the line K which has generic fiber a vector space ψ at any point on the positive real line and fiber ϕ at 0. There are two natural maps $v : \phi \to \psi$ and $u : \psi \to \phi$ which satisfy the condition that $\mathrm{Id}_\psi - vu$ is an automorphism, or equivalently $\mathrm{Id}_\phi - uv$ is invertible.

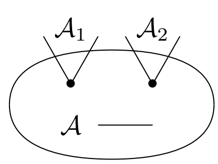
If we replace the vector spaces ϕ and ψ with pretriangulated dg categories Φ and Ψ then the map v becomes a functor $F : \Phi \to \Psi$. The condition that the map v exists and that $\mathrm{Id}_\phi - uv$ is an automorphism is analogous to claiming that F is spherical. The difference of morphisms becomes the cone of the unit $RF \to \mathrm{Id}_\Psi$, which is the twist of F. The sheaf Φ should be thought of as the "category of vanishing cycles" at 0, and Ψ should be thought of as the "category of nearby cycles".

A rough definition of perverse sheaf of categories is as follows.

Definition 2.1. *A perverse sheaf of categories on (M, S) is a constructible sheaf of categories on an appropriate skeleton (K, S_K) so that there are functors between stalks of this constructible sheaf which have properties which emulate the structure of $\mathcal{R}(\mathcal{F})$ for $\mathcal{F}$ a perverse sheaf on (M, S).*

We start with a definition. We shuffle our definition in order to study deformations of perverse sheaf of categories. We localize (K, S_K) and several smaller skeleta $\mathrm{Sch}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$. Our definition now looks like:

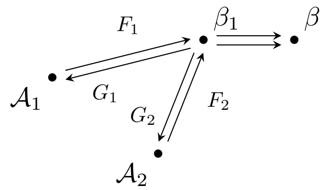
Definition 2.2 (Sheaves of categories over $\mathrm{Sch}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$)**.**



$\mathbb{C}$

$\mathcal{A}$ - *local sheaf of categories*
$\mathcal{A}_i$ - *sheaf of vanishing cycles*

$\mathcal{A}$, $\mathcal{A}_i$ - *2 categories*
F, G - *2 functors*
n, m - *2 natural transformations*



$$F_1 \underset{m_1}{\overset{n_1}{\rightleftarrows}} G_1$$

nearby vanishing cycles

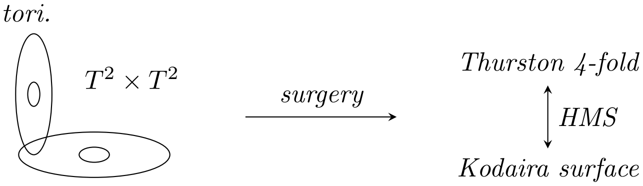
$$T_{G_1} = \beta_1 \operatorname{Id}_{G_1} - m_1 n_1$$

Theorem 2.3. *The deformations of* $\mathrm{Sch}(\mathcal{A}, \mathcal{A}_1, \dots, \mathcal{A}_n)$ *are described by :*

(1) *Adding a new category* β*;*
(2) *Changes in natural transformations* n_i*,* m_j*.*

We give some examples.

Example 2.4. *We start with a simple example* $T^2 \times T^2$ *- the product of two 2-dimensional tori.*

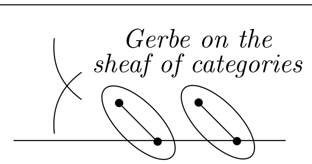
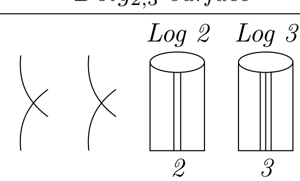


In [AAKO] the following theorem is proven.

Theorem 2.5. *The following categories are equivalent:*

$$\mathrm{D^b}(T^2 \times T^2, \textit{Gerbe}) \cong \mathrm{Fuk}(\textit{Thurston}) \cong \mathrm{D^b}(\textit{Kodaira}).$$

Example 2.6. *We generalize this construction to the case of LG models. The addition of gerbes should be an operation that is captured by perverse sheaves of categories and LG models, as described in the following example.*

LG model	*Dolg*$_{2,3}$ *surface*
	

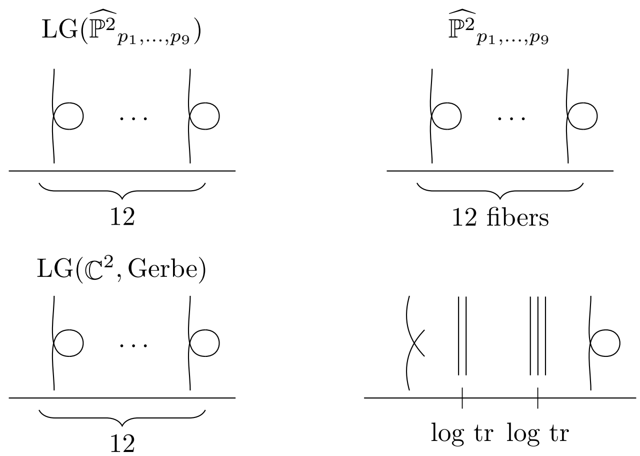
Recall: Dolg$_{2,3}$ *is the Dolgachev surface with multiple fibers of multiplicities 2 and 3, and is obtained from the rational elliptic surface* $\widehat{\mathbb{P}^2}_{p_1,\dots,p_9}$ *by applying 2 surgeries with order 2,3.*

The mirror of the surgery transforming the rational elliptic surface into the Dolgachev surface should be the addition of new fibers to the LG model of the mirror to the rational elliptic surface. The rational elliptic surface has mirror which is a rational elliptic surface with one smooth fiber removed and potential w the natural elliptic fibration over $\mathbb{C}$.

Theorem 2.7. *The mirror of $Dolg_{2,3}$ is obtained from the LG model of $\widehat{\mathbb{P}^2}_{p_1,\dots,p_9}$ by adding a gerbe G on it corresponding to a log transform. In other words:*

$$\mathrm{D^b}(Dolg_{2,3}) = \mathrm{FS}(LG(\widehat{\mathbb{P}^2}_{p_1,\dots,p_9}), G). \tag{2.1}$$

We indicate the proof of the theorem in the following diagram.



2.2. Some more examples. Consider a fibration $\mathcal{F} \xrightarrow{f} \mathbb{C}$ with a multiple n-fiber over 0.

$$\begin{array}{ccc} E \times \mathbb{C} & \xrightarrow[(\times l, \times \mathcal{E})]{n:1} & \mathcal{F} \xrightarrow{f} \mathbb{C} \\ \Big\downarrow \mathbb{Z}^n & & \\ \mathbb{C} & & nl = 0,\ \mathcal{E}^n = 1 \end{array}$$

The idea is that the addition of smooth fibers with multiplicity greater than 1 (by surgery) into an elliptic fibration over $\mathbb{C}$ should introduce quasi-phantoms into the Fukaya-Seidel category of the associated elliptic fibration. This is summarized in the following theorem.

Theorem 2.8. $\mathrm{MF}(\mathcal{F} \xrightarrow{f} \mathbb{C})$ *contains a quasi-phantom.*

Proof. Indeed $\mathrm{H}^*(\mathcal{F}, \text{vanishing cycles}) = 0$, since vanishing cycles are the elliptic curve E and $\mathrm{H}^*(E, L) = 0$, for any L - nontrivial rank 1 local system.

Also $\mathrm{K}(\mathrm{MF}(\mathcal{F} \to \mathbb{C})) = \mathbb{Z}_n$. □

One expects that phantom and quasi-phantom categories should be detectable via moduli spaces of objects. The following proposition provides evidence for this.

Proposition 2.9. *There exists a moduli space of stable objects on* $\mathrm{MF}(\mathcal{F} \to \mathbb{C})$.

Proof. Indeed these are the $\mathbb{Z}_n$-equivalent objects on $E \times \mathbb{C}$. For example, we have $M^{\mathrm{stab}} = E'$, E' - multiple fiber. □

In the next section, we consider more examples of deformations of perverse sheaves of categories.

3. Deformations of perverse sheaves of categories and Poisson deformations

Recall that deformations of perverse sheaves of categories are determined by three different types of deformations,

1. Deformations of the Stasheff polytope,
2. Deformations of the fiber categories,
3. Deformation of natural transformations.

Here we will give several examples of deformations of sheaves of categories which come from the second piece of data. We will show that noncommutative deformations of $\mathbb{P}^2$ and $\mathbb{P}^3$ may be obtained as globalization of a deformation of perverse sheaves of categories. We will describe an explicit realization of the following correspondence.

$$\left\{\begin{matrix}\text{Deformation of}\\ \text{natural transformations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Quantization of}\\ \text{Poisson deformations}\end{matrix}\right\}$$

In our case, we will recover quantizations of Poisson deformations for the simple reason that the deformation of perverse sheaves of categories that we produce comes with a deformation of the fiber category.

The results described below will appear in forthcoming work of the first two named authors [HK].

3.1. Warmup: Deformations of $\mathbb{P}^2$.

As a warmup we can analyze the case of $\mathbb{P}^2$. Here we will recover the classical noncommutative deformations of $\mathbb{P}^2$ as the deformations of a perverse sheaf of categories which is obtained by deforming the spherical functors and fixing the fiber category.

Recall that the following data determines a noncommutative deformation of $\mathbb{P}^2$. Let E be a smooth curve of genus 0, $\mathscr{L}$ be a line bundle on E of degree 3 and σ a translation automorphism of E. Then, according to Artin-Tate-van den Bergh [ATVdB91], the twisted coordinate ring of E associated to $(E, \mathscr{L}, \sigma)$ is the coordinate ring of a noncommutative deformation $\mathbb{P}^2_\mu$ of $\mathbb{P}^2$. Under the identification between $\bigwedge^2 T_{\mathbb{P}^2}$ and $-K_{\mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^2}(3)$, these deformations are associated to the choice of section of $\mathscr{O}_{\mathbb{P}^2}(3)$ which vanishes on the canonical image of E associated to $\mathscr{L}$.

The same data can be used to build a perverse schober on the disc Δ with three critical points p_1, p_2, p_3. This perverse schober can be used to reconstruct the derived category of the associated noncommutative deformation of $\mathbb{P}^2$. This fact was essentially noticed by Bondal-Polishchuk [BP93], but of course not in the language of perverse schobers.

We note that any line bundle $\mathscr{L}$ on E corresponds to a spherical functor $S_{\mathscr{L}} : D^b(k) \longrightarrow D^b(\operatorname{coh} E)$, and in particular, given the triple $(E, \mathscr{L}, \sigma)$, we can construct three spherical functors $S_0, S_{\mathscr{L}}, S_\sigma$ corresponding to line bundles $\mathscr{O}_E, \mathscr{L}$ and $\sigma^*\mathscr{L}^2$ respectively. If $\sigma = \mathrm{id}$ then this triple is precisely what one gets by restricting the strong exceptional collection $\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2)$ on $\mathbb{P}^2$ to the image of E under the embedding associated to $\mathscr{L}$.

We will let $\mathfrak{S}(E, \mathscr{L}, \sigma)$ be the perverse schober on the disc with three critical points associated to the spherical functors $S_0, S_{\mathscr{L}}$ and S_σ.

Proposition 3.1 (Harder-Katzarkov [HK])**.** *The category of global sections of the perverse schober $\mathfrak{S}(E, \mathscr{L}, \sigma)$ is $D^b(\mathrm{coh}\, \mathbb{P}^2_\mu)$ where $\mathbb{P}^2_\mu$ is the noncommutative deformation of $\mathbb{P}^2$ associated to the triple $(E, \mathscr{L}, \sigma)$.*

The fact that makes this possible is that we can deform spherical objects on an elliptic curve. By definition, if S is a spherical object on E, then $\mathrm{Ext}^1(S, S) = \mathbb{C}$. These infinitesimal deformations are obtained by pullback along an automorphism of E, though of course, deformation may be obstructed. Whether the corresponding perverse schober recovers $\mathbb{P}^2$ or not can be detected using the "monodromy at infinity". In essence, the action of spherical functors $S_0, S_{\mathscr{L}}, S_\sigma$ should be interpreted as monodromy around the degenerate fibers of the perverse sheaf of categories at points p_1, p_2 and p_3. The composition of the three monodromy functors should be interpreted as monodromy around the loop encompassing all three degenerate points. We have the following proposition.

Proposition 3.2. *The global sections of the perverse schober $\mathfrak{S}(E, \mathscr{L}, \sigma)$ is $D^b(\mathrm{coh}\, \mathbb{P}^2)$ if and only if $S_0 \cdot S_{\mathscr{L}} \cdot S_\sigma$ is the spherical twist associated to the line bundle $\mathscr{L}^3$.*

3.2. Noncommutative deformations of $\mathbb{P}^3$. Here we exhibit noncommutative deformations of $\mathbb{P}^3$ as coming from the deformations of a perverse sheaf of categories over a 1-dimensional base by deforming the structure of the category of nearby cycles, or in terms of a PSC over a 2-dimensional base by deforming the sheaves of vanishing cycles.

Polishchuk shows [Pol97] that there exist Poisson structures on $\mathbb{P}^3$ so that there are Poisson divisors which look like

1. A normal crossings union of two quadrics
2. A normal crossings union of a hyperplane and a cubic.

Thus we should be able to perform the construction of noncommutative deformations of $\mathbb{P}^2$ by replacing the smooth elliptic curve by either a normal crossings pair of quadrics $X_{2,2}$ or a normal crossings union of a hyperplane and a cubic surface, denoted $X_{1,3}$. For the sake of notation, we will only look at the case of $X_{2,2}$ in what follows, but all results hold for $X_{1,3}$ as well. We then obtain a schober over the disc with four singular points whose general fiber is $\mathrm{Perf}(X_{2,2})$. We then deform the schober, not by deforming the spherical functors S_0, S_1, S_2, S_3 and keeping $\mathrm{Perf}(X_{2,2})$ constant, but by taking non-commutative deformations of $\mathrm{Perf}(X_{2,2})$ which deform the spherical functors S_0, S_1, S_2, S_3.

Proposition 3.3 (Harder-Katzarkov [HK])**.** *One may construct noncommutative deformations $X_{2,2,\mu}$ of $X_{2,2}$ which preserve the spherical functors S_i for $i = 1, \ldots, 4$ corresponding to the data $(E, \mathscr{M}, \tau)$ where, as before, E is a smooth curve of genus 1 and τ is an automorphism of E, but now $\mathscr{M}$ is an ample line bundle on E of degree 2. There is a corresponding perverse schober over the disc with four singularities called $\mathfrak{T}(E, \mathscr{M}, \tau)$. This deformation has coordinate ring given by a quantization of the Sklyanin algebra of degree 4.*

The category $\mathrm{Perf}(X_{2,2})$ itself appears as global sections of a constructible sheaf of categories on a 2-dimensional complex as well. Taking E to be the elliptic curve that forms the singular locus of the union of smooth quadrics $X_{2,2}$, we take the skeleton $K_{X_{2,2}}$ of Δ with singularities in eight points,

To an edge of the skeleton above we take a dg extension of $D^b(\mathrm{coh}\, E)$. There is then a standard semiorthogonal decomposition of $D^b(\mathrm{coh}\, Q)$ for Q a generic quadric,

$$\langle \mathscr{O}_Q, \mathscr{O}_Q(1,0), \mathscr{O}_Q(0,1), \mathscr{O}_Q(1,1)\rangle$$

where the bundles above are determined by the natural identification of Q with $\mathbb{P}^1 \times \mathbb{P}^1$. If we let Q_1 and Q_2 be the two quadrics so that $X_{2,2} = Q_1 \cup Q_2$, then there are spherical functors $S_{i,(j,k)}$ associated to the pullback of $\mathscr{O}_{Q_i}(j,k)$ to $E = Q_1 \cap Q_2$. To the strata p_1 and q_4, we associate categories $\langle \mathscr{O}_{Q_1}\rangle$ and $\langle \mathscr{O}_{Q_2}(1,1)\rangle$ with the appropriate spherical functors to $D^b(\mathrm{coh}\, E)$. To the remaining 0-dimensional strata of the skeleton $K_{X_{2,2}}$, the appropriate categories are a bit less obvious. If $\mathcal{D}(k)$ is a dg extension of $D^b(k\mathrm{-mod})$ and $\mathcal{D}(\mathrm{coh}\, E)$ is a dg extension of $D^b(\mathrm{coh}\, E)$, then we can define $\mathcal{D}(k) \times_{S_{i,(j,k)}} \mathcal{D}(\mathrm{coh}\, E)$ to be the gluing of $\mathcal{D}(k)$ to $\mathcal{D}(\mathrm{coh}\, E)$ along the dg bimodule which assigns

$$(A, B) \in \mathrm{Ob}(\mathcal{D}(\mathrm{coh}\, E)) \times \mathrm{Ob}(\mathcal{D}(k)) \mapsto \mathrm{Hom}_{\mathcal{D}(\mathrm{coh}\, E)}(A, S_{i,(j,k)}(B)).$$

(see [KL15] for definitions). To p_2, p_3 and p_4 we assign the categories $\mathcal{D}(k) \times_{S_{i,(j,k)}} \mathcal{D}(\mathrm{coh}\, E)$ for $i = 1$ and (j,k) equal to $(1,0), (0,1)$ and $(1,1)$ respectively. Descriptions of appropriate functors will be given in [HK].

Proposition 3.4. *The dg category of global sections of the constructible sheaf of categories of $\mathcal{S}_{X_{2,2}}$ on $K_{X_{2,2}}$ is equivalent to a dg extension of* $\mathrm{Perf}(X_{2,2})$.

Therefore, we have that there is a constructible sheaf of categories whose sheaf of global sections gives the generic fiber of the constructible sheaf of categories which reconstructs $D^b(\mathrm{coh}\, \mathbb{P}^3)$. This suggests that perhaps there is a perverse sheaf of categories over $\Delta \times \Delta$ whose sheaf of global sections is $D^b(\mathrm{coh}\, \mathbb{P}^3)$. It is a somewhat remarkable fact that such a perverse sheaf of categories is provided by mirror symmetry.

Recall that the Landau-Ginzburg mirror of $\mathbb{P}^3$ is given by the pair $((\mathbb{C}^\times)^3, \mathsf{w})$ where w is the Laurent polynomial

$$\mathsf{w} = x + y + z + \frac{1}{xyz}.$$

Each monomial in this expression corresponds to a boundary divisor in $\mathbb{P}^3$, and the sum of these divisors is $-K_{\mathbb{P}^3}$. The decomposition of $-K_{\mathbb{P}^3}$ into a union of smooth quadrics then informally can be traced to a decomposition of this potential into the sum of a pair of functions,

$$\mathsf{w}_1 = x + y, \qquad \mathsf{w}_2 = z + \frac{1}{xyz}$$

This pair of functions give a map from $(\mathbb{C}^\times)^3$ to $\mathbb{C}^2$. The generic fiber of this map is a punctured elliptic curve, and this elliptic curve degenerates along the curve C_{deg}

$$\mathsf{w}_1\mathsf{w}_2(\mathsf{w}_1^2\mathsf{w}_2^2 - 16) = 0$$

The composition of the map $(x, y, z) \in (\mathbb{C}^\times)^3 \mapsto (\mathsf{w}_1, \mathsf{w}_2) \in \mathbb{C}^2$ with the map $(\mathsf{w}_1, \mathsf{w}_2) \in \mathbb{C}^2 \mapsto \mathsf{w}_1 + \mathsf{w}_2 \in \mathbb{C}$ recovers the map w. The map w has critical points over $4\sqrt{-1}^i$ for $i = 0, 1, 2, 3$. We state the following theorem. Details will appear in [HK].

Theorem 3.5 (Harder-Katzarkov, [HK])**.** *There is a singular (real) two dimensional skeleton K_2 of $\mathbb{C}^2$ whose singularities lie in C_{deg} which maps to a skeleton K*

of $\mathbb{C}$ with singularities at $4\sqrt{-1}^i$ for $i = 0,1,2,3$ under the map $(\mathsf{w}_1, \mathsf{w}_2) \mapsto \mathsf{w}_1 + \mathsf{w}_2$. On this skeleton, there is a constructible sheaf of categories whose category of global sections is a dg extension of $D^b(\operatorname{coh}\mathbb{P}^3)$.

The skeleton K is the skeleton associated to a perverse schober on $\mathbb{C}$ with four singular points. The structure of the skeleton K_2 is determined completely by the braid monodromy of the projection of the curve $C_{\deg}$ to $\mathbb{C}$ induced by the map $(\mathsf{w}_1, \mathsf{w}_2) \mapsto \mathsf{w}_1 + \mathsf{w}_2$. Finally, the following theorem holds.

Theorem 3.6 (Harder-Katzarkov [HK]). *There are deformations of the constructible sheaf of categories in Theorem 3.5, and these deformations correspond to quantizations of the Poisson deformations of $D^b(\operatorname{coh}\mathbb{P}^3)$ for which $X_{2,2}$ is a Poisson divisor.*

A similar theorem holds for the Poisson deformations of $\mathbb{P}^3$ for which $X_{1,3}$ remains a Poisson divisor – their quantizations may be recovered from deformations of a natural two-dimensional constructible sheaf of categories on a skeleton of $\mathbb{C}^2$ with singularities in a curve $D_{\deg}$, which is distinct from $C_{\deg}$.

3.3. Perverse sheaves of categories and elliptic curves. Here we describe 2-dimensional perverse sheaves of categories associated to the LG model whose Fukaya-Seidel category is equivalent to $D^b(\operatorname{coh}E)$. This approach should generalize to allow us to compute the derived category of an arbitrary curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us take a curve of degree $(2, n)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, then we build the following LG model

$$\mathsf{w} = x_1 + \frac{x_2^2}{x_1} + x_3 + \frac{x_2^n}{x_3}$$

which is a map $(\mathbb{C}^\times)^3$. The associated potential can be thought of as the composition of two maps. The first map sends $(x_1, x_2, x_3) \mapsto (x_2, \mathsf{w})$ and the second is a projection onto the second coordinate. Therefore, w is a fibration over $\mathbb{C}$ whose fibers are LG models of $\mathbb{P}^1 \times \mathbb{P}^1$ except the fiber over 0. The map (x_2, w) is a fibration over $\mathbb{C}^2$ with generic fiber a punctured elliptic curve. This can be partially compactified to an elliptic fibration written in Weierstrass form written as

$$Y^2 = X^3 - (2w_1^2 - w_2^2)X^2 + 4(w_2 - w_1)(w_1 + w_2)w_1^2 X + 4w_1^4(2w_1^2 + w_2^2).$$

This fibration degenerates along the curve

$$(w_2 - 4w_1)(4w_1 + w_2)w_2 w_1 = 0.$$

Blow up the base of this fibration at $(0,0)$ and call the result $\mathbb{C}^{\in}$. We can pull back the above elliptic fibration to get a fibration over $\widetilde{\mathbb{C}^2}$with exceptional divisor E. We can resolve singularities of this fibration over $\widetilde{\mathbb{C}^2}$ to obtain a smooth elliptic fibration Y. We can choose a chart $C_1 = \mathbb{C}^2$ on $\widetilde{\mathbb{C}^2}$ so that the map onto $\mathbb{C}$ is given by a quadratic map given in coordinates (t, s) as the function ts. Restricting the fibration of Y over $\widetilde{\mathbb{C}^2}$ to the chart C_1 and calling this elliptic fibration Y_1. This is written in Weierstrass form as

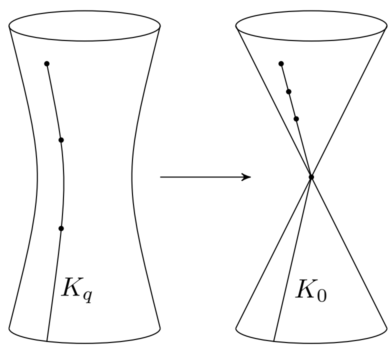
$$Y^2 = X^3 - (2t^2 - 4t + 1) - 4t(t-2)(t-1)^2 X + 4(t-1)^4(2t^2 - 4t + 3).$$

The discriminant curve of this fibration is given by the equation

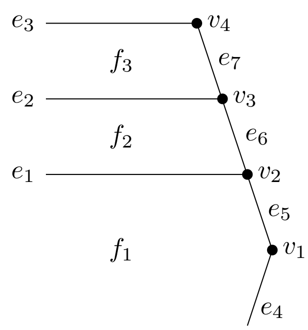
$$(4t - 5)(4t - 3)(t - 1) = 0$$

in terms of coordinates (t, s). The manifold Y_1 is then a smooth Calabi-Yau partial compactification of the LG model of the elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ whose Fukaya-Seidel category should be equivalent to $D^b(\operatorname{coh}E)$.

We can build a natural complex on $\mathbb{C}^2$ near the fiber $ts = 0$ and equip it with a perverse sheaf of categories whose global sections category should be $D^b(\mathrm{coh}E)$. All of the data involved in this perverse sheaf of categories comes from the elliptic fibration above. This complex is built as follows. Take a straight path γ in a disc around 0 in $\mathbb{C}$ going from the boundary to 0. Fibers over points q in this path are smooth conics if $q \neq 0$, and a pair of copies of $\mathbb{C}$ meeting in a single point if $q = 0$. In each fiber over a point in γ, we can draw a skeleton K_q, and over the point 0 with singularities in the intersection of the discriminant curve in $\mathbb{C}^2$ with the fiber over q, we draw the skeleton K_0, which has singularities at $(0, 0)$ as well as at the intersection of the fiber over 0 with the discriminant curve. These skeleta are drawn as in the following diagram



Putting all of these fiberwise complexes into a two-dimensional complex, we get a complex which looks as follows –



The constructible sheaf of categories on this complex is given by a set of categories assigned to each vertex, edge and face, and a sequence of functors $F_{s \to t} : \mathscr{C}_s \to \mathscr{C}_t$ for pairs of strata t and s so that $s \subseteq \bar{t}$ which satisfy the natural relations, i.e. that $F_{t \to q} \cdot F_{s \to t} = F_{s \to q}$. Our categories are:

$$
\begin{aligned}
&\mathscr{C}_{f_1} = \mathscr{C}_{f_2} = \mathscr{C}_{f_3} = \mathscr{C}_{v_1} = \mathscr{D}(\mathrm{coh}E) \\
&\mathscr{C}_{e_4} = \mathscr{C}_{e_5} = \mathscr{C}_{e_6} = \mathscr{C}_{e_7} = \mathscr{C}_{v_2} = \mathscr{C}_{v_3} = \mathscr{C}_{v_4} = 0 \\
&\mathscr{C}_{e_3} = \mathscr{C}_{k-\mathrm{dgm}} \\
&\mathscr{C}_{e_1} = \mathscr{C}_{k-\mathrm{dgm}} \times_{\Phi_2} \mathscr{D}(\mathrm{coh}E) \\
&\mathscr{C}_{e_2} = \mathscr{A} \times_{\Phi_1} \mathscr{D}(\mathrm{coh}E)
\end{aligned}
$$

Here, $\mathscr{D}(\mathrm{coh}E)$ is a pretriangulated dg extension of $D^b(\mathrm{coh}E)$. If we have a functor $\Phi : \mathscr{C} \to \mathscr{D}(\mathrm{coh}E)$, then the category $\mathscr{C} \times_\Phi \mathscr{D}(\mathrm{coh}E)$ is the dg category of pairs

(A,B,μ) where $A \in \mathscr{C}$ and $B \in \mathscr{D}(\mathrm{coh}E)$ and $\mu \in \mathrm{Hom}^0_{\mathscr{D}(\mathrm{coh}E)}(\Phi_2(A),B)$ and closed (see Kuznetsov-Lunts [KL15] for definition). There are functors Φ_0, Φ_1 and Φ_2 given as follows. There's a semiorthogonal decomposition

$$D^b(\mathrm{coh}\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}, \mathcal{A}, \mathcal{O}(1,1)\rangle$$

thus there are spherical functors

$$\begin{aligned} \phi_0 &: D^b(k) \longrightarrow D^b(\mathrm{coh}E) \\ \phi_1 &: \mathcal{A} \longrightarrow D^b(\mathrm{coh}E) \\ \phi_2 &: D^b(k) \longrightarrow D^b(\mathrm{coh}E) \end{aligned}$$

which have dg lifts of these functors

$$\begin{aligned} \Phi_0 &: \mathscr{C}_{k-\mathrm{dgm}} \longrightarrow \mathscr{D}(\mathrm{coh}E) \\ \Phi_1 &: \mathscr{A} \longrightarrow \mathscr{D}(\mathrm{coh}E) \\ \Phi_2 &: \mathscr{C}_{k-\mathrm{dgm}} \longrightarrow \mathscr{D}(\mathrm{coh}E). \end{aligned}$$

There are two well-defined functors from $\mathscr{A} \times_\Phi \mathscr{D}(\mathrm{coh}E)$ to $\mathscr{D}(\mathrm{coh}E)$ then there are two functors F^+ and F^-, given by the map sending (A,B,μ) to B and $\mathrm{Cone}(\mu)[1]$ respectively. The functor $F_{e_i \to f_i}$ is given by the corresponding functor F^- for $i = 1, 2$, the functor $F_{e_i \to f_{i+1}}$ is the corresponding functor F^+ for $i = 1, 2$. The functor $F_{e_3 \to f_3}$ is the functor Φ_0. The category $F_{v_1 \to f_1}$ is the identity functor. It's easy to check that the global sections of this sheaf of categories is $\mathscr{D}(\mathrm{coh}E)$.

The restriction of this perverse sheaf of categories to the 1-dimensional skeleton K_q in each fiber for $q \neq 0$ has global sections category which is a dg extension of $D^b(\mathrm{coh}\,\mathbb{P}^1 \times \mathbb{P}^1)$ [HK], which is equivalent to the Fukaya-Seidel category of the Landau-Ginzburg model associated to each fiber of w over $q \neq 0$.

Remark 3.7 (Cubic fourfolds with two planes and K3 surfaces). A similar structure should arise in the case of the cubic fourfold containing a pair of planes. It is well known that the cubic containing two planes is $\mathbb{P}^4$ blown up at the transversal intersection S of a cubic and a quadric hypersurface.

According to mirror symmetry, there should be a pair of potentials w_1 and w_2 on the LG model of this cubic (defined on some open subset of the relatively compactified LG model) corresponding to the divisor classes D_1 and D_2 of the cubic and quadrics in $\mathbb{P}^4$ containing S. This gives rise to a fibration over $\mathbb{C}^2$ which can then be viewed as the composition of a fibration over $\mathbb{C}^2$ blown up at $(0,0)$ and the contraction of the exceptional divisor. This fibration over the blown up plane should have relative dimension 2 and the fibers over a general point should be a K3 surface mirror to the intersection of a cubic and a quadric in $\mathbb{P}^4$. Furthermore, the fibers over the exceptional divisor should be generically smooth, as should the fibers over the locus $\mathsf{w}_1 + \mathsf{w}_2 = 0$.

Associated to this fibration, there should be a perverse sheaf of Fukaya categories over the blown-up plane. Near the intersection of the exceptional divisor and the proper transform of $\mathsf{w}_1 + \mathsf{w}_2 = 0$, this perverse sheaf of Fukaya categories should localize along a skeleton to look exactly like the construction above, except instead of having generic fibers the Fukaya category of an elliptic curve, we should have generic fibers the Fukaya category of the mirror to the complete intersection in $\mathbb{P}^4$ of a cubic and a quadric. The global section of this constructible sheaf of categories should have category of global sections equal to the derived category of the intersection of the cubic and the quadric in $\mathbb{P}^4$, which is reflected in the fact that the cubic fourfold containing two planes has, as a semiorthogonal summand of its derived category the derived category of the complete intersection of the cubic and the quadric in $\mathbb{P}^4$.

4. Landau-Ginzburg model computations for threefolds

In this section we connect our program to birational geometry and the theory of LG models. The main goal of this section is to emphasize our program is connected to Voisin's approach. In terms of deformations of perverse sheaves of categories, the LG model gives a PSC whose global sections recover the derived category of a Fano variety. We will degenerate one of the categories of vanishing cycles of this PSC in order to produce a category which has nontrivial "Brauer group". The approach to degeneration that we take is standard in symplecic geometry and goes back at least to Seidel [Sei01], and involves removing closed subvarieties.

We recall some inspiration from birational geometry stemming from the work of Voisin [Voi15], Colliot-Thélène and Pirutka [CTP14]. A variety X is called stably non-rational if $X \times \mathbb{P}^n$ is non-rational for all n. It is known that if a variety over $\mathbb{C}$ is stably rational then for any field L containing $\mathbb{C}$, the Chow group $\mathrm{CH}_0(X_L)$ is isomorphic to $\mathbb{Z}$. Under this condition, $\mathrm{CH}_0(X)$ is said to be universally trivial. Voisin has shown that universal nontriviality of $\mathrm{CH}_0(X)$ can be detected by deformation arguments, in particular [Voi15, Theorem 1.1] says that if we have a smooth variety $\mathcal{X}$ fibered over a smooth curve B so that a special fiber $\mathcal{X}_0$ has only mild singularities and a very general fiber $X := \mathcal{X}_b$ has universally trivial $\mathrm{CH}_0(X)$ then so does any projective model of $\mathcal{X}_0$. If V is a threefold, then one can detect failure of universal $\mathrm{CH}_0(X)$-triviality by showing that there exists torsion in $H^3(V, \mathbb{Z})$ (i.e. there exists torsion in the Brauer group). As an example, we may look at the classical Artin-Mumford example [AM72] which takes a degeneration of a quartic double solid to a variety which is a double cover of $\mathbb{P}^3$ ramified along a quartic with ten nodes. It is then proven in [AM72] that the resolution of singularities of this particular quartic double solid V has a $\mathbb{Z}/2$ in $H^3(V, \mathbb{Z})$. Voisin uses this to conclude that a general quartic double solid is not stably rational, whereas Artin and Mumford could only conclude from this that their specific quartic double solid is not rational.

The main idea that we explore in this section is that the approach of Voisin to stable non-rationality should have a generalization to deformations or degenerations of $\mathrm{D}^b(\mathrm{coh}\, X)$. Via mirror symmetry, this should translate to a question about deformations or degenerations of sheaves of categories associated to the corresponding LG model of X. Mirror symmetry for Fano threefolds should exchange

$$H^{\mathrm{even}}(X, \mathbb{Z}) \cong H^{\mathrm{odd}}(\mathrm{LG}(X), S; \mathbb{Z})$$
$$H^{\mathrm{odd}}(X, \mathbb{Z}) \cong H^{\mathrm{even}}(\mathrm{LG}(X), S; \mathbb{Z})$$

where S is a smooth generic fiber of the LG model of X. See [KKP14] for some justification for this relationship. This is analogous to the case where X is a Calabi-Yau threefold (see [Gro01, Gro98]). The degenerations of the sheaf of categories associated to $\mathrm{LG}(X)$ that we will produce are not necessarily degenerations of LG models in the usual geometric sense, but they are produced by blowing up or excising subvarieties from X, as described in section 3. We then show that we find torsion in $H^2(U, S; \mathbb{Z})$ for U our topologically modified LG model. We propose that this torsion is mirror dual to torsion in the K_0 of some deformation of the corresponding category. By the relation above, the torsion groups appearing in the following subsections should be mirror categorical obstructions to stable rationality of the quartic double solid and the cubic threefold.

4.1. The LG model of a quartic double solid. Here we review a description of the LG models of several Fano threefolds in their broad strokes. We begin with the following situation. Let X be a Fano threefold of one of the following types. Recall that V_7 denotes the blow-up of $\mathbb{P}^3$ at a single point.

(1) X is a quartic double solid.
(2) X is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$.
(3) X is a double cover of V_7 with branch locus an anticanonical divisor.
(4) X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with ramification locus of of degree $(2, 2, 2)$.

Then the singular fibers of the LG model of X take a specific form which is independent of X. The construction described here appears in [DHNT15] for the case of quartic double solids. There are several fibers of each LG model which are simply nodal K3 surfaces, and there is one fiber which is a more complicated. We assume the complicated fiber is the fiber over 0 in $\mathbb{C}$ and we will denote it Y_0. Monodromy about this complicated fiber has order 2, and the fiber itself has a single smooth rational component with multiplicity 2 and a number of rational components with multiplicity 1. We will henceforward denote the LG model by Y, and it will be equipped with a regular function w.

A natural way to understand Y_0 is to take base-change along the map $t = s^2$ where s is a parameter on the base $\mathbb{C}_t$ of the original LG model Y. Performing this base-change and taking normalization, we obtain a (possibly) singular family of K3 surfaces $\widehat{Y}$ with a map $\widehat{\mathsf{w}} : \widehat{Y} \to \mathbb{C}_s$. The (possible) singularities of $\widehat{Y}$ are contained in the fiber $\widehat{\mathsf{w}}^{-1}(0) = \widehat{Y}_0$, which is a K3 surface with several A_1 singularities.

Furthermore, there is an involution ι on $\widehat{Y}$ from which we may recover the original LG model Y. This quotient map sends no fiber to itself except for $\widehat{Y}_0$. On this fiber, the automorphism ι acts as a non-symplectic involution on $\widehat{Y}_0$ and fixes a number of rational curves.

In the Landau-Ginzburg model Y, given as the resolved quotient of $\widehat{Y}/\iota$, the fiber Y_0 is described as follows. In the quotient $\widehat{Y}/\iota$, the fiber over 0 is scheme-theoretically 2 times the preimage of 0 under the natural map. Furthermore, there are a number of curves of cA_1 singularities. We resolve these singularities by blowing up along these loci in sequence, since there is nontrivial intersection between them. This blow-up procedure succeeds in resolving the singularities of $\widehat{Y}/\iota$ and that the relative canonical bundle of the resolved threefold is trivial. Let $E_1, \ldots, E_n$ denote the exceptional divisors obtained in Y under this resolution of singularities.

4.2. Torsion in cohomology of the LG model. We will now denote by U the manifold obtained from Y by removing components of Y_0 with multiplicity 1, in other words, $U = Y \setminus (\cup_{i=1}^n E_i)$ where $E_1, \ldots, E_n$ are the exceptional divisors described in the previous paragraph. Another way to describe this threefold is as follows. Take the threefold $\widehat{Y}$ described above, and excise the fixed locus of ι, calling the resulting threefold $\widehat{U}$. Note that this is the complement of a union of smooth codimension 2 subvarieties. The automorphism ι extends to a fixed-point free involution on $\widehat{U}$ and the quotient $\widehat{U}/\iota$ is U. Let us denote by w_U the restriction of w to U. Our goal is to show that if S is a generic smooth fiber of w_U, then there is $\mathbb{Z}/2$ torsion in $H^2(U, S; \mathbb{Z})$.

The group $H^2(U, S; \mathbb{Z})$ should be part of the K-theory of some quotient category of the Fukaya-Seidel category of $\mathrm{LG}(X)$ equipped with an appropriate integral structure.

Proposition 4.1. *The manifold $\widehat{U}$ is simply connected.*

Proof. First, let $\widetilde{Y}$ be a small analytic resolution of singularities of $\widehat{Y}$ and let $\widetilde{\mathsf{w}}$ be the natural map $\widetilde{\mathsf{w}} : \widetilde{Y} \to \mathbb{A}^1_s$. Then, since the fixed curves of ι contain the singular points of $\widehat{Y}$, the variety $\widehat{U}$ can be written as the complement in $\widetilde{Y}$ of the union of the exceptional curves of the resolution $\widetilde{Y} \to \widehat{Y}$ and the proper transform of the fixed locus of the involution ι on $\widehat{Y}$. This is all to say that $\widehat{U}$ is the complement of a codimension 2 subvariety of the smooth variety $\widetilde{Y}$. Thus it follows by general theory that $\pi_1(\widehat{U}) = \pi_1(\widehat{Y})$, and so it is enough to show that $\pi_1(\widehat{Y})$ is simply connected.

At this point, we may carefully apply the van Kampen theorem and the fact that ADE singular K3 surfaces are simply connected to prove that $\widetilde{Y}$ is simply connected. Begin with a covering $\{V_i\}_{i=1}^m$ of $\mathbb{A}^1$ so that the following holds:

1. Each V_i is contractible,
2. Each $\widetilde{\mathsf{w}}^{-1}(V_i)$ contains at most one singular fiber of $\widetilde{\mathsf{w}}$,
3. For each pair of indices i, j, the intersection $V_i \cap V_j$ is contractible, connected,
4. For each triple of indices i, j, k, the intersection $V_i \cap V_j \cap V_k$ is empty.

(it is easy to check that such a covering can be found). Then the Clemens contraction theorem tells us that $Y_i := \widetilde{\mathsf{w}}^{-1}(V_i)$ is homotopic to the unique singular fiber (if V_i contains no critical point, then Y_i is homotopic to a smooth K3 surface). Since ADE singular K3 surfaces are simply connected, then Y_i is simply connected. The condition that $V_i \cap V_j$ is connected then allows us to use the Seifert–van Kampen theorem to conclude that $\widetilde{Y}$ is simply connected. □

As a corollary to this proposition, we have that

Corollary 4.2. *The free quotient $U = \widehat{U}/\iota$ has fundamental group $\mathbb{Z}/2$ and hence $H^2(U, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}^n$ for some positive integer n.*

Now, finally, we show that this implies that there is torsion $\mathbb{Z}/2$ in the cohomology group $H^2(U, S; \mathbb{Z})$.

Theorem 4.3. *We have an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some positive integer m.*

Proof. We compute using the long exact sequence in relative cohomology,

$$\dots \to H^1(S, \mathbb{Z}) \to H^2(U, S; \mathbb{Z}) \to H^2(U, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to \dots$$

Since S is a smooth K3 surface, we know that $H^1(S, \mathbb{Z}) = 0$, and that the subgroup $\mathbb{Z}/2$ of $H^2(U, \mathbb{Z})$ must be in the kernel of the restriction map $H^2(U, \mathbb{Z}) \to H^2(S, \mathbb{Z})$. Thus it follows that there is a copy of $\mathbb{Z}/2$ in $H^2(U, S; \mathbb{Z})$, and furthermore, that $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some integer m. □

4.3. The cubic threefold. A very similar construction can be performed in the case of the LG model of the cubic threefold with some minor modifications. The details of the construction of the LG model of the cubic threefold that are relevant are contained in [GKR12][1]. There is a smooth log Calabi-Yau LG model of the cubic threefold, which we denote (Y, w) with the following properties:

(1) The generic fiber is a K3 surface with Picard lattice $\mathrm{M}_6 = E_8^2 \oplus U \oplus \langle -6 \rangle$.
(2) There are three fibers with nodes.
(3) The fiber over 0 which is a union of 6 rational surfaces whose configuration is described in [GKR12]. Monodromy around this fiber is of order 3.

By taking base change of Y along the map $g : \mathbb{C} \to \mathbb{C}$ which assigns λ to μ^3, and resolving g^*Y, we obtain a threefold $\hat{Y}$ which is K3 fibered over $\mathbb{C}$, but now has only 6 singular fibers, each with only a node. This means that there is a birational automorphism ι on $\hat{Y}$ of order 3 so that $\hat{Y}/\iota$ is birational to Y. Explicitly, in [GKR12] it is shown that the automorphism ι is undefined on nine pairs of rational curves, each pair intersecting in a single point and all of these pairs of curves are in the fiber of $\hat{Y}$ over 0. We can contract these A_2 configurations of rational curves to get a threefold $\widetilde{Y}$ on which ι acts as an automorphism, but which is singular. The automorphism ι fixes six rational curves in the fiber of $\widetilde{Y}$ over 0. After blowing up sequentially along these six rational curves to get $\widetilde{Y}'$, the automorphism ι continues to act biholomorphically, and no longer has fixed curves. The quotient $\widetilde{Y}'/\iota$ is smooth, according to [GKR12], and there are seven components, the image of the six exceptional divisors, and a single component $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ of multiplicity three. The rational surfaces coming from exceptional divisors meet R along three vertical and three horizontal curves. The divisor R can be contracted onto either one of its $\mathbb{P}^1$ factors. Performing one of these two contractions, we recover Y.

Now let $U = (\widetilde{Y}'/\iota) \setminus \{S_1, \dots, S_6\}$. Note that this can be obtained by blowing up Y in the curve which is the intersection of three components of the central fiber and removing all of the other components. Then a proof almost identical to that of Theorem 4.3 shows that, if S is a generic fiber of w, then

Theorem 4.4. *There is an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}^m$ for some positive integer m.*

Therefore, if X is the cubic threefold, then there should exist a non-commutative deformation of $\mathrm{D}^b(\operatorname{coh} X)$ with torsion in its periodic cyclic cohomology obstructing stable rationality of X.

4.4. The quartic double fourfold. Here we will look at the LG models of the quartic double fourfold. There is an analogy between the LG model of the quartic double fourfold and the LG model of the cubic threefold.

Here we will give a model which describes the LG model of the quartic double fourfold, which we call X. Recall that we may write such a variety as a hypersurface in $\mathbb{WP}(1, 1, 1, 1, 1, 2)$ of degree 4. Therefore, following the method of Givental, we may write the LG model of X as a hypersurface in $(\mathbb{C}^\times)^5$ cut out by the equation

$$x_4 + x_5 + \frac{1}{x_1 x_2 x_3 x_4 x_5^2} = 1$$

[1]In the most recent versions of [GKR12], these details have been removed, so we direct the reader to versions 1 and 2 of [GKR12] on the arXiv

equipped with a superpotential

$$\mathsf{w} = x_1 + x_2 + x_3.$$

Call this hypersurface Y^0. We may write this superpotential as the sum of three superpotentials,

$$\mathsf{w_i} = x_i \text{ for } i = 1, 2, 3.$$

There's then a map from $\mathrm{LG}(X)$ to $\mathbb{C}^3$ given by the restriction of the projection

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3).$$

The fibers of this projection map are open elliptic curves which can be compactified in $\mathbb{C}^2$ to

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 x_4 x_5^2(x_4 + x_5 - 1) + 1 = 0$$

We may then write this threefold in Weierstrass form as

$$y^2 = x(x^2 + \mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3^2 x + 16\mathsf{w}_1^3\mathsf{w}_2^3\mathsf{w}_3^3)$$

This elliptic fibration over $\mathbb{C}^3$ has smooth fibers away from the coordinate axes. We will resolve this threefold to get an appropriate smooth resolution of Y^0. We do this by blowing up the base of the elliptic fibration and pulling back until we can resolve singularities by blowing up the resulting fourfold in fibers.

First, we blow up $\mathbb{C}^3$ at $(0,0,0)$, and we call the resulting divisor E_0. Then we blow up the resulting threefold base at the intersection of E_0 and the strict transforms of $\{\mathsf{w}_i = 0\}$, calling the resulting exceptional divisors $E_{i,0}$. We then blow up the intersections of the strict transforms of $\mathsf{w}_i = \mathsf{w}_j = 0$ five times (in appropriate sequence) and call the resulting divisors $E_{ij,k}, k = 1, \ldots, 5$. There is now a naturally defined elliptic fibration over this blown-up threefold. Over an open piece in each divisor in the base, the fibers of this elliptic fibration and their resolutions can be described by Kodaira's classification. Identifying E_0 and $E_{i,0}$ with their proper transforms in R, we have:

- Fibers of type III over points in E_0.
- Fibers of type III^* over points in $\{\mathsf{w}_i = 0\}$.
- Fibers of type I_0^* over points in $E_{ij,3}$.
- Fibers of type III over $E_{ij,2}$ and $E_{ij,4}$
- Fibers of type I_1 along some divisor which does not intersect any other divisor in the set above.

and smooth fibers everywhere else. We may now simply blow up appropriately to resolve most singularities in the resulting elliptic fourfold over R. We are left with singularities in fibers over $E_{ij,2} \cap E_{ij,3}$ and $E_{ij,4} \cap E_{ij,3}$. These singularities admit a small resolution by work of Miranda. Thus we obtain a smooth resolution of our elliptic fourfold.

We will call this resolved fourfold $\mathrm{LG}(X)$. The map w can be extended to a morphism from $\mathrm{LG}(X)$ to $\mathbb{C}$ by simply composing the elliptic fibration map from $\mathrm{LG}(X)$ to R with the contraction map from R onto $\mathbb{C}$ and the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$. The fiber over any point in $\mathbb{C}$ away from 0 is irreducible, and the fiber over 0 is composed of the preimages of E_0 and $E_{i,0}$ in the elliptic fibration, along with the strict transform of the preimage of $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3 = 0$ in Y^0, which is simply a smooth elliptically fibered threefold.

Therefore, the fiber over 0 is composed of 6 divisors with multiplicity 1. However, this is not normal crossings, since the preimage of E_0 in the elliptic fibration on

LG(X) is a pair of divisors which intersect with multiplicity 2 in the fiber over each point in E_0.

4.5. Base change and torsion. Just as in the case of the cubic threefold, we may blow-up the LG model (Y, w) of the quartic double fourfold to get a fibration over $\mathbb{A}^1$ which we call $(\widetilde{Y}, \widetilde{\mathsf{w}})$ and remove divisors from $\widetilde{\mathsf{w}}^{-1}(0)$ to get a fibration over $\mathbb{A}^1$ which we denote $(Y_{np}, \mathsf{w}_{np})$ so that there is torsion in $H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z})$ for s a regular value of w.

We outline this construction, ignoring possible birational maps which are isomorphisms in codimension 1. We note that over the fibration E_0 in the LG model (Y, w) expressed as an elliptic fourfold over a blow-up of $\mathbb{C}^3$ as described in the previous section is a fibration by degenerate elliptic curves of Kodaira type III. Each fiber then, over a Zariski open subset of E_0 is a pair of rational curves meeting tangentially in a single point. The preimage of E_0 in Y is then a pair of divisors D_1 and D_2 in Y which intersect with multiplicity 4 along a surface. Blowing up Y in this surface of intersection of D_1 and D_2 which is isomorphic to E_0 produces a rational threefold D' in the blow up (which we call $\widetilde{Y}$), whose multiplicity in the fiber over 0 of the inherited fibration over $\mathbb{C}$ is four.

Taking base change of $\widetilde{Y}$ along the map $t \mapsto s^4$ is the same as taking the fourfold cover of $\widetilde{Y}$ ramified along the fiber over 0. After doing this, the multiplicity of the preimage of D' is 1 and all components of the fiber over 0 except for the preimage of D' can be smoothly contracted to produce a fibration (Y', w') over $\mathbb{C}$.

The upshot of this all is that Y' admits a birational automorphism σ of order 4 so that Y'/σ is birational to $\widetilde{Y}$. In fact, if we excise the (codimension ≥ 2) fixed locus of σ and take the quotient, calling the resulting threefold Y_{np}, then Y_{np} is just $\widetilde{Y}$ with all components of the fiber over 0 which are not equal to D' removed. The fibration map on Y_{np} over $\mathbb{C}$ will be called w_{np}, and we claim that $H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z})$ has order four torsion. To do this, one uses arguments identical to those used in the case of the quartic double solid.

Proposition 4.5. *Letting Y_{np} and w_{np} be as above, and let s be a regular value of w. Then*

$$H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}^a$$

for some positive integer a.

Therefore, the deformation of the Fukaya-Seidel category of $(\widetilde{Y}, \mathsf{w})$ obtained by removing cycles passing through the components of $\widetilde{\mathsf{w}}^{-1}(0)$ of multiplicity 1 should have 4-torsion in its K_0. This torsion class, under mirror symmetry should be an obstruction to the rationality of the quartic double fourfold.

4.6. Cubic fourfolds and their mirrors. This section does not relate directly to deformations of perverse sheaves of categories, though it continues to explore the relationship between rationality and symplectic invariants of corresponding LG models.

In this section, we will look at the LG models of cubic fourfolds and cubic fourfolds containing one or two planes. Since cubic fourfolds containing one or two planes are still topologically equivalent to a generic cubic fourfold, this is a somewhat subtle problem which we avoid by instead obtaining LG models for cubic fourfolds containing planes which are blown up in the relevant copies of $\mathbb{P}^2$.

It is known (see [Kuz10]) that a general cubic has bounded derived category of coherent sheaves $\mathrm{D}^b(X)$ which admits a semi-orthogonal decomposition

$$\langle \mathcal{A}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3)\rangle.$$

When X contains a plane, $\mathcal{A}_X = \mathrm{D}^b(S,\beta)$ is the bounded derived category of β twisted sheaves on a K3 surface S for β an order 2 Brauer class. It is known [Has99, Lemma 4.5] that the lattice T in $H^4(X,\mathbb{Z})$ orthogonal to the cycles $[H]^2$ and $[P]$ where H is the hyperplane class and P is the plane contained in X, is isomorphic to

$$E_8^2 \oplus U \oplus \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

which is not the transcendental lattice of any K3 surface. It is expected that such cubic fourfolds are non-rational. When X contains two planes, it is known that X is then rational. According to Kuznetsov [Kuz10], we then have that the category $\mathcal{A}$ is the derived category of a K3 surface S, and by work of Hassett [Has99], we have that the orthogonal complement of the classes $[H]^2, [P_1], [P_2]$ where P_1 and P_2 are the planes contained in X is isomorphic to

$$U \oplus E_8^2 \oplus \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix},$$

which is the transcendental lattice of a K3 surface S, and generically $\mathcal{A}_X = \mathrm{D}^b(\mathrm{coh}\, S)$ and $S^{[2]}$ is the Fano variety of lines in X.

Our goal in this section is to describe the mirror side of this story. In particular, we want to observe in the three cases above, how rationality and non-rationality can be detected using symplectic characteristics of LG models. We will construct smooth models of smooth models of

(1) The LG model of a cubic fourfold (which we call Z_0).
(2) The LG model of a cubic fourfold containing a plane P blown up in P (which we call Z_1).
(3) The LG model of a cubic fourfold containing a pair of disjoint planes P_1 and P_2 blown up in $P_1 \cup P_2$ (which we call Z_2).

According to a theorem of Orlov [Orl92], the bounded derived categories of Z_1 and Z_2 admit semi-orthogonal decompositions with summands equal to the underlying cubics. Therefore, homological mirror symmetry predicts that the derived categories of coherent sheaves of the underlying cubics should be visible in the Fukaya-Seidel (or directed Fukaya) categories of the LG models of Z_1 and Z_2. In particular, we should be able to see $\mathrm{D}^b(\mathrm{coh}\, S, \beta)$ in the Fukaya-Seidel category of $\mathrm{LG}(Z_1)$ and $\mathrm{D}^b(\mathrm{coh}\, Z_2)$ in the Fukaya-Seidel category of $\mathrm{LG}(Z_2)$.

It is conjectured by Kuznetsov [Kuz10] that a cubic fourfold X is rational if and only if $\mathcal{A}_X$ is the bounded derived category of a geometric K3 surface, thus in the case where X contains a single plane, the gerbe β is an obstruction to rationality of X. Such gerbes arise naturally in mirror symmetry quite commonly. If we have a special Lagrangian fibration on a manifold M over a base B, and assume that there is a special Lagrangian multisection of π and no special Lagrangian section, then mirror symmetry is expected assign to a pair (L,∇) in the Fukaya category of M a complex of α-twisted sheaves on the mirror for α some nontrivial gerbe. We will see this structure clearly in the LG models of Z_0, Z_1 and Z_2.

4.7. The general cubic fourfold. Let us now describe the LG model of the general cubic fourfold in a such a way that a nice smooth resolution becomes possible. Givental [Giv98] gives a description of constructions of mirrors of toric complete intersections. A more direct description of Givental's construction is described in [HD15].

We begin with the polytope Δ corresponding to $\mathbb{P}^5$ given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Using Givental's construction, we get a LG model with total space

$$Y^0 = \{z + w + u = 1\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu}.$$

We will express Y^0 as a fibration over $\mathbb{C}^3$ by elliptic curves. Then we will use work of Miranda [Mir83] to resolve singularities of this fibration and thus obtain a smooth model of Y^0. This is necessary, since there are singularities "at infinity" in the LG model provided by Givental. A more uniform construction of smooth compactifications of the LG models constructed by Givental can be found in [Har16, Chapter 3].

To carry do this, we decompose w into three different functions

$$\mathsf{w}_1 = x, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

Then Y^0 is birational to a variety fibered by affine curves written as

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw(z + w - 1) - 1 = 0$$

where $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3$ are treated as coordinates on $\mathbb{C}^3$. This is can be rearranged into Weierstrass form as

$$y^2 = x^3 + \mathsf{w}_2^2\mathsf{w}_1^2\mathsf{w}_3^2 x^2 + 8\mathsf{w}_3^3\mathsf{w}_2^3\mathsf{w}_1^3 x + 16\mathsf{w}_1^4\mathsf{w}_2^4\mathsf{w}_3^4.$$

The discriminant locus of this fibration over $\mathbb{C}^3$ has four components, and for a generic point in each component we can give a description of the structure of the resolution of singularities over that point in terms of Kodaira's classification of the singular fibers of elliptic fibrations.

- Singular fibers of type IV^* along $\{\mathsf{w}_i = 0\}$ for $i = 1, 2, 3$,
- Singular fibers of type I_1 along the divisor cut out by the equation $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0$.

The loci $\mathsf{w}_i = 0$ intersect each other of course, but D_{I_1} does not intersect any $\{w_i = 0\}$, thus we must only worry about singularities at $(0, 0, 0)$ and $\mathsf{w}_i = \mathsf{w}_j = 0$ for $i, j = 1, 2, 3$ and $i \neq j$. We blow up sequentially at these loci and describe the fibers over the exceptional divisors. We will use Kodaira's conventions for describing the minimal resolution of singular fibers of an elliptic fibration.

- Blow up the base $\mathbb{C}^3$ at $(0,0,0)$. Call the associated blow-up map $f_1 : T_1 \to \mathbb{C}^3$ and call the exceptional divisor Q. As before, if π_1 is the induced elliptic fibration on T_1, then on Q there are just smooth fibers away from the intersection of the strict transform of $\{\mathsf{w}_i = 0\}$.
- Blow up the intersections $\{\mathsf{w}_i = \mathsf{w}_j = 0\}$ for $i, j = 1, 2, 3$ and $i \neq j$. Call the associated map $f_2 : T_2 \to T_1$ and call the exceptional divisors $E_{i,j}$. Let π_2 be the induced elliptic fibration on T_2. The fibration π_2 has fibers with resolutions of type IV over R_{ij}.
- Blow up at the intersections of R_{ij} and the strict transforms of $\{\mathsf{w}_i = 0\}$ and $\{\mathsf{w}_j = 0\}$. Call the associated map $f_3 : T_3 \to T_2$ and call the exceptional divisors $R_{ij,i}$ and $R_{ij,j}$ respectively. Let π_3 be the induced elliptic fibration over T_3, then the fibration π_3 has smooth fibers over the divisors $R_{ij,i}$ and $R_{ij,j}$.

Thus we have a fibration over T_3 with discriminant locus a union of divisors, and none of these divisors intersect one another. Thus we may resolve singularities of the resulting Weierstrass form elliptic fourfold by simply blowing up repeatedly the singularities along these loci. Call this fourfold $\mathrm{LG}(Z_0)$. By composing the elliptic fibration π_3 of $\mathrm{LG}(Z_0)$ over T_3 with the contraction of T_3 onto $\mathbb{C}^3$ we get a map which we call $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_2$ from $\mathrm{LG}(Z_0)$ to $\mathbb{C}^3$. We will describe explicitly the fibers over points of $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$.

- If p is a point in the complement of the strict transform of
$$\{\mathsf{w}_1 = 0\} \cup \{\mathsf{w}_2 = 0\} \cup \{\mathsf{w}_3 = 0\} \cup \{\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0\}$$
then the fiber over p is smooth.
- If p is in $\{\mathsf{w}_1 = 0\}, \{\mathsf{w}_2 = 0\}$, or $\{\mathsf{w}_3 = 0\}$, then the fiber over p is of type IV*. If p is a point in $\{\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0\}$, then the fiber over p is a nodal elliptic curve.
- If $p \in \{\mathsf{w}_1 = \mathsf{w}_2 = 0\}, \{\mathsf{w}_1 = \mathsf{w}_3 = 0\}$ or $\{\mathsf{w}_2 = \mathsf{w}_3 = 0\}$, then the fiber over p is of dimension 2.
- If $p = (0,0,0)$, then the fiber is a threefold. This threefold is precisely the restriction of the fibration π_3 to the strict transform of the exceptional $\mathbb{P}^2$ obtained by blowing up $(0,0,0)$.

Now we will let $\mathrm{LG}(Z_0)$ be the smooth resolution of the elliptically fibered threefold over T_3 described above. We compose the fibration map π_3 with the map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from $\mathbb{C}^3$ to $\mathbb{C}$, then we recover the map w on the open set that $\mathrm{LG}(Z_0)$ and Y^0 have in common. Then we obtain a nice description of the fiber in $\mathrm{LG}(Z_0)$ of w over 0 as a union of two elliptically fibered threefolds, one component being the threefold fiber over $(0,0,0)$ in Y, and the other being the natural elliptically fibered threefold obtained by taking the preimage of the line $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3 = 0$ in $\mathrm{LG}(Z_0)$ under the elliptic fibration map. These two threefolds intersect along a surface S which is naturally elliptically fibered. This surface can be described by taking the subvariety of the exceptional divisor $Q = \mathbb{P}^2$ given by a the natural fibration over a hyperplane in $\mathbb{P}^2$. This is an elliptically fibered surface over $\mathbb{P}^2$ with three singular fibers of type IV* and a order 3 torsion section.

Proposition 4.6. *The smooth K3 surface S of Picard rank 20 with transcendental lattice isomorphic to the (positive definite) root lattice A_2, which has Gram matrix*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

This can be proved using the techniques described in [HT15].

4.8. Cubic fourfolds blown up in a plane. We will apply a similar approach to describe the LG model of the cubic fourfold blown up in a plane. We start by expressing this as a toric hypersurface. Blowing up $\mathbb{P}^5$ in the intersection of three coordinate hyperplanes is again a smooth toric Fano variety $\mathbb{P}_\Delta$ which is determined by the polytope Δ with vertices given by points $\rho_1, \dots, \rho_7$ given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The vertices of this polytope (determined by the columns of the above matrix) determine torus invariant Cartier divisors in $\mathbb{P}_\Delta$, and the cubic blown up in a plane is linearly equivalent to $D_{\rho_3} + D_{\rho_4} + D_{\rho_5}$. Thus, following the prescription of Givental [Giv98] (or more precisely, [HD15]), one obtains the Landau-Ginzburg model with

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} = 1 \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with potential given by restriction of

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu}$$

to Y^0. We may decompose w into the three potentials

$$\mathsf{w}_1 = x, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

so that $\mathsf{w} = \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$. Therefore, if we take the map $\pi : Y^0 \to \mathbb{C}^3$ given by $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3)$, this can be compactified to a family of elliptic curves with fiber

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw(z + w - 1) + 1 + a\mathsf{w}_3 w = 0.$$

This can be written as a family of elliptic curves in Weierstrass form as

$$y^2 = x^3 + \mathsf{w}_1\mathsf{w}_2^2\mathsf{w}_3(\mathsf{w}_1\mathsf{w}_3 - 4a)x^2 + 8\mathsf{w}_1^3\mathsf{w}_2^3\mathsf{w}_3^3 x + 16\mathsf{w}_1^4\mathsf{w}_2^4\mathsf{w}_3^4.$$

Away from $(0, 0, 0)$, the singularities of this fibration can be resolved.

- I_1^* along $\mathsf{w}_1 = 0$ and $\mathsf{w}_2 = 0$
- IV^* along $\mathsf{w}_3 = 0$
- I_1 along

$$(a\mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3^2 - 8a^2\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3^2 + \mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3 + 16a^3\mathsf{w}_3^2 - 36a\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27\mathsf{w}_1\mathsf{w}_2) = 0$$

We first blow up the base $\mathbb{C}^3$ at $(0,0,0)$ to obtain a fibration with smooth fibers over the exceptional divisor. We cannot yet resolve singularities of this fibration, since the fibers over the intersection of any two coordinate hyperplanes do not have known resolutions. Following work of Miranda [Mir83], we may blow up the base of this fibration again several times in order to produce a fibration over a threefold which has a fiber-wise blow-up which resolves singularities.

We blow up the base along the lines $R_{ij} = \{\mathsf{w}_i = \mathsf{w}_j = 0\}$ to get three exceptional surfaces R_{ij} over which there are singular fibers generically of type IV. Blowing up again in all lines of intersection between R_{ij} and $\mathsf{w}_j = 0$ and R_{ij} and $\mathsf{w}_i = 0$, calling the resulting exceptional divisors $R_{ij,j}$ and $R_{ij,i}$, we get an elliptic fibration over this blown up threefold so that:

- I_1^* along $\mathsf{w}_1 = 0$ and $\mathsf{w}_2 = 0$
- IV^* along $\mathsf{w}_3 = 0$
- IV along R_{ij}.
- I_0 (i.e. smooth) along $R_{ij,j}$ and $R_{ij,i}$.
- I_1 along some divisor which does not intersect $\mathsf{w}_1 = 0, \mathsf{w}_2 = 0, \mathsf{w}_3 = 0$ or $R_{ij} = 0$.

Therefore, one may simply resolve singularities of this fibration in the same way as one would in the case of surfaces – blowing up repeatedly in sections over divisors in the discriminant locus. Let us refer to this elliptically fibered fourfold as $\mathrm{LG}(Z_1)$. There is an induced map from $\mathrm{LG}(Z_1)$ to $\mathbb{C}$ which we call w essentially comes from the composition of the fibration on $\mathrm{LG}(Z_1)$ by elliptic curves with its contraction onto $\mathbb{C}^3$ along with the addition map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from $\mathbb{C}^3$ to $\mathbb{C}$. This is the superpotential on $\mathrm{LG}(Z_1)$, and $\mathrm{LG}(Z_1)$ is a partially compactified version of the Landau-Ginzburg model of the cubic fourfold blown up in a plane.

The fiber of w over 0 is the union of two elliptically fibered smooth threefolds, one being the induced elliptic fibration over the proper transform of the exceptional divisor obtained when we blew up $(0,0,0)$ in $\mathbb{C}^3$. The other is the proper transform in $\mathrm{LG}(Z_1)$ of the induced elliptic fibration over the surface $z_1 + z_2 + z_3 = 0$ in $\mathbb{C}^3$.

These two threefolds meet transversally along a smooth K3 surface S. This K3 surface is equipped naturally with an elliptic fibration structure over $\mathbb{P}^1$ and inherits two singular fibers of type I_1^*, a singular fiber of type IV^* and two singular fibers of type I_1.

Proposition 4.7. *The orthogonal complement of the Picard lattice in $H^2(S,\mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

To prove this, one uses a concrete model of S and shows that there is another elliptic fibration on S so that the techniques in [HT15] can be applied to show that there is a lattice polarization on a generic such S by the lattice

$$E_8^2 \oplus \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}. \tag{4.1}$$

Then one shows that the complex structure on the surface S varies nontrivially as the parameter a varies, thus a generic such S has Picard lattice equal to exactly the lattice in Equation (4.1). Then applying standard results of Nikulin [Nik80], one obtains the proposition.

4.9. Cubic threefolds blown up in two planes. Here we begin with the toric variety $\mathbb{P}^5$ blown up at two disjoint planes, which is determined by the polytope Δ with vertices at the columns $\rho_1, \dots, \rho_8$ of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

The cubic blown up along two disjoint planes is then linearly equivalent to the torus invariant divisor $D_{\rho_3} + D_{\rho_4} + D_{\rho_5} + D_{\rho_7}$, therefore, by the prescription of Givental, we may write the associated LG model as

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu} + bxyz.$$

We split this into the sum of three functions,

$$\mathsf{w}_1 = x + bxyz, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

The fibers of the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3)$ from Y to $\mathbb{C}^3$ are written as a family of affine cubics

$$(z + w - 1)\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw + (1 + b\mathsf{w}_2 z)(1 + a\mathsf{w}_3 w) = 0$$

which are open elliptic curves. We may write this in Weierstrass form and use Tate's algorithm to show that, the singular fibers of this fibration are of types:

- I_1^* along $\mathsf{w}_3 = 0$ and $\mathsf{w}_2 = 0$
- I_5 along $\mathsf{w}_1 = 0$
- I_1 along a divisor determined by a complicated equation in $\mathsf{w}_1, \mathsf{w}_2$ and w_3.

Elsewhere, the fibers of this map can be compactified to smooth elliptic curves.

In order to obtain a smooth model of this fibration, we will first blow up $\mathbb{C}^3$ at $(0, 0, 0)$. The induced elliptic fibration is generically smooth over this exceptional divisor, which we call Q. In order to obtain a model of this elliptic fibration which we may resolve by sequentially blowing up in singular fibers, we must now blow up along the line $\mathsf{w}_2 = \mathsf{w}_3 = 0$. We will call the exceptional surface under this blow-up R_{23}. We obtain a singular elliptically fibered fourfold over this new threefold base so that the fibers over the divisor R_{23} are generically of Kodaira type IV. Blowing up again at the intersections of R_{23} and $\mathsf{w}_2 = 0$ and at the intersection of R_{23} and $\mathsf{w}_3 = 0$ (calling the exceptional divisors $R_{23,2}$ and $R_{23,3}$ respectively) we obtain a fibration which can be resolved by blowing up curves of divisors in the fibers over $R_{23}, \mathsf{w}_1 = 0, \mathsf{w}_2 = 0$ and $\mathsf{w}_3 = 0$, and by taking resolution over curves in $\mathsf{w}_1 = \mathsf{w}_2 = 0$ and $\mathsf{w}_1 = \mathsf{w}_3 = 0$ (following [Mir83, Table 14.1]). Call the resulting fibration $\mathrm{LG}(Z_2)$ and let π be the fibration map onto the blown up threefold. We have singular fibers of types:

- I_1^* along $\mathsf{w}_3 = 0$ and $\mathsf{w}_2 = 0$

- I_5 along $\mathsf{w}_1 = 0$
- IV along R_{23}
- Fibers over $\mathsf{w}_1 = \mathsf{w}_2 = 0$ and $\mathsf{w}_1 = \mathsf{w}_3 = 0$ of the type determined by Miranda [Mir83] and described explicitly in [Mir83, Table 14.1].
- I_1 along a complicated divisor which does not intersect any of the divisors above.

and smooth fibers otherwise.

The variety $\mathrm{LG}(Z_2)$ admits a non-proper elliptic fibration over $\mathbb{C}^3$ obtained by composing π with the blow-up maps described above. Then the fiber in $\mathrm{LG}(Z_2)$ over $(0, 0, 0)$ is an elliptic threefold over a blown-up $\mathbb{P}^2$ base. Composing this non-proper elliptic fibration with the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$ from $\mathbb{C}^3$ to $\mathbb{C}$ recovers the potential w. The fiber over 0 of the map w from $\mathrm{LG}(Z_2)$ to $\mathbb{C}$ has two components, each an elliptically fibered threefold meeting along a smooth K3 surface. This K3 surface, which we call S, admits an elliptic fibration over $\mathbb{P}^1$ canonically with two singular fibers of type I_1^*, a singular fiber of type I_5 and five singular fibers of type I_1.

Proposition 4.8. *The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

Again, this result is obtained by finding an appropriate alternative elliptic fibration on S and demonstrating that an appropriate lattice embeds into its Picard lattice, then combining results of Nikulin [Nik80] and the fact that there is a non-trivial 2-dimensional deformation of S obtained by letting the parameters a and b vary to see that indeed, this is the transcendental lattice of a generic such S.

Remark 4.9. In the last three sections, we have glossed over the issue of providing an appropriate relative compactification of our LG models with respect to w. Indeed, one wants to produce a relatively compact partial compactification of the LG models above whose total space is smooth and has at least trivial canonical class. In the cases that we have described above, this can be done by taking a relative compactification of $\mathbb{C}^3$ with respect to the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$ and writing $\mathrm{LG}(Z_i)$ as an elliptically fibered fourfold over this variety. Performing the same procedure as above (blowing up the base of this fibration until a global resolution can be obtained by simply blowing up in fibers or taking small resolutions as described by Miranda [Mir83], one can produce a partial compactification of $\mathrm{LG}(Z_i)$ so that the fibers of w are compact. Using the canonical bundle formula in [Mir83], one can then show that this compactification is indeed appropriate. We note that, strictly speaking, Miranda's work only applies to three dimensional elliptic fibrations. However, since we do not have to deal with intersections of more than two divisors in our discriminant locus, and all of our intersections are transverse, the arguments of [Mir83] still may be applied.

4.10. Special Lagrangian fibrations. In the case of hyperkähler surfaces, special Lagrangian fibrations can be constructed with relatively little difficulty. The procedure is outlined in work of Gross and Wilson [GW97]. We review their work in the following section and apply it to our examples.

Definition 4.10. *A K3 surface S is lattice polarized by a lattice L if there is a primitive embedding of L into* $\mathrm{Pic}(S)$ *whose image contains a pseudo-ample class.*

For a given lattice L of signature $(1, \rho - 1)$ for $\rho \leq 20$ which may be embedded primitively into $H^2(S, \mathbb{Z})$ for a K3 surface, there is a $(20 - \rho)$-dimensional space of complex structures on S corresponding to K3 surfaces which admit polarization by L. A generic L-polarized K3 surface will then be a general enough choice of complex structure in this space.

We will follow the notation of Gross and Wilson [GW97] from here on. We choose I to be a complex structure on a K3 surface S and let g be a compatible Kähler-Einstein metric. Since S is hyperkähler, there is an S^2 of complex structures on S which are compatible with g. We will denote by I, J and K the complex structures from which all of these complex structures are obtained. The complex 2-form associated to the complex structure I is written as $\Omega(u, v) = g(J(u), v) + ig(K(u), v)$ for u and v sections of T_S. The associated Kähler form is given, as usual, by $\omega(u, v) = g(I(u), v)$. Similarly, one may give formulas for the holomorphic 2-form and Kähler forms associated to the complex structures J and K easily in terms of the real and imaginary parts of Ω and ω as described in [GW97, pp. 510].

A useful result that Gross and Wilson attribute to Harvey and Lawson [HL82, pp. 154] is:

Proposition 4.11 ([GW97, Proposition 1.2])**.** *A two-dimensional submanifold Y of S is a special Lagrangian submanifold of S with respect to the complex structure I if and only if it is a complex submanifold with respect to the complex structure K.*

Using the same notation as in [GW97], we will let S_K be the complex K3 surface with complex structure K, which then has holomorphic 2-form given by $\Omega_K = \mathrm{Im}\Omega + i\omega$ where ω and Ω are as before. If this vanishes when restricted to a submanifold E of S, then we must have $\omega_{|E} = 0$ as well. If ω is chosen generically enough in the Kähler cone of S (so that $\omega \cap L = 0$) then this forces E to be in $L^\perp$. One can show that a complex elliptic curve E on a K3 surface satisfies $[E]^2 = 0$ therefore, since $L^\perp$ has no isotropic elements, S_K cannot contain any complex elliptic curves and thus S has no special Lagrangian fibration. Therefore, we have proven that:

Proposition 4.12. *If L is a lattice so that $L^\perp$ contains no isotropic element, then a generic L-polarized K3 surface with a generic choice of Kähler-Einstein metric g has no special Lagrangian fibration.*

We will use this to prove a theorem regarding K3 surfaces which appeared in the previous sections. Let us recall that the transcendental lattices of the K3 surface appearing as the intersection of the pair of divisors in $\mathrm{LG}(Z_0), \mathrm{LG}(Z_1)$ and $\mathrm{LG}(Z_2)$ are

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

In the first case, it is clear that the lattice is positive definite, therefore it cannot represent 0, and thus Proposition 4.12 shows that in this case there is no special Lagrangian fibration on this specific K3 surface. In the third case, we can use [GW97, Proposition 1.3] to see that there is a special Lagrangian fibration with numerical special Lagrangian section for a generic choice of Kähler-Einstein metric g.

In the second case, the discriminant of the lattice (which we will call M) is -8, and its discriminant group, which is just $M^\vee/M$, is isomorphic to $\mathbb{Z}/8$ and has generator with square $3/8$. Using a result of Nikulin [Nik80], it follows that this is not equivalent to the lattice $\langle -8\rangle \oplus U$. At the same time, one can conclude that this is not the lattice $\langle -2\rangle \oplus U(2)$, and therefore, we cannot directly apply [GW97, Proposition 1.3] to obtain a special Lagrangian fibration on such a K3 surface.

However, applying the method used in the proofs of [GW97, Proposition 1.1] and [GW97, Proposition 1.3], one obtains a special Lagrangian fibration on S for a generic choice of g so that there is no special Lagrangian section, but there is a numerical special Lagrangian 2-section. To do this, we use the fact that $(1,-1,1)$ is isotropic in this lattice.

Putting all of this together, we obtain the following theorem:

Theorem 4.13. *Let S be a generic K3 surface appearing as the intersection of the two components of the fiber over 0 of the LG models of a generic cubic Z_0, a cubic blown up in a plane Z_1, and a cubic blown up in two disjoint planes Z_2. Let ω be a generic Kähler class on S and Ω the corresponding holomorphic 2-form on S. Then:*

(1) In the case where $S \subseteq \mathrm{LG}(Z_0)$, then S admits no special Lagrangian torus fibration.

(2) In the case where $S \subseteq \mathrm{LG}(Z_1)$, then S admits a special Lagrangian torus fibration with no Lagrangian section but a (numerical) Lagrangian 2-section.

(3) In the case where $S \subseteq \mathrm{LG}(Z_2)$, then S admits a special Lagrangian torus fibration with a (numerical) Lagrangian section.

The first statement in Theorem 4.13 is mirror dual to the fact that the subcategory $\mathcal{A}_X$ of $\mathrm{D}^b(\mathrm{coh}\, X)$ for X a generic cubic fourfold is not the derived category of a K3 surface. The second statement corresponds to the fact that $\mathcal{A}_X \cong \mathrm{D}^b(S,\beta)$ for β an order 2 Brauer class on S for X a general cubic fourfold containing a plane. The third case corresponds to the fact that when X contains two disjoint planes, $\mathcal{A}_X \cong \mathrm{D}^b(S)$ for S a K3 surface.

According to [AAK12, Corollary 7.8], there is an embedding of the (derived) Fukaya category of the K3 surface S appearing in Theorem 4.13 as a subcategory of the derived version of the Fukaya-Seidel category of the LG model of Z_0, Z_1 and Z_2 respectively. The objects in the Fukaya-Seidel category of an LG model are so-called admissible Lagrangians, which are, roughly, Lagrangian submanifolds L of the LG model with (possible) boundary in a fiber V of w. In the case where w is a Lefschetz fibration, it is well-known (see [Sei01]) that such Lagrangians (so-called Lagrangian thimbles) can be produced by taking appropriate paths between V and p for p a critical value of w and tracing the image of the vanishing cycle at $\mathsf{w}^{-1}(p)$ along this path.

This embedding works as follows. The central fiber of our degeneration is simply a union of two smooth varieties meeting transversally in a K3 surface, so the vanishing cycle is simply an S^1 bundle over the critical locus of the degenerate fiber.

In our case, this is simply an S^1 bundle over a K3 surface, which is then homotopic to $S^1 \times \mathrm{K3}$. Thus, along any straight path approaching 0 in $\mathbb{C}$, we have a vanishing thimble homotopic to $D^2 \times \mathrm{K3}$ where D^2 is the two-dimensional disc. This, of course, cannot be a Lagrangian in $\mathrm{LG}(Z_i)$ for dimension reasons, but if instead we take all points in $D^2 \times \mathrm{K3}$ which converge to a Lagrangian ℓ in the K3 surface (in some appropriate sense), then there exists a Lagrangian thimble L_ℓ whose restriction to $\mathsf{w}^{-1}(0)$ is ℓ. In this way, Lagrangians in S extend to admissible Lagrangians in $\mathrm{LG}(X)$ and in particular induce a faithful A_∞-functor from the Fukaya category of S into the Fukaya-Seidel category of $\mathrm{LG}(Z_i)$, both with appropriate symplectic forms. In particular, we have that

(1) There is no admissible Lagrangian L in $\mathrm{LG}(Z_0)$ so that $L_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian torus.
(2) There is no pair of admissible Lagrangians L_1 and L_2 in $\mathrm{LG}(Z_1)$ so that $(L_1)_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian torus and $(L_2)_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian section of a special Lagrangian fibration on S.

These statements should be viewed as interpretations of Theorem 4.13 in terms of the Fukaya-Seidel category of Z_0, Z_1 and Z_2. As claimed in section 3, the non-existence of a family appropriate Lagrangians in the LG models of Z_0 and Z_1 therefore corresponds to the conjectural fact that Z_0 and Z_1 are non-rational.

5. Hybrid models and filtrations

In this section, we introduce a perverse sheaf of categories analog of unramified cohomology - hybrid models [Pir16]. We will associate with this hybrid model a Hodge type filtration - this is the invariant discussed in the main conjecture. Our consideration can be considered as generalizations of classical degenerations in Hodge theory.

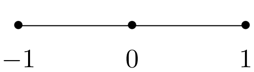
5.1. Filtration. Let $\mathcal{A}$ be an Artinian category and $Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R}$ an additive homomorphism.

Theorem 5.1. *For any object E in $\mathcal{A}$, there exists a filtration F_λ with the following properties:*

(1) $\bigcap_{\lambda\in\mathbb{R}} F_{\leqslant\lambda} = 0$*;*
(2) $\bigcup_{\lambda\in\mathbb{R}} F_{\leqslant\lambda} = E$*;*
(3) $F_{\lambda+1}/F_\lambda = \bigoplus G_\alpha$ *is semisimple and splits for every* λ*.*

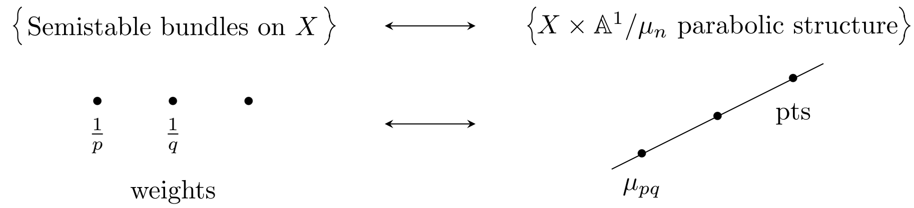
Example 5.2.

(1) (A_3) *Ob*$=\mathbb{C}[x]/x^3$.



The filtration here is $-1, 0, 1$.
(2) (A_7) *The filtration here is* $-3, -2, -1, 0, 1, 2, 3$.

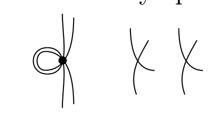
The above filtrations can be given the following interpretation by parabolic structures.



The multiplicity of the divisor over 0 is equal to the common multiple of all denominators. The points on this divisor determine the jumps of the filtration. This geometric interpretation suggests:

Theorem 5.3. *Let Cone($a \xrightarrow{\varphi} b$) be the cone of a and b with respect to the functor φ, then Filt(Cone($a \xrightarrow{\varphi} b$)) = superposition(Filt a, Filt b).*

One example with such filtration is the symplectic Lefschetz pencils.

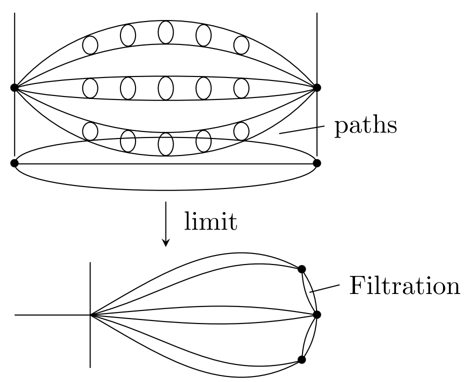


We have a symplectic pencil $(X, [\omega])$ for X a four dimensional compact symplectic manifold. Here $[\omega]$ is the symplectic form on the pencil. A symplectic Lefschetz pencil is defined by a word in the mapping class group.

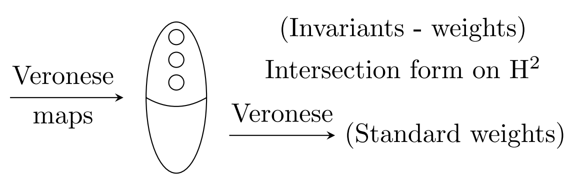
$$\mu : \pi_1(\mathbb{P}^1/\{p_1, \ldots, p_v\}) \to Map(g)$$

Here $Map(g)$ is the mapping class group of Riemann surfaces of genus g.

We consider a symplectic Lefschetz pencil as a perverse sheaf of categories over $\mathbb{P}^1$. An object in the Fukaya category of this symplectic pencil gives a graph Γ in the base along with a choice of singular Lagrangian in each smooth fiber over Γ. For example:

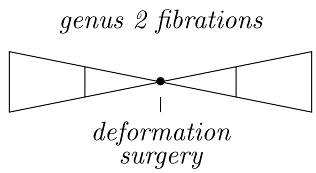


The asymptotic behavior of the above semistable Lagrangian under the mean curvature flow determines a filtration.



The asymptotic behavior of semistable Lagrangians added after the Veronese embedding reduces standard weights and does not affect initial symplectic invariants.

Conjecture 5.4. *The intersection form on* $\mathrm{H}^2(X)$ *determines the filtration.*

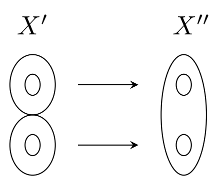


$$(\sigma_1, \ldots, \sigma_5)^{24} = 1 \qquad \longleftrightarrow \qquad (\sigma_1, \ldots, \sigma_6)^{20} = 1$$

Each equation determines a semistable Lagrangian. The filtrations associated with the two words in the mapping class group are different. This suggests that the above genus 2 Lefschetz pencils are not symplectomorphic. This is the A side application of our construction.

Our filtrations share many properties with classical weight filtrations. In particular we have the following strictness property.

Theorem 5.5.



Consider a fully faithful functor

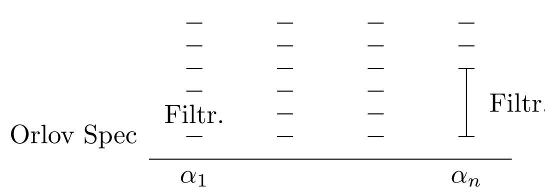
$$\mathcal{F} : \mathrm{FS}(X') \to \mathrm{FS}(X'')$$

so that the induced map $\mathrm{Ext}^1(o') \to \mathrm{Ext}^1(o'')$ *is injective. Here* o' *and* o'' *are semistable objects in* $\mathrm{FS}(X')$ *and* $\mathrm{FS}(X'')$*. Then we have a compatibility of filtrations under the functor* $\mathcal{F}$*. (Here* $\mathrm{FS}(X')$*,* $\mathrm{FS}(X'')$ *are 1-dimensional FS categories.)*

Corollary 5.6. *The filtration of FS*(X') *determines the filtration of FS*(LP)*.*

As a consequence of theorem 5.5, we can define a filtration for any generator of a category. In fact, we can associate a filtration with a generator corresponding to an element in the Orlov spectrum of a category.

For generator α, $\mathrm{Cone}(\alpha \xrightarrow{F} T) \longrightarrow$ sequence of filtrations on α.



Question 5.7. *Does this sequence of filtrations determine a categorical invariant?*

Now we consider a B side example, [MP16a] and [MP16b]. Let X be a smooth projective variety and D a divisor on it. Following [MP16a] and [MP16b], we define an object in $\mathrm{D^b}(X)$, $F_k\omega_x(*D)$.

$$j : U = X/D \hookrightarrow X$$
$$F_k\omega_x(*D) = \omega_x(k+1)D \otimes I_k(D) \quad \forall k >> 0.$$

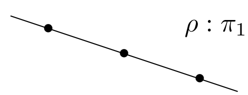
This is an example of filtrations discussed above.

Theorem 5.8. *The above filtration satisfies the cone and functoriality properties.*

Indeed let $H \subset X$ be a hypersurface. So $I_k(D_H) \leq I_k(D) \cdot \mathcal{O}_H$. We also have the cone property:

$$I_k(D_1 + D_2) \subseteq \sum_{i+j=k} I_i(D_1)I_j(D_2) \cdot \mathcal{O}_X(-jD_1 - jD_2).$$

5.2. Hybrid models. In this section, we take a brief look at the results of Pirutka, [Pir16]. Our considerations suggest that there are two new ways of constructing filtrations. Classically we can use the degenerations of cohomologies in order to obtain filtrations.



$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{GL}(\mathrm{H}^3)$$

Nilpotent degenerations produce classical filtrations.

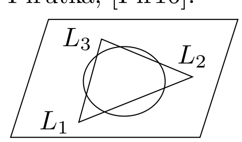
The examples of previous section suggest that we can extend the applications of this method from

$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{GL}(\mathrm{H}^3)$$

to

$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{Aut}(\mathrm{D^b}(\mathcal{F}_t)).$$

We propose a new possible way to create "interesting filtrations". We generalize the procedure suggested by A. Pirutka, [Pir16].

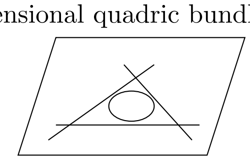


In her approach, Pirutka expresses the existence of nontrivial Brauer group via the combinatorics of the base of the nontrivial Del Pezzo fibration.

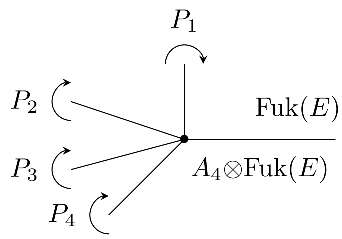
Our considerations in section 4 suggests the following:

Proposition 5.9. *The Pirutka condition can be represented as a filtration on semistable objects.*

Now we will look at 4-dimensional quadric bundles. We have a base:



with trivial nonramified cohomology. On the fiber we have a perverse sheaf of cohomology groups (see section 3).



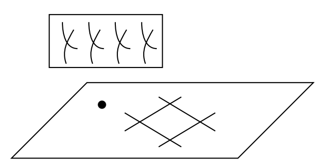
According to section 3, noncommutative deformations can be determined by changing spherical functors. One way of approaching rationality of quadric bundles could be to take a noncommutative deformation of the quadric bundle and compute its invariants. To deform a quadric bundle, one might consider a noncommutative deformation of the quadrics themselves, as described in Section 4. We can then try to understand what the unramified cohomology of such an object looks like to deduce non-rationality of the original quadric bundle.

Question 5.10. *Can we find an example of sheaf of noncommutative quadrics such that*

(1) Pirutka's invariant (unramified cohomology) is trivial;
(2) We have nontrivial filtrations on some semistable generator.

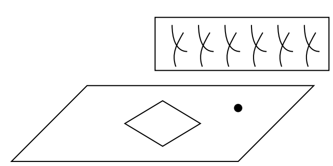
There are two important cases where this approach might bear fruit. These cases correspond to cubics containing extra algebraic cycles, for instance the quadric containing a plane described in section 4.4.

(1) Sheaves of quadrics over $\mathbb{P}^2$.



Question 5.11. *Can we find a deformation of* $\mathrm{D^b}(\mathcal{F}_t)$ *so that non-abelian Pirutka invariant is nontrivial?*

(2) Sheaves of Del Pezzo surfaces.

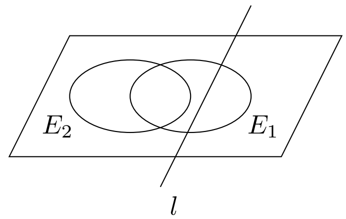


We get a hybrid model over $\mathbb{P}^2$ with fiber $\mathrm{D^b}(\mathcal{F}_t)$ - category of Del Pezzo surfaces.

Question 5.12. *Can we find a deformation of* $\mathrm{D^b}(\mathcal{F}_t)$ *so a noncommutative version of Pirutka's invariant is nontrivial?*

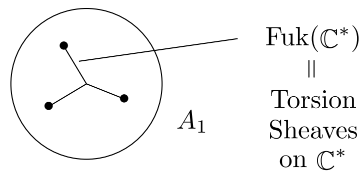
5.3. Artin-Mumford example. We can also look at the Artin-Mumford example [AM72] from the perspective of perverse sheaves of categories.

Recall that the classical Artin-Mumford example is a conic bundle over $\mathbb{P}^2$ with curves of degeneration $C = E_1 \cup E_2$, where E_1 and E_2 are smooth degree 3 curves.

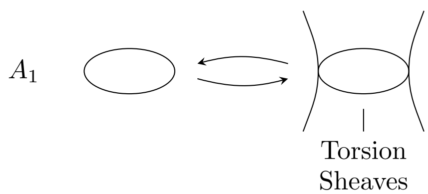


Let l be a line in $\mathbb{P}^2$. Over l we have a conic bundle. This conic bundle itself defines a perverse sheaf of categories as described below.

(⋆) A_2^{CY2}



The spherical functors are functors from A_1 to $\mathrm{Fuk}(\mathbb{C}^*)$, which is just the category of torsion sheaves on $\mathbb{C}^*$.



In terms of representations, we have classically:

$$\rho : \pi_1(\mathbb{P}^1/\,\mathrm{pts}) \to \mathrm{GL}\,\mathrm{H}^1(\mathbb{C}^*).$$

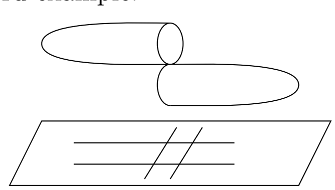
Categorically, our sets of spherical functors give

$$\rho : \pi_1(\mathbb{P}^1/\,\mathrm{pts}) \to \mathrm{Aut}\,\mathrm{Fuk}(\mathbb{C}^*),$$

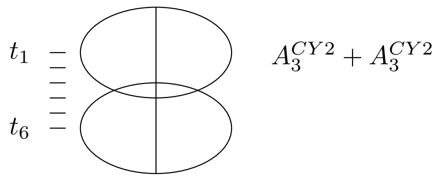
compared with sections 3.2 and 3.3. The second representation, along with the braid group representation of monodromy of the curve of degeneration of the Artin-Mumford threefold contains a wealth of information regarding the topology of the Artin-Mumford threefold. Since it is the topology of this threefold which determines its non-rationality, we should be able to recover the main theorem of [AM72] from this perverse sheaf of categories.

This gives us possibilities for non-commutative deformations. We start with:

(1) Classical Artin-Mumford example.

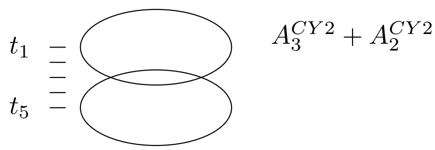


The Pirutka type configuration leads to nontrivial torsion in H^3, [Pir16] (see also [HT16], [HPT16a], [HPT16b], [AHTVA16]). Artin-Mumford's construction can be reproduced using the technique of PSC. Instead of the classical monodromy, we use the spherical functors in the PSC to construct a cycle with linking number $\frac{1}{2}$. Here A_3^{CY2} are 2-dimensional CY categories constructed in ($\star$).



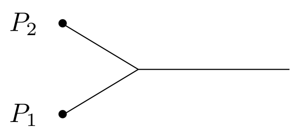
This amounts to a semistable Lagrangian with strictly quasi-unipotent monodromy (asymptotics).

(2) Smooth cubic, compared with section 4.3. In this case we start with the hybrid model described below:



The conic bundle has a curve of degeneration consisting of a quadric and a cubic in $\mathbb{P}^2$. The linking number is 0. So the monodromy is strictly unipotent.

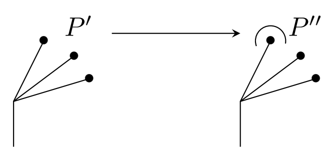
(3) Let us consider now the PSC A_2^{CY2} associated with a conic.



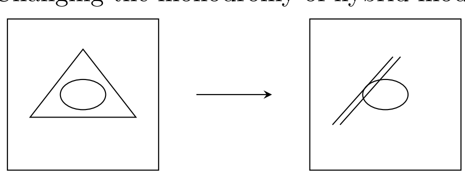
We deform this PSC so that the spherical functor in $\mathbb{P}^2$ does not belong to $\mathrm{GL}(\mathrm{H}^1)$. In such a way we produce a strictly non-unipotent filtration for the noncommutative deformation of the PSC associated with the quadric. This leads to a nontrivial torsion in H^3, compared with section 4.3.

5.4. Conclusions. In conclusion, we can say the following: the construction of hybrid models gives new directions of deforming PSC .

(1) (Monodromy 1) Deforming PSC of the fiber of hybrid model, see section 3.

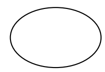


(2) (Monodromy 2) Changing the monodromy of hybrid models, see section 4.



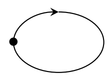
We have a categorical version of unramified cohomology - hybrid models with monodromies and filtrations. The main conjecture states that these filtrations produce new birational invariants. More details will be given elsewhere.

5.5. Final example. We give one more example. It is known, see [AAE+13], that $\mathrm{F}_{\mathrm{wrap}}(\mathbb{C}^*) = \mathrm{D}^{\mathrm{b}}(\mathbb{C}^*)$. The object $\mathbb{C}(t)/(t-a)^n$ corresponds to a loop with holonomy:



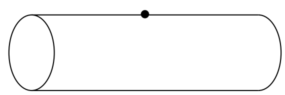
$$\begin{pmatrix} 0 & 1 & \\ 0 & & 1 \\ n & & & 0 \end{pmatrix}$$

On the B side we have a quiver with a relation $l^n = 1$:



The flow of (E, h) creates a filtration of E.

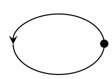
$L = na$



Here $E = \mathrm{H}^0(na)$ and the filtration on E is coming from the action of

$$\begin{pmatrix} 0 & 1 & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

On the A side we have

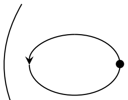


$\mathrm{HF}(L, L') = \mathrm{H}(na)$

with holonomy

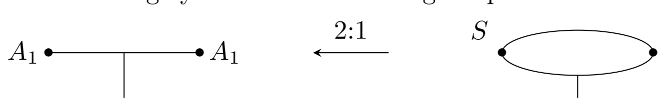
$$\begin{pmatrix} 0 & 1 & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The above cycle

$S = \Big($  $,\ \begin{pmatrix} 0 & 1 & \\ & & 1 & \\ & & & 0 \end{pmatrix} \Big)$

$l^n = 1$

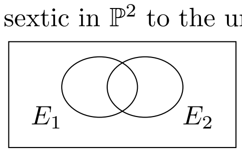
can be seen as a vanishing cycle of the base change of perverse sheaf of categories.



Instead of the vanishing cycle A_1, we have a vanishing cycle S.

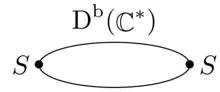
Based on that we propose now a hybrid model associated with the construction in section 4.4 - 4-dimensional cubic containing a plane.

(1) We degenerate the sextic in $\mathbb{P}^2$ to the union of two elliptic curves $E_1 \cup E_2$.

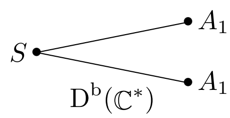


$\deg E_i = 3$

(2) We put a sheaf of categories over $\mathbb{P}^2$. Over a point on E_i we put the following category:



and over generic point we put the following:

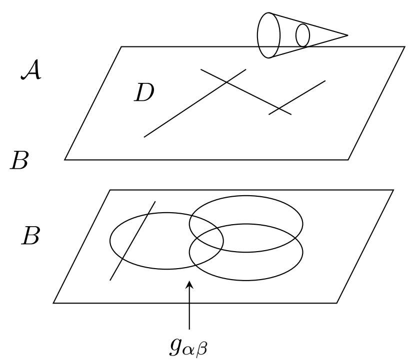


It is rather clear that the hybrid model above produces nontrivial filtration. It is an intriguing question to use Artin-Mumford's idea in the case of the above hybrid model in order to prove the non-rationality of 4-dim cubics.

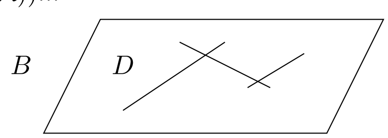
In what follows we make a parallel between

$$\left\{\begin{matrix}\text{Unramified}\\ \text{cohomology}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Cohomologies of}\\ \text{sheaves of categories}\end{matrix}\right\}.$$

We start with a PSC with base B and ramification divisor D, complex of dim $\geqslant 2$.



We associate a cycle $\{g_{\alpha\beta}\}$ with PSC in $\mathbb{H}^0(Base, \mathrm{Ext}^1(\mathcal{A},\mathcal{A}))$, $\mathbb{H}^2(Base, \mathrm{Ext}^{-1}(\mathcal{A},\mathcal{A}))$ and $\mathbb{H}^3(Base, \mathrm{Ext}^{-2}(\mathcal{A},\mathcal{A}))$...



Similarly we have $\mathbb{H}^i(\mathrm{D}, \mathrm{Ext}^{-(i-1)}(\mathcal{A},\mathcal{A}))$.

Conjecture 5.13. *To a semistable object we correspond:*

(1) Harmonic sections of metrized objects;
(2) Cohomology classes $\mathbb{H}^i(Base, Ext^{-(i-1)}(\mathcal{A},\mathcal{A}))$*;*
(3) Cohomology classes $\mathbb{H}^i(D, Ext^{-(i-1)}(\mathcal{A},\mathcal{A}))$.

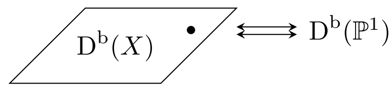
Conjecture 5.14. *Let* $\mathcal{F}$ *be a deformation of PSC associated with a quadric bundle* X*. If there exists an object* $\mathcal{A}$ *in* $\mathcal{F}$ *s.t.*

(1) $\mathbb{H}^i(Base, Ext^{-(i-1)}(\mathcal{A},\mathcal{A})) = 0$*;*
(2) $\mathbb{H}^i(D, Ext^{-(i-1)}(\mathcal{A},\mathcal{A})) = 0$.

Then X *is rational.*

The examples from section 4 and section 5 seem to suggest that monodromy (the filtration) is an obstruction to the existence of such a section. In such a way these examples serve as some justification for conjecture 1.1.

Remark 5.15. Instead of perverse sheaves of categories, on can consider holomorphic fibrations. For example, we can consider



$\mathbb{P}^1$-fibration over $\mathrm{D^b}(X)$. (We will call such fibrations Schulers.)

The result is a fibration of categories which looks like the quiver of $D^b(\mathbb{P}^1)$ but with vertexes $D^b(X)$.

$$\mathrm{D^b}(X) \rightrightarrows \mathrm{D^b}(X)$$

One can also combine the PSC construction with the above construction and increase the possibility for deformations.

References

[AAE+13] Mohammed Abouzaid, Denis Auroux, Alexander I. Efimov, Ludmil Katzarkov, and Dmitri Orlov. Homological mirror symmetry for punctured spheres. *J. Amer. Math. Soc.*, 26(4):1051–1083, 2013.

[AAK12] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov. Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces. *arXiv preprint arXiv:1205.0053*, 2012.

[AAKO] Mohammed Abouzaid, Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. In preparation.

[AHTVA16] Nicolas Addington, Brendan Hassett, Yuri Tschinkel, and Anthony Várilly-Alvarado. Cubic fourfolds fibered in sextic del pezzo surfaces. *arXiv preprint arXiv:1606.05321*, 2016.

[AM72] Michael Artin and David Mumford. Some elementary examples of unirational varieties which are not rational. In *Proc. London math. soc.(3)*, volume 25, page 3, 1972.

[ATVdB91] M. Artin, J. Tate, and M. Van den Bergh. Modules over regular algebras of dimension 3. *Invent. Math.*, 106(2):335–388, 1991.

[BP93] A. I. Bondal and A. E. Polishchuk. Homological properties of associative algebras: the method of helices. *Izv. Ross. Akad. Nauk Ser. Mat.*, 57(2):3–50, 1993.

[CTP14] Jean-Louis Colliot-Thélène and Alena Pirutka. Hypersurfaces quartiques de dimension 3: non rationalit\'e stable. *arXiv preprint arXiv:1402.4153*, 2014.

[DHNT15] Charles F Doran, Andrew Harder, Andrey Y Novoseltsev, and Alan Thompson. Calabi-yau threefolds fibred by mirror quartic k3 surfaces. *arXiv preprint arXiv:1501.04019*, 2015.

[Giv98] Alexander Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics*, pages 141–175. Springer, 1998.

[GKR12] Mark Gross, Ludmil Katzarkov, and Helge Ruddat. Towards mirror symmetry for varieties of general type. *arXiv preprint arXiv:1202.4042*, 2012.

[Gro98] Mark Gross. Special lagrangian fibrations ii: geometry. *arXiv preprint math/9809072*, 1998.

[Gro01] Mark Gross. Special lagrangian fibrations i: Topology. *AMS IP STUDIES IN ADVANCED MATHEMATICS*, 23:65–94, 2001.

[GW97] Mark Gross and PMH Wilson. Mirror symmetry via 3-tori for a class of calabi-yau threefolds. *Mathematische Annalen*, 309(3):505–531, 1997.

[Har16] Andrew Harder. The geometry of landau-ginzburg models. *PhD thesis, University of Alberta*, 2016.

[Has99] Brendan Hassett. Some rational cubic fourfolds. *Journal of Algebraic Geometry*, 8:103–114, 1999.

[HD15] Andrew Harder and Charles F Doran. Toric degenerations and the laurent polynomials related to givental's landau-ginzburg models. *arXiv preprint arXiv:1502.02079*, 2015.

[HK] Andrew Harder and Ludmil Katzarkov. Perverse sheaves of categories and noncommutative deformations. In preparation.

[HKKP] Fabian Haiden, Ludmil Katzarkov, Maxim Kontsevich, and Pranav Pandit. In preparation.

[HL82] Reese Harvey and H Blaine Lawson. Calibrated geometries. *Acta Mathematica*, 148(1):47–157, 1982.

[HPT16a] Brendan Hassett, Alena Pirutka, and Yuri Tschinkel. Stable rationality of quadric surface bundles over surfaces. *arXiv preprint arXiv:1603.09262*, 2016.

[HPT16b] Brendan Hassett, Alena Pirutka, and Yuri Tschinkel. A very general quartic double fourfold is not stably rational. *arXiv preprint arXiv:1605.03220*, 2016.

[HT15] Andrew Harder and Alan Thompson. The geometry and moduli of k3 surfaces. In *Calabi-Yau Varieties: Arithmetic, Geometry and Physics*, pages 3–43. Springer, 2015.

[HT16] Brendan Hassett and Yuri Tschinkel. On stable rationality of fano threefolds and del pezzo fibrations. *arXiv preprint arXiv:1601.07074*, 2016.

[KKP14] Ludmil Katzarkov, Maxim Kontsevich, and Tony Pantev. Bogomolov-tian-todorov theorems for landau-ginzburg models. *arXiv preprint arXiv:1409.5996*, 2014.

[KLa] Ludmil Katzarkov and Yijia Liu. Categorical base loci and spectral gaps, via okounkov bodies and nevanlinna theory. *String-Math 2013 88 (2014): 47.*

[KLb] Ludmil Katzarkov and Yijia Liu. Sheaf of categories and categorical donaldson theory. *In preparation.*

[KL15] Alexander Kuznetsov and Valery A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, (13):4536–4625, 2015.

[KNPS13] Ludmil Katzarkov, Alexander Noll, Pranav Pandit, and Carlos Simpson. Harmonic Maps to Buildings and Singular Perturbation Theory. *arXiv preprint arXiv:1311.7101*, 2013.

[KNPS15] Ludmil Katzarkov, Alexander Noll, Pranav Pandit, and Carlos Simpson. Harmonic maps to buildings and singular perturbation theory. *Communications in Mathematical Physics*, 336(2):853–903, 2015.

[KS16] Mikhail Kapranov and Vadim Schechtman. Perverse sheaves and graphs on surfaces. *arXiv preprint arXiv:1601.01789*, 2016.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, pages 219–243. Springer, 2010.

[Mir83] Rick Miranda. Smooth models for elliptic threefolds. *The birational geometry of degenerations (Cambridge, Mass., 1981)*, 29:85–133, 1983.

[MP16a] Mircea Mustata and Mihnea Popa. Hodge ideals. *arXiv preprint arXiv:1605.08088*, 2016.

[MP16b] Mircea Mustata and Mihnea Popa. Restriction, subadditivity, and semicontinuity theorems for hodge ideals. *arXiv preprint arXiv:1606.05659*, 2016.

[Nik80] Vyacheslav Valentinovich Nikulin. Integral symmetric bilinear forms and some of their applications. *Izvestiya: Mathematics*, 14(1):103–167, 1980.

[Orl92] Dmitri Olegovich Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 56(4):852–862, 1992.

[Pir16] Alena Pirutka. Varieties that are not stably rational, zero-cycles and unramified cohomology. *arXiv preprint arXiv:1603.09261*, 2016.

[Pol97] A. Polishchuk. Algebraic geometry of Poisson brackets. *J. Math. Sci. (New York)*, 84(5):1413–1444, 1997. Algebraic geometry, 7.

[Sei01] Paul Seidel. Vanishing cycles and mutation. In *European Congress of Mathematics*, pages 65–85. Springer, 2001.

[Voi15] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

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