Cohomological Hall algebras, semicanonical bases and Donaldson-Thomas invariants for 2-dimensional Calabi-Yau categories

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to Maxim Kontsevich for his 50th anniversary

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1 Introduction

Representations of a quiver Q give rise in general to the category of cohomological dimension one. In this sense quivers are analogous to curves.

In a similar vein representations of the preprojective algebra Π_Q give rise in general to the category of cohomological dimension two. Since preprojective algebras are obtained as simplectic reductions, we can say that they are analogous to Higgs bundles on a curve. Likewise the deformed preprojective algebras Π^λ are analogous to algebraic vector bundles with connections on a curve. The interplay between algebraic and geometric sides of this dictionary is a subject of many papers (here are few random samples: [2], [7], [17]).

The above categories can be "upgraded" to 3-dimensional Calabi-Yau categories. On the algebraic side it can be achieved by constructing a "triple" quiver $\widehat{Q}$ endowed with a cubic potential W, see e.g. [16] or Section 2 below. On the geometric side one takes the total space of an appropriate rank two vector bundle on a curve or the total space of the anticanonical bundle on a surface. From the point of view of, say, gauge theory, the lower-dimensional categories can be thought of as "dimensional reductions" of the corresponding 3-dimensional Calabi-Yau categories.

The framework of 3-dimensional Calabi-Yau categories is appropriate for the theory of motivic Donaldson-Thomas invariants (see [12], [13]). The above-mentioned dimensional reduction gives rise to the corresponding theory in a lower dimension. It is natural to ask about the meaning of the objects arising as a result of such dimensional reduction. Our paper illustrates this philosophy in two examples.

In our first example we discuss semicanonical bases (see [14], [15]) from the point of view of Cohomological Hall algebras (see [13]). In our second example we formulate a conjecture about an analog of the Kac polynomial of a 2-dimensional Calabi-Yau category. The conjecture is motivated by [16] in which the motive of the stack of indecomposable representations of a quiver (Kac polynomial) was expressed in terms of the motives of stacks of representations of the corresponding preprojective algebra and the Donaldson-Thomas invariants of the corresponding $3CY$ category.

It is known after Lusztig (see [14], [15]) that semicanonical bases can be interpreted in terms of top-dimensional components of the stack of nilpotent representations of the preprojective algebra Π_Q associated with the quiver Q. On the other hand, the critical loci of $Tr(W)$ contains the stack of representations of Π_Q. This gives an idea that the semicanoniccal basis can be derived from the pair $(\widehat{Q}, W)$.

Indeed, the pair $(\widehat{Q}, W)$ gives rise to the corresponding Cohomological Hall algebra (COHA for short), see [13]. It is natural to transport the associative product from the COHA to the vector space generated by the above-mentioned top-dimensional components (in a more invariant way one can speak about the Galois-invariant part of COHA, cf. [13]). We will show that in this way we obtain an associative algebra endowed with a basis which enjoys the properties of the semicanonical one.

From the point of view of the above dictionary between algebra and geometry, the preprojective algebra Π_Q should be replaced in general by a 2-dimensional Calabi-Yau category ($2CY$ category for short), while the pair $(\widehat{Q}, W)$ should be replaced by a 3-dimensional Calabi-Yau category ($3CY$ category for short). The "dimensional reduction" of this $3CY$ categories to a 2-dimensional category (as well as the relation between the corresponding cohomology groups of stacks of objects) was described in [13], Section 4.8. One can expect a similar story in the general categorical framework.

In particular we expect that the notion of semicanonical basis has intrinsic categorical meaning for $2CY$ categories. A class of such categories is proposed in Section 3.2. The categories from our class are parametrized by quivers.

One can also hope that the reduction from $3CY$ categories to 2-dimensional categories will help to understand the relation between motivic Donaldson-Thomas theory (see [12], [13]) and some invariants of $2CY$ categories, e.g. Kac polynomials.[1]

It is not surprising that some notions introduced for Kac-Moody algebras can be interpreted in terms of $2CY$ categories, since Kac-Moody algebras arise naturally in the framework of $2CY$ categories generated by spherical collections a'la [13]. Spherical objects generate a t-structure of a $2CY$ category. Its heart corresponds to a Borel subalgebra, and the spherical objects themselves correspond to simple roots. In a similar vein Schur objects correspond to positive roots, etc. Change of the t-structure corresponds to the change of the cone of positive roots (and hence to the change of Borel subalgebra in the corresponding Kac-Moody algebra). Reflections at spherical objects correspond to the action of the Weyl group. Space of stability functions (i.e. central charges) on a fixed t-structure corresponds to the Cartan subalgebra. It contains the lattice which is dual to the K-theory classes

[1] The relationship between motivic DT-invariants and Kac polynomials was studied in many papers, including [6], [16], [8].

of spherical objects. The Euler bilinear form on the K-group of the category (it is symmetric due to the $2CY$ condition) corresponds to the Killing form on the Cartan subalgebra. The moduli stack of objects of the heart of the t-structure is symplectic, and it contains a Lagrangian substack (in fact subvariety). [2]

Notice that every $2CY$ category gives rise to a $3CY$ category with the trivial Euler form. Geometrically, the underlying Calabi-Yau 3-fold is the product $S \times \mathbb{A}^1$, where S is a Calabi-Yau surface. Upgrade from Π_Q to $(\widehat{Q}, W)$ illustrates the algebraic side of the dictionary (in general the $3CY$ category should have a t-structure with the heart containing pairs (E, f), where E is an object of the heart of a fixed t-structure of our $2CY$ category and $f \in Hom(E, E)$).

Motivic Donaldson-Thomas invariants of such "upgraded" $3CY$ categories do not change inside of a connected component of the space of stability conditions. As a result, the DT-invariants are in fact invariants of the t-structure of the underlying $2CY$ category.

Finally, we remark that there are $2CY$ categories which have purely geometric origin (e.g. the category of coherent sheaves on a $K3$ surface). The above questions about semicanonical basis or Kac polynomial make sense in the geometric case as well.

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2 COHA and semicanonical basis

We start by recalling the definition of the product on the critical COHA proposed in [13]. For the convenience of the reader we will closely follow the

[2]This Lagrangian subvariety should be compared with Lusztig's Lagrangian nilpotent cone, whose irreducible components correspond to positive roots. In the spirit of the analogy with Higgs bundles it is similar to the Laumon nilpotent cone.

very detailed exposition from [5] which contains proofs of several statements sketched in [13] as well as several useful improvements of the loc.cit.

2.1 Reminder on the critical cohomology

Let Y be a complex manifold, and $f : Y \to \mathbb{C}$ a holomorphic function. We define the vanishing cycles functor φ_f as follows: $\varphi_f \mathcal{F}[-1] := (R\Gamma_{\{Re(f)\leq 0\}}\mathcal{F})_{f^{-1}(0)}$. This is a nonstandard definition of this functor, which is equivalent to the usual one in the complex case. There is an isomorphism $\mathbb{Q}_Y \otimes \mathbb{T}^{\dim Y} \to D\mathbb{Q}_Y$, where $\mathbb{T}$ is the Tate motive, and D is the Verdier duality. The above isomorphism induces an isomorphism

$$\varphi_f \mathbb{Q}_Y \otimes \mathbb{T}^{\dim Y} \to \varphi_f D\mathbb{Q}_Y. \tag{1}$$

If $g : Y' \to Y$ is a map between manifolds, then the natural transformation of functors $\Gamma_{\{Re(f)\leq 0\}} \to g_* \Gamma_{\{Re(fg)\leq 0\}} g^*$ induces a natural transformation

$$\varphi_f \to g_* \varphi_{fg} g^*. \tag{2}$$

Assume that g is an affine fibration, then by [5, Cor. 2.3] there is a natural equivalence

$$\varphi_f g_! g_* \to g_! \varphi_{fg} g_*. \tag{3}$$

Definition 2.1. *For any submanifold $Y^{sp} \subset Y$, the critical cohomology with compact support $H_c^{\bullet,crit}(Y^{sp}, f)$ is defined as the cohomology of the following object in $\mathscr{D}^b(\boldsymbol{MMHS})$ ($\boldsymbol{MMHS}$ denotes the category of monodromic mixed Hodge structures):*

$$(\mathbb{C}^* \to \mathbb{A}^1)_! (Y^{sp} \times \mathbb{C}^* \to \mathbb{C}^*)_! (Y^{sp} \times \mathbb{C}^* \to Y \times \mathbb{C}^*)^* \varphi_{\frac{f}{u}} \mathbb{Q}_{Y \times \mathbb{C}^*},$$

where u is the coordinate on $\mathbb{C}^*$.

Let $Y = X \times \mathbb{A}^n$ be the total space of the trivial vector bundle, endowed with the $\mathbb{C}^*$-action that acts trivially on X and with weight one on $\mathbb{A}^n$. Let $f : Y \to \mathbb{A}^1$ be a $\mathbb{C}^*$-equivariant holomorphic function, where $\mathbb{C}^*$ acts with weight one on $\mathbb{A}^1$. Then $f = \sum_{k=1}^{k=n} f_k x_k$, where $\{x_k, k = 1, ..., n\}$ is a linear coordinate system on $\mathbb{A}^n$, and f_k are functions on X. Let $Z \subset X$ be the reduced scheme which is the vanishing locus of all functions f_k. Then Z is independent of the choice of x_k. Let $\pi : Y \to X$ be the natural projection, and $i : Z \to X$ be the closed inclusion.

Theorem 2.2. *(see* [5, Cor. A.6]*) There is a natural isomorphism of functors in $\mathscr{D}^b(\boldsymbol{MHM}(X))$:*

$$\pi_! \varphi_f \pi^* \xrightarrow{\sim} \pi_! \pi^* i_* i^*.$$

In particular,

$$H_c^{\bullet,crit}(Y, f) \simeq H_c^\bullet(Z \times \mathbb{A}^n, \mathbb{Q}) \simeq H_c^\bullet(Z, \mathbb{Q}) \otimes \mathbb{T}^n.$$

Here $\boldsymbol{MHM}(X)$ denotes the category of mixed Hodge modules on X.

If $Y_i = X_i \times \mathbb{A}^{n_i}$ with equivariant functions f_i satisfy the above conditions for $i = 1, 2$, then we have

Theorem 2.3. *(see* [5, Prop. A.5]*)*
The following diagram of isomorphisms commutes:

$$\begin{array}{ccc} H_c^{\bullet,crit}(Y_1 \times Y_2, f_1 \boxplus f_2) & \xrightarrow{TS} & H_c^{\bullet,crit}(Y_1, f_1) \otimes H_c^{\bullet,crit}(Y_2, f_2) \\ \downarrow & & \downarrow \\ H_c^\bullet(Z_1 \times Z_2 \times \mathbb{A}^{n_1+n_2}, \mathbb{Q}) & \xrightarrow{Ku} & H_c^\bullet(Z_1 \times \mathbb{A}^{n_1}, \mathbb{Q}) \otimes H_c^\bullet(Z_2 \times \mathbb{A}^{n_2}, \mathbb{Q}) \end{array}$$

Here TS denotes the Thom-Sebastiani isomorphism, and Ku the Künneth isomorphism (see loc.cit.).

Corollary 2.4. *(see* [5, Cor. A.7]*)*
Let $X^{sp} \subset X$ be a subvariety of X and $Y^{sp} = X^{sp} \times \mathbb{A}^n$, $Z^{sp} = Z \cap X^{sp}$. There is a natural isomorphism in ***MMHS***

$$H_c^{\bullet,crit}(Y^{sp}, f) \simeq H_c^\bullet(Z^{sp} \times \mathbb{A}^n, \mathbb{Q}).$$

The above statements also hold in equivariant case. Let us recall that framework. Assume that Y is a G-equivariant vector bundle over X, where G is an algebraic group embedded in $GL(n, \mathbb{C})$, and $f : Y \to \mathbb{A}^1$ is G-invariant. Let $fr(n, N)$ be the space of n-tuples of linearly independent vectors in $\mathbb{C}^N$ for $N \geqslant n$, and $\overline{(Y, G)}_N := Y \times_G fr(n, N)$. We denote the induced function by $f_N : \overline{(Y, G)}_N \to \mathbb{A}^1$. For a G-invariant closed subset $Y' \subset Y$, we define the equivariant cohomology with compact support by $H_{c,G}^{\bullet,crit}(Y', f) := \lim_{N\to\infty} H_c^{\bullet,crit}(Y'_N, f_N) \otimes \mathbb{T}^{-dim(fr(n,N))}$, where $Y'_N \subset \overline{(Y, G)}_N$ is the subspace of points projected to Y'.

Theorem 2.5. *(see [5, Cor. A.8]) Let $Y^{sp} = X^{sp} \times \mathbb{A}^n$ be the total space of a sub G-bundle. Then there is an isomorphism in* ***MMHS***

$$H^{\bullet,crit}_{c,G}(Y^{sp}, f) \simeq H^{\bullet}_{c,G}(Z^{sp} \times \mathbb{A}^n, \mathbb{Q}).$$

Moreover, the following diagram of isomorphisms commutes:

$$\begin{array}{ccc} H^{\bullet,crit}_{c,G}(Y_1^{sp} \times Y_2^{sp}, f_1 \boxplus f_2) & \xrightarrow{TS} & H^{\bullet,crit}_{c,G}(Y_1^{sp}, f_1) \otimes H^{\bullet,crit}_{c,G}(Y_2^{sp}, f_2) \\ \downarrow & & \downarrow \\ H^{\bullet}_{c,G}(Z_1^{sp} \times Z_2^{sp} \times \mathbb{A}^{n_1+n_2}, \mathbb{Q}) & \xrightarrow{Ku} & H^{\bullet}_{c,G}(Z_1^{sp} \times \mathbb{A}^{n_1}, \mathbb{Q}) \otimes H^{\bullet}_{c,G}(Z_2^{sp} \times \mathbb{A}^{n_2}, \mathbb{Q}) \end{array}$$

2.2 COHA and preprojective algebras

Let Q be a quiver with the set of vertices I and the set of arrows Ω. We denote by $a_{ij} \in \mathbb{Z}_{\geqslant 0}$ the number of arrows from i to j for $i, j \in I$.

One constructs the double quiver $\overline{Q}$, the preprojective algebra Π_Q, and the triple quiver with potential $(\widehat{Q}, W)$ as follows. For any arrow $a : i \to j \in \Omega$, we add an inverse arrow $a^* : j \to i$ to Q to get $\overline{Q}$, then $\Pi_Q = \mathbb{C}Q/\sum_{a\in\Omega}[a, a^*]$. Adding loops $l_i : i \to i$ at each vertex $i \in I$ to $\overline{Q}$ gives us the "triple" quiver $\widehat{Q}$. It is endowed with cubic potential $W = \sum_{a\in\Omega}[a, a^*]l$, where $l = \sum_{i\in I} l_i$. For any dimension vector $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I_{\geqslant 0}$ we have the following algebraic varieties:

a) the space $\mathbf{M}_{\overline{Q},\gamma}$ of representations of the double quiver $\overline{Q}$ in coordinate spaces $(\mathbb{C}^{\gamma^i})_{i\in I}$;

b) the similar space of representations $\mathbf{M}_{\Pi_Q,\gamma}$ of Π_Q;

c) the similar space of representations $\mathbf{M}_{\widehat{Q},\gamma}$ of $\widehat{Q}$.

All these spaces of representations are endowed with the action by conjugation of the complex algebraic group $\mathbf{G}_\gamma = \prod_{i\in I} GL(\gamma^i, \mathbb{C})$.

Let $\chi_Q(\gamma_1, \gamma_2) = \chi(Ext^\bullet(x_1, x_2)) = -\sum\limits_{i,j\in I} a_{ij}\gamma_1^i\gamma_2^j + \sum\limits_{i\in I}\gamma_1^i\gamma_2^i$ be the Euler form on the K_0 group of the category of finite dimensional representations of Q, where x_1 and x_2 are arbitrary representations of Q of dimension vectors γ_1 and γ_2 respectively.

In the context of the previous section, let $X = \mathbf{M}_{\overline{Q},\gamma}$, $Y = \mathbf{M}_{\widehat{Q},\gamma} = \mathbf{M}_{\overline{Q},\gamma} \times \mathbb{A}^{\gamma\cdot\gamma}$ (dot denotes the inner product), and $f = Tr(W)_\gamma = \sum\limits_{i\in I, k=1,\dots,(\gamma^i)^2} f_{ik}x_{ik}$, where f_{ik} are functions on $\mathbf{M}_{\overline{Q},\gamma}$, and $\{x_{ik}\}$ is a linear coordinate system on $\mathbb{A}^{\gamma\cdot\gamma}$. Then $Z = \mathbf{M}_{\Pi_Q,\gamma}$. Denote by $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ the space of representations of Π_Q in coordinate spaces of dimension $\gamma_1+\gamma_2$ such that the standard coordinate subspaces of dimension γ_1 form a subrepresentation. The group $\mathbf{G}_{\gamma_1,\gamma_2} \subset \mathbf{G}_\gamma$ consisting of transformations preserving subspaces $(\mathbb{C}^{\gamma_1^i} \subset \mathbb{C}^{\gamma^i})_{i\in I}$ acts on $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$. Suppose that we are given a collection of $\mathbf{G}_\gamma$-invariant closed subsets $\mathbf{M}^{sp}_{\overline{Q},\gamma} \subset \mathbf{M}_{\overline{Q},\gamma}$ satisfying the following condition: for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of representations of $\overline{Q}$ with dimension vectors $\gamma_1, \gamma := \gamma_1+\gamma_2, \gamma_2$ respectively, $E \in \mathbf{M}^{sp}_{\overline{Q},\gamma}$ if and only if $E_1 \in \mathbf{M}^{sp}_{\overline{Q},\gamma_1}$, and $E_2 \in \mathbf{M}^{sp}_{\overline{Q},\gamma_2}$. Then $\mathbf{M}^{sp}_{\widehat{Q},\gamma} = \mathbf{M}^{sp}_{\overline{Q},\gamma} \times \mathbb{A}^{\gamma\cdot\gamma}$.

The COHA of $(\widehat{Q}, W)$ (see [13]) induces the coproduct on the vector space $\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q})$ as follows:

- $H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \to H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q})$, which is the pullback associated to the closed embedding of groups $\mathbf{G}_{\gamma_1,\gamma_2} \to \mathbf{G}_\gamma$ with proper quotient. The projections $pr_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_\gamma)}_N$ induce natural transformations of functors $\varphi_{\gamma,N} \to (pr_{\gamma_1,\gamma_2,N})_! \varphi_{\gamma,\gamma_1,\gamma_2,N} (pr_{\gamma_1,\gamma_2,N})^*$ by (2) and properness of $pr_{\gamma_1,\gamma_2,N}$, thus give us

  $$(\pi_{\gamma,N})_! \varphi_{\gamma/u,N} (\pi_{\gamma,N})^*[-1] \to (\pi_{\gamma,N})_! (pr_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma,\gamma_1,\gamma_2)/u,N} (pr_{\gamma_1,\gamma_2,N})^* (\pi_{\gamma,N})^*[-1].$$

  Here $\varphi_{\gamma,N} = \varphi_{Tr(W)_{\gamma,N}}$ is the vanishing cycles functor of the function $tr(W)_{\gamma,N}$ on $\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_\gamma)}_N$, and $\varphi_{\gamma,\gamma_1,\gamma_2,N}$ corresponds to $Tr(W)_{\gamma,\gamma_1,\gamma_2,N}$ on $\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N$. (Note that in subscript of $\varphi_{\gamma,\gamma_1,\gamma_2,N}$, γ indicates the dimension vector of $\mathbf{M}_{\widehat{Q},\gamma}$, and γ_1, γ_2 indicate those of $\mathbf{G}_{\gamma_1,\gamma_2}$. We will use similar notations in the subsequent steps.)

  Since the following diagram commutes:

  $$\begin{array}{ccc}
\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{pr_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_\gamma)}_N \times \mathbb{C}^* \\
\downarrow{\scriptstyle \pi_{\gamma,\gamma_1,\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma,N}} \\
\overline{(\mathbf{M}_{\overline{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{pr_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma}, \mathbf{G}_\gamma)}_N \times \mathbb{C}^*
\end{array}$$

  we have

  $$\begin{array}{rl}
& (\pi_{\gamma,N})_! (pr_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma,\gamma_1,\gamma_2)/u,N} (pr_{\gamma_1,\gamma_2,N})^* (\pi_{\gamma,N})^*[-1] \\
\simeq & (pr_{\overline{Q},\gamma_1,\gamma_2,N})_! (\pi_{\gamma,\gamma_1,\gamma_2,N})_! \varphi_{(\gamma,\gamma_1,\gamma_2)/u,N} (\pi_{\gamma,\gamma_1,\gamma_2,N})^* (pr_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1].
\end{array}$$

By Theorem 2.2, we have two isomorphisms:

$$(\pi_{\gamma,N})_!\varphi_{\gamma/u,N}(\pi_{\gamma,N})^*[-1] \simeq (\pi_{\gamma,N})_!(\pi_{\gamma,N})^*(i_{\gamma,N})_*(i_{\gamma,N})^*$$

and

$$\begin{array}{rl} & (pr_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(pr_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (pr_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(i_{\gamma,\gamma_1,\gamma_2,N})_*(i_{\gamma,\gamma_1,\gamma_2,N})^*(pr_{\overline{Q},\gamma_1,\gamma_2,N})^*. \end{array}$$

Here $i_{\gamma,N}$ and $i_{\gamma,\gamma_1,\gamma_2,N}$ are inclusions, and the subscripts have the same meaning as the vanishing cycles functors above.

Pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma,N} \times \mathbb{C}^*$ gives us the commutative diagram

$$\begin{array}{ccc} H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) & \longrightarrow & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) \\ \downarrow\wr & & \downarrow\wr \\ H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma} & \longrightarrow & H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma} \end{array}$$

- $H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \to H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$ $\simeq H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$, where $\mathbf{M}^{sp}_{\Pi_Q,\gamma_1,\gamma_2} = \mathbf{M}^{sp}_{\Pi_Q,\gamma} \cap \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$, and $\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2} \subset \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ is the pullback of $\mathbf{M}^{sp}_{\Pi_Q,\gamma_1} \times \mathbf{M}^{sp}_{\Pi_Q,\gamma_2}$ under the projection $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2} \to \mathbf{M}_{\Pi_Q,\gamma_1} \times \mathbf{M}_{\Pi_Q,\gamma_2}$. This is the pullback associated to the closed embedding $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2} \to \mathbf{M}_{\Pi_Q,\gamma}$. The inclusions $j_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N$ induce natural transformations of functors $\varphi_{\gamma,\gamma_1,\gamma_2,N} \to (j_{\gamma_1,\gamma_2,N})_*\varphi_{\gamma_1,\gamma_2,N}(j_{\gamma_1,\gamma_2,N})^*$ by (2). So we have

$$\begin{array}{rl} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \to & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!(j_{\gamma_1,\gamma_2,N})_*\varphi_{(\gamma_1,\gamma_2)/u,N}(j_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1]. \end{array}$$

By the commutative diagram

$$\begin{array}{ccc} \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{j_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \\ \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma,\gamma_1,\gamma_2,N}} \\ \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{j_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \end{array}$$

we have

$$\begin{array}{rl} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!(j_{\gamma_1,\gamma_2,N})_*\varphi_{(\gamma_1,\gamma_2)/u,N}(j_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (j_{\overline{Q},\gamma_1,\gamma_2,N})_*(\pi_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,N})^*(j_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1]. \end{array}$$

Then the isomorphisms

$$\begin{array}{rl} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(i_{\gamma,\gamma_1,\gamma_2,N})_*(i_{\gamma,\gamma_1,\gamma_2,N})^* \end{array}$$

and

$$\begin{array}{rl} & (j_{\overline{Q},\gamma_1,\gamma_2,N})_*(\pi_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,N})^*(j_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (j_{\overline{Q},\gamma_1,\gamma_2,N})_*(\pi_{\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,N})^*(i_{\gamma_1,\gamma_2,N})_*(i_{\gamma_1,\gamma_2,N})^*(j_{\overline{Q},\gamma_1,\gamma_2,N})^* \end{array}$$

obtained from the theorem give us the commutative diagram by pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma,\gamma_1,\gamma_2,N} \times \mathbb{C}^*$:

$$\begin{array}{ccc} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_\gamma) \xrightarrow{\sim} H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_\gamma) & \\ \nearrow & & \searrow \\ H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) & & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2}) \\ \downarrow\wr & & \downarrow\wr \\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma} & \longrightarrow & H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{l_1} \end{array}$$

where $l_1 = \gamma \cdot \gamma - \gamma_1 \cdot \gamma_2$.

- $H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \xrightarrow{\sim} H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$. The affine fibrations $q_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N$ induce isomorphisms $\varphi_{(\gamma_1,\gamma_2)/u,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}$ $\xrightarrow{\sim} (q_{\gamma_1,\gamma_2,N})_*\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*})$. By applying Verdier duality we get $\varphi_{(\gamma_1,\gamma_2)/u,N}((q_{\gamma_1,\gamma_2,N})_!D\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*} \xrightarrow{\sim} D\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*})$. Then $\varphi_{(\gamma_1,\gamma_2)/u,N}((q_{\gamma_1,\gamma_2,N})_!\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*} \xrightarrow{\sim} \mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}\otimes$ $\mathbb{T}^{\gamma_1\cdot\gamma_2})$ by (1), and $(q_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}$

$\simeq (q_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N} (q_{\gamma_1,\gamma_2,N})^* \mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times\mathbb{C}^*}$
$\xrightarrow{\sim} \varphi_{(\gamma_1,\gamma_2)/u,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times\mathbb{C}^*} \otimes \mathbb{T}^{\gamma_1\cdot\gamma_2})$ by (3). Then we have isomorphisms

$$\begin{array}{cl} & (\pi_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2)/u,N} (\pi_{\gamma_1,\gamma_2,N})^* (\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times\mathbb{C}^*} \otimes \mathbb{T}^{\gamma_1\cdot\gamma_2}) \\ \to & (\pi_{\gamma_1,\gamma_2,N})_! (q_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N} (q_{\gamma_1,\gamma_2,N})^* (\pi_{\gamma_1,\gamma_2,N})^* \mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times\mathbb{C}^*}. \end{array}$$

The commutative diagram

$$\begin{array}{ccc} \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{q_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \\ \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,N}} \\ \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{q_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \end{array}$$

gives us isomorphisms

$$\begin{array}{cl} & (\pi_{\gamma_1,\gamma_2,N})_! (q_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N} (q_{\gamma_1,\gamma_2,N})^* (\pi_{\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (q_{\overline{Q},\gamma_1,\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N} (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^* (q_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1]. \end{array}$$

Theorem 2.2 implies isomorphisms

$$(\pi_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2)/u,N} (\pi_{\gamma_1,\gamma_2,N})^*[-1] \simeq (\pi_{\gamma_1,\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,N})^* (i_{\gamma_1,\gamma_2,N})_* (i_{\gamma_1,\gamma_2,N})^*$$

and

$$\begin{array}{cl} & (q_{\overline{Q},\gamma_1,\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N} (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^* (q_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\ \simeq & (q_{\overline{Q},\gamma_1,\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^* (i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_* (i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^* (q_{\overline{Q},\gamma_1,\gamma_2,N})^*. \end{array}$$

Pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma_1,\gamma_2,N} \times \mathbb{C}^*$ gives us the commutative diagram

$$\begin{array}{ccc} H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2}) & \xrightarrow{\sim} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2} \\ \downarrow\wr & & \downarrow\wr \\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma-\gamma_1\cdot\gamma_2} & \xrightarrow{\sim} & H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma-2\gamma_1\cdot\gamma_2} \end{array}$$

- $H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q}) \xrightarrow{\sim} H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\chi_{\widetilde{Q}}(\gamma_2,\gamma_1)}$.
  Similar as the previous step, the affine fibrations $p_{\gamma_1,\gamma_2,N}:\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N$ induce isomorphisms
  $(p_{\gamma_1,\gamma_2,N})_!\varphi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}(p_{\gamma_1,\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}$
  $\xrightarrow{\sim}\varphi_{\gamma_1\boxplus\gamma_2,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^l)$,where $l=\sum_{a:i\to j\in\widetilde{Q}_1}\gamma_1^j\gamma_2^i$.
  Then we have isomorphisms

$$\begin{aligned}&(\pi_{\gamma_1\times\gamma_2,N})_!(p_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(p_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1\times\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\\ &\xrightarrow{\sim}(\pi_{\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1\boxplus\gamma_2)/u,N}(\pi_{\gamma_1\times\gamma_2,N})^*(\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^l).\end{aligned}$$

  The commutative diagram

$$\begin{array}{ccc}\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{p_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*\\ \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma_1\times\gamma_2,N}}\\ \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{p_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*\end{array}$$

  implies isomorphisms

$$\begin{aligned}&(\pi_{\gamma_1\times\gamma_2,N})_!(p_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(p_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1\times\gamma_2,N})^*[-1]\\ \simeq\ &(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1].\end{aligned}$$

  By Theorem 2.2, we have
  $$(\pi_{\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1\boxplus\gamma_2)/u,N}(\pi_{\gamma_1\times\gamma_2,N})^*[-1]\simeq(\pi_{\gamma_1\times\gamma_2,N})_!(\pi_{\gamma_1\times\gamma_2,N})^*(i_{\gamma_1\times\gamma_2,N})_*(i_{\gamma_1\times\gamma_2,N})^*$$
  and

$$\begin{aligned}&(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1]\\ \simeq\ &(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*.\end{aligned}$$

  By pulling back to $\overline{(\mathbf{M}^{sp}_{\overline{Q},\gamma_1}\times\mathbf{M}^{sp}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*$, we have

$$\begin{array}{ccc}H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2},W_{\gamma_1,\gamma_2}) & \xrightarrow{\sim} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma_1}\times\mathbf{M}^{sp}_{\widehat{Q},\gamma_2},W_{\gamma_1}\boxplus W_{\gamma_2})\otimes\mathbb{T}^l\\ \downarrow\wr & & \downarrow\wr\\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{l_1} & \xrightarrow{\sim} & H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{l_2}\end{array}$$

  where $l_1=\gamma\cdot\gamma-\gamma_1\cdot\gamma_2$, and $l_2=\gamma_1\cdot\gamma_1+\gamma_2\cdot\gamma_2+l$.

- $H^\bullet_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q}) \xrightarrow{\sim} H^\bullet_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})\otimes H^\bullet_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})$. This is the Künneth isomorphism compatible with the Thom-Sebastiani isomorphism by Theorem 2.3.

The above computations can be summarized for convience of the reader in the form of the following statement.

Proposition 2.6. *The coproduct making the vector space* $\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} H^\bullet_{c,\boldsymbol{G}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})$ *into a coalgebra is given by the composition of the maps*

$$\begin{array}{ll} & H^\bullet_{c,\boldsymbol{G}_\gamma}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \to H^\bullet_{c,\boldsymbol{G}_{\gamma_1,\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \\ \to & H^\bullet_{c,\boldsymbol{G}_{\gamma_1,\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\gamma_1\cdot\gamma_2} \simeq H^\bullet_{c,\boldsymbol{G}_{\gamma_1,\gamma_2}}(\widetilde{\boldsymbol{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\gamma_1\cdot\gamma_2} \\ \xrightarrow{\sim} & H^\bullet_{c,\boldsymbol{G}_{\gamma_1}\times\boldsymbol{G}_{\gamma_2}}(\widetilde{\boldsymbol{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-2\gamma_1\cdot\gamma_2} \\ \xrightarrow{\sim} & H^\bullet_{c,\boldsymbol{G}_{\gamma_1}\times\boldsymbol{G}_{\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_1}\times\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)} \\ \xrightarrow{\sim} & H^\bullet_{c,\boldsymbol{G}_{\gamma_1}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})\otimes H^\bullet_{c,\boldsymbol{G}_{\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)}. \end{array}$$

Now let $\mathcal{H}_\gamma := H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma,\gamma)}$, and $\mathcal{H} = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}\mathcal{H}_\gamma$. Then the above coproduct makes $\mathcal{H}$ an associative algebra with product

$$\begin{array}{ll} & \mathcal{H}_{\gamma_1}\otimes\mathcal{H}_{\gamma_2} = H^\bullet_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)}\otimes H^\bullet_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma_2,\gamma_2)} \\ = & H^\bullet_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^\vee\otimes H^\bullet_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)} \\ \to & H^\bullet_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)}\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)} \\ = & H^\bullet_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma_1+\gamma_2,\gamma_1+\gamma_2)} = \mathcal{H}_{\gamma_1+\gamma_2}. \end{array}$$

Definition 2.7. *The associative algebra $\mathcal{H}$ is called the Cohomological Hall algebra of the preprojective algebra Π_Q associated with the quiver Q.*

Remark 2.8. *In the framework of equivariant K-theory a similar notion was introduced in [19].*

Corollary 2.9. *This product preserves the modified cohomological degree, thus the zero degree part* $\mathcal{H}^0 = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}\mathcal{H}^0_\gamma = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} H^{-2\chi_Q(\gamma,\gamma)}_{c,\boldsymbol{G}_\gamma}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^\vee\otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)}$ *is a subalgebra of $\mathcal{H}$.*

We can reformulate the definition of COHA of Π_Q using language of stacks. The natural morphism of stacks $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}/\mathbf{G}_{\gamma_1,\gamma_2}\to\mathbf{M}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma$ is proper, hence it induces the pushforward map on $\mathcal{H}$. Composting it with the pullback by the morphism $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}/\mathbf{G}_{\gamma_1,\gamma_2}\to\mathbf{M}_{\Pi_Q,\gamma_1}/\mathbf{G}_{\gamma_1}\times\mathbf{M}_{\Pi_Q,\gamma_2}/\mathbf{G}_{\gamma_2}$, we obtain the product.

2.3 Lusztig's seminilpotent Lagrangian subvariety

In this subsection we work in the framework close to the one from [1].

Let Q be a quiver (possibly with loops) with vertices I and arrows Ω, and denote by Ω_i the set of loops at $i \in I$. We call i imaginary if the number of loops $\omega_i = |\Omega_i| \geqslant 1$, and real if $\omega_i = 0$. Let I^{im} be the set of imaginary vertices and I^{re} real vertices.

Definition 2.10. *A representation $x \in \boldsymbol{M}_{\overline{Q},\gamma}$ is seminilpotent if there is an I-graded filtration $W = (W_0 = V_\gamma \supset \ldots \supset W_r = \{0\})$ of the representation space $V_\gamma = (V_i)_{i\in I}$, such that $x_{a^*}(W_\bullet) \subseteq W_{\bullet+1}$, and $x_a(W_\bullet) \subseteq W_\bullet$ for $a \in \Omega$.*

Remark 2.11. *Our definition of seminilpotency is slightly different from that in* [1]*. We put nilpotent condition on the dual arrows a^* rather than a. But main results of* [1] *hold in our situation as well.*

We denote by $\mathbf{M}^{sp}_{\overline{Q},\gamma}$ the space of seminilpotent representations of dimension γ. Then by [1, Th. 1.15], the space of seminipotnet representations of Π_Q of dimension γ, $\mathbf{M}^{sp}_{\Pi_Q,\gamma} \subset \mathbf{M}^{sp}_{\overline{Q},\gamma}$, is a Lagrangian subvariety of $\mathbf{M}_{\overline{Q},\gamma}$.

Let $\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l} = \{x \in \mathbf{M}^{sp}_{\Pi_Q,\gamma} | \mathrm{codim}(\bigoplus_{j\neq i, a:j\to i \mathrm{in} \overline{Q}} \mathrm{Im} x_a) = l\}$. Then $\mathbf{M}^{sp}_{\Pi_Q,\gamma} = \bigcup_{i\in I, l\geqslant 1} \mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}$ by the seminilpotency condition. There is a one to one correspondence of the sets of irreducible components (see [1, Prop.1.14])

$$\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma-le_i,i,0}) \times \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i}), \tag{4}$$

where $e_i = (\delta_{ij})_{j\in I}$. For any vertex i, we have $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}) = \bigsqcup_{l\geqslant 0} \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l})$. If $i \in I^{re}$ then $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$ consists of only one element, namely the zero representation. We denote by $Z_{i,l}$ the only element in $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$. If $i \in I^{im}$, then there are two cases. If the number of loops $\omega_i = 1$, then $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$ is parametrized by $\mathfrak{C}_{i,l} = \{c = (c_k)\}$, the set of partitions of l (i.e., $\sum_k c_k = l$, $c_k > 0, \forall k$, and $c_{k+1} \geqslant c_k$). If $\omega_i > 1$, then it is parametrized by the set of compositions also denoted by $\mathfrak{C}_{i,l}$ (i.e., $\sum_k c_k = l$, $c_k > 0, \forall k$).

We put $|c| = \sum_k c_k$ for $c \in \mathfrak{C}_{i,l}$, and denote by $Z_{i,c} \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$ the irreducible component corresponding to c. Let $Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})$, then there exists $i \in I$ and $l \geqslant 1$ such that $Z \bigcap \mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}$ is dense in Z. We denote by $\varepsilon_i(Z)$ the corresponding partition or composition if $i \in I^{im}$, and $\varepsilon_i(Z) = l$ if $i \in I^{re}$, via the one to one correspondence (4).

Now let $\mathscr{M}_\gamma$ be the $\mathbb{Q}$-vector space of constructible functions $f : \mathbf{M}^{sp}_{\Pi_Q,\gamma} \to \mathbb{Q}$ which are constant on any $\mathbf{G}_\gamma$-orbit, and $\mathscr{M} = \bigoplus_\gamma \mathscr{M}_\gamma$. Then one can define a product $*$ on $\mathscr{M}$ in the way which is analogous to the definition of Lusztig for nilponent case in [14, Section 12].

More precisely, let us denote by $\mathbf{M}^{sp}_{\Pi_Q,V}$ the space of seminilpotent representations of Π_Q with I-graded vector space V, and $\mathscr{M}_V$ the $\mathbb{Q}$-vector space of constructible functions $f : \mathbf{M}^{sp}_{\Pi_Q,V} \to \mathbb{Q}$ constant on any $\mathbf{G}_\gamma$-orbit. Let V_1, V_2 and V be I-graded vector spaces of dimensions γ_1, γ_2 and $\gamma = \gamma_1 + \gamma_2$ respectively, and $f_i \in \mathscr{M}_{V_i}, i = 1, 2$. Then $f_1 * f_2 \in \mathscr{M}_V$ is defined using the diagram

$$\mathbf{M}^{sp}_{\Pi_Q,V_1} \times \mathbf{M}^{sp}_{\Pi_Q,V_2} \xleftarrow{p_1} \mathbf{F}' \xrightarrow{p_2} \mathbf{F}'' \xrightarrow{p_3} \mathbf{M}^{sp}_{\Pi_Q,V}$$

where the notations are as follows: F'' is the variety of pairs (x, U) with $x \in \mathbf{M}^{sp}_{\Pi_Q,V}$ and U an x-stable I-graded subspace of V with dimension γ_2; F' is the variety of quadruples (x, U, R'', R') where $(x, U) \in F''$, $R'' : V_2 \xrightarrow{\sim} U$ and $R' : V_1 \xrightarrow{\sim} V/U$. The maps $p_1(x, U, R'', R') = (x_1, x_2)$ where $xR' = R'x_1$ and $xR'' = R''x_2$, $p_2(x, U, R'', R') = (x, U)$, and $p_3(x, U) = x$. Note that p_2 is a $\mathbf{G}_{V_1} \times \mathbf{G}_{V_2}$-princepal bundle and p_3 is proper. Let $f(x_1, x_2) = f_1(x_1)f_2(x_2)$, then there is a unique function $f_3 \in \mathscr{M}_{F''}$ such that $p_1^* f = p_2^* f_3$. Finally, define $f_1 * f_2 = (p_3)_!(f_3)$. By identifying the vector spaces $\mathscr{M}_V$ for various V with $\mathscr{M}_\gamma$ in a coherent way ($\dim(V) = \gamma$), we define the product $*$ on $\mathscr{M}$, making it an associative $\mathbb{Q}$-algebra.

One can also reformulate this product using the diagram of stacks

$$\mathbf{M}_{\Pi_Q,\gamma_2}/\mathbf{G}_{\gamma_2} \times \mathbf{M}_{\Pi_Q,\gamma_1}/\mathbf{G}_{\gamma_1} \leftarrow \mathbf{M}_{\Pi_Q,\gamma_2,\gamma_1}/\mathbf{G}_{\gamma_2,\gamma_1} \to \mathbf{M}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma.$$

We denote by $1_{i,c}$ (resp. $1_{i,l}$) the characteristic function of $Z_{i,c}$ (resp. $Z_{i,l}$), and $\mathscr{M}_0 \subseteq \mathscr{M}$ the subalgebra generated by $1_{i,(l)}$ and $1_{i,1}$. For any $Z \in \operatorname{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})$ and $f \in \mathscr{M}_\gamma$, let $\rho_Z(f) = c$ if $Z \bigcap f^{-1}(c)$ is open dense in Z.

Theorem 2.12. *(see [1, Prop. 1.18]) For any $Z \in \operatorname{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma})$ there exists $f_Z \in \mathscr{M}_{0,\gamma} = \mathscr{M}_0 \cap \mathscr{M}_\gamma$ such that $\rho_Z(f_Z) = 1$, and $\rho_{Z'}(f_Z) = 0$ for $Z' \neq Z$.*

2.4 Generalized quantum group

We recall some definitions and facts about generalized quantum group introduced in [1].

Let $(\bullet,\bullet)$ be the symmetric Euler form on $\mathbb{Z}^I$ defined by $(i,j) = 2\delta_{ij} - a_{ij} - a_{ji}$, and $(\iota, j) = l(i,j)$ if $\iota = (i,l) \in I_\infty = (I^{re} \times \{1\}) \bigcup (I^{im} \times \mathbb{N}_{\geqslant 1})$ and $j \in I$.

Definition 2.13. *Let F be the $\mathbb{Q}(v)$-algebra generated by $(E_\iota)_{\iota \in I_\infty}$, $\mathbb{N}^I$-graded by $|E_\iota| = li$ for $\iota = (i,l)$. If $A \subseteq \mathbb{N}^I$, then let $\mathrm{F}[A] = \{E \in \mathrm{F} \,||E| \in A\}$.*

For any $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I$, let $\mathrm{ht}(\gamma) = \sum_i \gamma^i$ be its height, and $v_\gamma = \prod_i v_i^{\gamma^i}$, where $v_i = v^{(i,i)/2}$. We endow F with a coproduct $\delta(E_{i,l}) = \sum\limits_{l_1+l_2=l} v_i^{l_1 l_2} E_{i,l_1} E_{i,l_2}$, where $E_{i,0} = 1$. Then for any family $(v_\iota)_{\iota \in I_\infty} \subseteq \mathbb{Q}(v)$, there is a bilinear form $\{\bullet,\bullet\}$ on F such that

- $\{E, E'\} = 0$ if $|E| \neq |E'|$,
- $\{E_\iota, E_\iota\} = v_\iota$, $\forall \iota \in I_\infty$,
- $\{EE', E''\} = \{E \otimes E', \delta(E'')\}$, $\forall E, E', E'' \in \mathrm{F}$.

It turns out that $\sum\limits_{l_1+l_2=-(\iota,j)+1} (-1)^{l_1} \frac{E_{j,1}^{l_1}}{l_1!} E_\iota \frac{E_{j,1}^{l_2}}{l_2!}$ is in the radical of $\{\bullet,\bullet\}$.

Definition 2.14. *Let $\widetilde{\mathcal{U}}^+$ be the quotient of F by the ideal generated by the above element and the commutators $[E_{i,l}, E_{i,k}]$ for $\omega_i = 1$. Then $\{\bullet,\bullet\}$ is well-defined on $\widetilde{\mathcal{U}}^+$. Let $\mathcal{U}^+$ be the quotient of $\widetilde{\mathcal{U}}^+$ by the radical of $\{\bullet,\bullet\}$.*

Theorem 2.15. *(see [1, Th. 3.34]) There is an isomorphism of algebras*

$$\begin{aligned} \phi: \ & \mathcal{U}^+_{v=1} \to \mathscr{M}_0, \\ & E_{i,(l)} \mapsto 1_{i,(l)}, \quad i \in I^{im}, \\ & E_{i,1} \mapsto 1_{i,1}, \quad i \in I^{re}. \end{aligned}$$

Definition 2.16. *The semicanonical basis of $\mathcal{U}^+_{v=1}$ is $\phi^{-1}(\{f_Z | Z \in \mathrm{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q})\})$.*

2.5 Semicanonical basis of $\mathcal{H}^0$

We have already seen that for an appropriate subspace $\mathbf{M}^{sp}_{\overline{Q},\gamma} \subset \mathbf{M}_{\overline{Q},\gamma}$, the degree 0 part $\mathcal{H}^0$ of COHA is a subalgebra. In particular, we can take $\mathbf{M}^{sp}_{\overline{Q},\gamma}$ to be the space of seminilpotent representations of $\overline{Q}$. Then $\mathbf{M}^{sp}_{\Pi_Q,\gamma}$ is the space of seminilpotent representations in $\mathbf{M}_{\Pi_Q,\gamma}$, and $\dim(\mathbf{M}^{sp}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma) = -\chi_Q(\gamma,\gamma)$, so the classes of irreducible components $\{[Z] | Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})\}$ lie in $\mathcal{H}^0$. In fact, these classes form a basis of $\mathcal{H}^0$ by the following theorem.

Theorem 2.17. *Let X be a scheme with top dimensional irreducible components $\{C^k\}$, and a connected algebraic group G acts on it. Then $H^{2top}_{c,G}(X)$ has a basis one to one corresponding to $\{C^k\}$, where top is the dimension of the stack X/G.*

Proof. Choose an embedding $G \hookrightarrow GL(n, \mathbb{C})$. Let $fr(n, N)$ be the space of n-tuples of linearly independent vectors in $\mathbb{C}^N$ for $N \geqslant n$. Then $X \times fr(n, N)$ has irreducible components $\{C^k \times fr(n, N)\}$, thus $X \times_G fr(n, N) = (X \times fr(n, N))/G$ has irreducible components $\{\overline{C^k}\}$ one to one corresponding to $\{C^k\}$ since G is irreducible. Then the Borel-Moore homology $H^{BM}_{2\bullet}(X \times_G fr(n, N))$ has a basis $\{[\overline{C^k}]\}$, where $\bullet = \dim(X) + \dim(fr(n, N)) - \dim G$, implying that $H^{2\bullet}_c(X \times_G fr(n, N))^\vee = H^{BM}_{2\bullet}(X \times_G fr(n, N))$ has basis one to one corresponding to $\{C^k\}$ (For details of Borel-Moore homology, see [4, Section 2.6]). Then $H^{2top}_{c,G}(X) = \lim\limits_{N\to\infty} H^{2\bullet}_c(X \times_G fr(n, N)) \otimes \mathbb{T}^{-\dim fr(n,N)}$ has basis one to one corresponding to $\{C^k\}$, where $top = \bullet - \dim(fr(n, N)) = \dim(X/G)$. $\square$

Definition 2.18. *We call the basis defined above the semicanonical basis of the subalgebra $\mathcal{H}^0$.*

Given an element $\mathcal{F}$ in $\mathscr{D}^b(X)$ with constructible cohomology, and $x \in X$, the function $\chi(\mathcal{F})(x) = \chi(\mathcal{F}_x) = \sum_i (-1)^i \dim(H^i(\mathcal{F}_x))$ is constructible. Moreover, the standard operations (pullback, pushforward, etc.) in $\mathscr{D}^b(X)$ and the corresponding operations on constructible functions are compatible.

Recall the family of constructible functions $\{f_Z | Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q})\}$. Then $U_Z = f_Z^{-1}(1)$ is constructible. Let $f_{Z,N}$ be the characteristic function of $\overline{(U_Z, \mathbf{G}_\gamma)_N}$, and $\mathbb{Q}_{Z,N}$ be the constant sheaf on $\overline{(U_Z, \mathbf{G}_\gamma)_N}$. Since the operations on constructible functions and constructible sheaves agree, there is an isomorphism of algebras $\Psi : \mathcal{H}^0 \to \mathscr{M}_0^{op}, [Z] \mapsto f_Z$. It is obtained by taking the dual of compactly supported cohomology and passing to the limit.

Furthermore, notice that $\mathcal{H}^0 \simeq (\mathcal{U}^+_{v=1})^{op}$, and that Lusztig's product $*$ is opposite to the product of COHA (see the end of Section 2.3).

The semicanonical basis of $\mathcal{H}^0$ is compatible with a certain filtration. More precisely, we have the following result.

Theorem 2.19. *Fix $d = (d_i) \in \mathbb{Z}^I_{\geqslant 0}$. Then the subspace spanned by $\{[Z] | \exists i, s.t. |\varepsilon_i(Z)| \geqslant d_i\}$ coincides with $\sum\limits_{i\in I, |c|=d_i} \mathcal{H}^0[Z_{i,c}]$, where $Z_{i,c} \in \mathrm{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q, le_i})$ is the irreducible component corresponding to c (defined in Section 2.3), and $c = l$ if $i \in I^{re}$.*

Proof. By definitions, $\sum_{i\in I,|c|=d_i} \mathcal{H}^0[Z_{i,c}]$ is contained in the subspace spanned by $\{[Z]|\exists i, s.t.|\varepsilon_i(Z)| \geqslant d_i\}$. To prove the reverse inclusion it suffices to show that for any $i \in I$, $\gamma \in \mathbb{Z}^I_{\geqslant 0}$, and $[Z] \in \mathcal{H}^0$ such that $Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})$ and $|\varepsilon_i(Z)| = l$, we have $[Z] \in \sum_{|c|=l} \mathcal{H}^0[Z_{i,c}]$. We use descending induction on $l \leqslant \gamma^i$. For above Z, we have $\gamma - le_i \in \mathbb{N}^I$, and by the proof of [1, Pro. 1.18], there exists a unique $Z' \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma-le_i})$ and $Z_{i,c} \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$ such that $|\varepsilon_i(Z')| = 0$ and $[Z'][Z_{i,c}] = Z + \sum_{|\varepsilon_i(\widetilde{Z})|>l} a_{\widetilde{Z}}[\widetilde{Z}]$ for some $a_{\widetilde{Z}} \in \mathbb{Q}$. By applying the induction hypothesis to $\widetilde{Z}$ we have that the subspace spanned by $\{[Z]|\exists i, s.t.|\varepsilon_i(Z)| \geqslant d_i\}$ is contained in $\sum_{i\in I,|c|=d_i} \mathcal{H}^0[Z_{i,c}]$. Thus the two subspaces coincide. $\square$

The dual of representations of Π_Q induces a bijection $* : \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}) \to \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}), Z \mapsto Z^*$, thus an antiautomorphism of $\mathcal{H}^0$. Then the dual of the above theorem holds:

Theorem 2.20. *The subspace spanned by* $\{[Z]|\exists i, s.t.|\varepsilon_i(Z^*)| \geqslant d_i\}$ *coincides with* $\sum_{i\in I,|c|=d_i} [Z_{i,c}]\mathcal{H}^0$.

3 $2CY$ categories and Donaldson-Thomas series

In Section 2 we discussed the semicanonical basis obtained as a result of the dimensional reduction from $3CY$ category to the 2-dimensional category. In this section we are going to discuss DT-series for $2CY$ categories.

3.1 Motivic stack functions and motivic Hall algebras: reminder

Let X be a constructible set over a field $\mathbf{k}$ of characteristic zero, G an affine algebraic group acting on X. In this section we are going to recall the definition of the abelian group of stack functions $Mot_{st}((X,G))$ following [12, Section 4] (see also [9] for a different exposition).

Let us consider the following 2-category of constructible stacks over $\mathbf{k}$. Objects are pairs (X, G), where X is a constructible set, and G is an affine algebraic group acting on it. The category of 1-morphisms $\mathrm{Hom}((X_1, G_1), (X_2, G_2))$ consists of pairs (Z, f), where Z is a $G_1 \times G_2$-constructible set such that $\{e\} \times G_2$ acts freely on Z in such a way that we have the induced G_1-equivariant isomorphism $Z/G_2 \simeq X_1$, and $f : Z \to X_2$ is a $G_1 \times G_2$-equivariant map with trivial action of G_1 on X_2. Furthermore, objects of $\mathrm{Hom}((X_1, G_1), (X_2, G_2))$ form naturally a groupoid. The 2-category of constructible stacks carries a direct sum operation induced by disjoint union of stacks.

After the above preliminaries we define the group of motivic stack functions $Mot_{st}((X, G))$ as the abelian group generated by isomorphism classes of 1-morphisms of stacks $[(Y, H) \to (X, G)]$ with the fixed target (X, G), subject to the relations

- $[((Y_1, H_1) \sqcup (Y_2, H_2)) \to (X, G)] = [(Y_1, H_1) \to (X, G)] + [(Y_2, H_2) \to (X, G)]$,
- $[(Y_2, H) \to (X, G)] = [(Y_1 \times \mathbb{A}^d_k, H) \to (X, G)]$ if $Y_2 \to Y_1$ is an H-equivariant constructible vector bundle of rank d.

One can define pullbacks, pushforwards and fiber products of elements of $Mot_{st}((X, G))$ in the natural way (see loc.cit.).

Let $\mathcal{C}$ be an ind-constructible locally regular (e.g. locally Artin) triangulated A_∞-category over a field $\mathbf{k}$ (see [12]). Then the stack of objects admits a countable decomposition into the union of quotient stacks $\mathcal{O}b(\mathcal{C}) = \sqcup_{i \in I}(Y_i, GL(N_i))$, where Y_i is a reduced algebraic scheme acted by the group $GL(N_i)$.

Definition 3.1. *(cf. [12]) The motivic Hall algebra $H(\mathcal{C})$ is the $Mot(Spec(\boldsymbol{k}))-$module $\bigoplus_{i \in I} Mot_{st}(Y_i, GL(N_i))[\mathbb{L}^n, n < 0]$ (i.e. we extend the direct sum of the groups of motivic stack functions by adding negative powers of the Lefschetz motive $\mathbb{L}$), endowed with the product defined below.*

The product is defined as follows. Let us denote $\dim Ext^i(E, F)$ by $(E, F)_i$, and use the truncated Euler characteristic $(E, F)_{\le N} = \sum_{i \le N} (-1)^i (E, F)_i$. Let $[\pi_i : Y_i \to \mathcal{O}b(\mathcal{C})], i = 1, 2$ be two elements of $H(\mathcal{C})$, then for any $n \in \mathbb{Z}$ we have constructible sets

$$W_n = \{(y_1, y_2, \alpha) | y_i \in Y_i, \alpha \in Ext^1(\pi_2(y_2), \pi_1(y_1)), (\pi_2(y_2), \pi_1(y_1))_{\le 0} = n\}.$$

Then $[tot((\pi_1 \times \pi_2)^*(\mathcal{EXT}^1)) \to \mathcal{O}b(\mathcal{C})] = \sum_{n\in\mathbb{Z}}[W_n \to \mathcal{O}b(\mathcal{C})]$. Define the product $[Y_1 \to \mathcal{O}b(\mathcal{C})] \cdot [Y_2 \to \mathcal{O}b(\mathcal{C})] = \sum_{n\in\mathbb{Z}}[W_n \to \mathcal{O}b(\mathcal{C})]\mathbb{L}^{-n}$, where the map $W_n \to \mathcal{O}b(\mathcal{C})$ is given by $(y_1, y_2, \alpha) \mapsto Cone(\alpha : \pi_2(y_2)[-1] \to \pi_1(y_1))$.

Theorem 3.2. *(see* [12, Prop. 10]*) The algebra $H(\mathcal{C})$ is associative.*

For a constructible stability condition on $\mathcal{C}$ with an ind-constructible class map $cl : K_0(\mathcal{C}) \to \Gamma$, a central charge $Z : \Gamma \to \mathbb{C}$, a strict sector $V \subset \mathbb{R}^2$ and a branch Log of the logarithm function on V, we have (see [12]) the category $\mathcal{C}_V := \mathcal{C}_{V,Log}$ generated by semistables with the central charge in V. Then we define the corresponding completed motivic Hall algebra $\widehat{H}(\mathcal{C}_V) := \prod_{\gamma\in(\Gamma\cap C(V,Z,Q))\cup\{0\}} H(\mathcal{C}_V \cap cl^{-1}(\gamma))$. It contains an invertible element $A_V^{Hall} = 1 + \cdots = \sum_{i\in I} \mathbf{1}_{(\mathcal{O}b(\mathcal{C}_V)\cap Y_i, GL(N_i))}$, where 1 comes from the zero object. The element A_V corresponds (roughly) to the sum over all isomorphism classes of objects of $\mathcal{C}_V$, each counted with the weight given by the inverse to the motive of the group of automorphisms.

Theorem 3.3. *(see* [12, Prop. 11]*) The elements A_V^{Hall} satisfy the Factorization Property:*

$$A_V^{Hall} = A_{V_1}^{Hall} \cdot A_{V_2}^{Hall}$$

for a strict sector $V = V_1 \sqcup V_2$ (decomposition in the clockwise order).

Let's fix the following data:

(1) a triple $(\Gamma, \langle \bullet, \bullet \rangle, Q)$ consisting of a free abelian group Γ of finite rank endowed with a bilinear form $\langle \bullet, \bullet \rangle : \Gamma \otimes \Gamma \to \mathbb{Z}$, and a quadratic form Q on $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$,

(2) an ind-constructible , $Gal(\overline{\mathbf{k}}/\mathbf{k})-$ equivariant homomorphism $cl_{\overline{\mathbf{k}}} : K_0(\mathcal{C}(\overline{\mathbf{k}})) \to \Gamma$ compatible with the Euler form of $\mathcal{C}$ and the bilinear form $\langle \bullet, \bullet \rangle$,

(3) a constructible stability condition $\sigma \in Stab(\mathcal{C}, cl)$ compatible with the quadratic form Q in the sense that $Q|_{Ker(Z)} < 0$ and $Q(cl_{\overline{\mathbf{k}}}(E)) \geq 0$, $\forall E \in \mathcal{C}^{ss}(\overline{\mathbf{k}})$.

We define the quantum torus $\mathcal{R}_{\Gamma,R}$ over a given commutative unital ring R containing an invertible symbol $\mathbb{L}^{\frac{1}{2}}$ as an R-linear associative algebra $\mathcal{R}_{\Gamma,R} := \bigoplus_{\gamma\in\Gamma} R \cdot \widehat{e}_\gamma$, where the generators $\widehat{e}_\gamma, \gamma \in \Gamma$ satisfy the relations $\widehat{e}_{\gamma_1}\widehat{e}_{\gamma_2} = \mathbb{L}^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle}\widehat{e}_{\gamma_1+\gamma_2}$, $\widehat{e}_0 = 1$. For any strict sector $V \subset \mathbb{R}^2$, we define the quantum torus associated with V by $\mathcal{R}_{V,R} := \prod_{\gamma\in\Gamma\cap C_0(V,Z,Q)} R\cdot\widehat{e}_\gamma$, where $C_0(V,Z,Q) := C(V,Z,Q)\cup\{0\}$, and $C(V,Z,Q)$ is the convex cone generated by $S(V,Z,Q) = \{x\in\Gamma_{\mathbb{R}}\setminus\{0\}|Z(x)\in V, Q(x)\geq 0\}$.

In the case when $\mathcal{C}$ is a $3CY$ category, one can define a homomorphism from the algebra $\widehat{H}(\mathcal{C}_V)$ to an appropriate motivic quantum torus (the word "motivic" here means that the coefficient ring R is a certain ring of motivic functions). This homomorphism was defined in [12] via the motivic Milnor fiber of the potential of the $3CY$ category. The notion of motivic DT-series was also introduced in the loc.cit.

It was later shown in [13] that in the case of quivers with potential one can define motivic DT-series differently, using equivariant critical cohomology (cf. our Section 2). In that case instead of the motivic Hall algebra one uses COHA.

3.2 A class of $2CY$ categories

Let us consider a class of 2-dimensional Calabi-Yau categories $\mathcal{C}$ which are:

1) Ind-constructible and locally ind-Artin in the sense of [12].

2) Endowed with a constructible homomorphism of abelian groups (class map) $cl: K_0(\mathcal{C})\to\Gamma$, where $\Gamma\simeq\mathbb{Z}^I$ carries a symmetric integer bilinear form $\langle\bullet,\bullet\rangle$, and the class map cl satisfies $\langle cl(E), cl(F)\rangle = \chi(E,F) := \sum_{i\in\mathbb{Z}}(-1)^i \mathrm{dim} Ext^i(E,F)$.

3) Generated by a spherical collection $\mathcal{E}=(E_i)_{i\in I}$ in the sense of loc. cit. such that $cl(E_i)\in\Gamma_+\simeq\mathbb{Z}^I_{\geq 0}$. This means that $Ext^\bullet(E_i,E_i)\simeq H^\bullet(S^2)$, and that $Ext^m(E_i,E_j)$ can be non-trivial for $m=1$ only as long as $i\neq j$.

4) For any $\gamma\in\Gamma_+$ the stack $\mathcal{C}_\gamma(\mathcal{E})$ of objects F of the heart of the t-structure corresponding to $(E_i)_{i\in I}$ such that $cl(F)=\gamma$ is a countable disjoint union of Artin stacks of dimensions less or equal than $-\frac{1}{2}\langle\gamma,\gamma\rangle$.

5) For any strict sector $V\subset\mathbb{R}^2$ with the vertex at $(0,0)$, and a constructible stability central charge $Z:\Gamma\to\mathbb{C}$ such that $\mathrm{Im}(Z(E_i)) := Z(cl(E_i))\in V, i\in I$, the stack of objects of the category $\mathcal{C}_V$ generated by semistable objects with central charges in V is a finite union of Artin stacks satisfying the inequality of 4) above.

With the category from our class one can associate a symmetric quiver (vertices correspond to spherical objects E_i and arrows correspond to a basis in $Ext^1(E_i, E_j)$). Similarly to [12], Section 8 one can prove a classification theorem for our categories in terms of Ginzburg algebras associated with quivers. Many $2CY$ categories which appear in "nature" belong to our class. For example, if Q is not an ADE quiver, then the derived category of finite-dimensional representations of Π_Q belongs to our class. Without any restrictions on Q one can construct a $2CY$ category as the category of dg-modules over the corresponding Ginzburg algebra.

3.3 Stability conditions and braid group action

Assume that $\mathcal{C}$ is a $2CY$ category from our class. We consider an open subset of the space $Stab(\mathcal{C})$ of stability conditions which is defined as $U := \prod_{i\in I}(Im\, z_i > 0)$, i.e. it is a product of upper-half planes. A point $Z = (z_i)_{i\in I} \in U$ defines the central charge $Z : \Gamma := \mathbb{Z}^I \to \mathbb{C}$ which maps classes of spherical generators to the open upper-half plane (hence the stability condition is determined by Z and the t-structure in $\mathcal{C}$ generated by $(E_i)_{i\in I}$).

Recall that with every $i_0 \in I$ we can associate an autoequivalence of $\mathcal{C}$ (called *reflection functor*) by the formula

$$R_{E_{i_0}} : F \mapsto Cone(Ext^\bullet(E_{i_0}, F) \otimes F \to F).$$

Then $R_{E_{i_0}}(E_{i_0}) = E_{i_0}[-1]$, and $R_{E_{i_0}}(E_j), j \neq i_0$ is determined as the middle term in the extension

$$0 \to E_j \to R_{E_{i_0}}(E_j) \to E_{i_0} \otimes Ext^1(E_{i_0}, E_j) \to 0.$$

The inverse reflection functor $R^{-1}_{E_{i_0}}$ is given by

$$R^{-1}_{E_{i_0}}(E_{i_0}) = E_{i_{i_0}}[1],$$

$$0 \to E_{i_0} \otimes Ext^1(E_{i_0}, E_j) \to R^{-1}_{E_{i_0}}(E_j) \to E_j \to 0.$$

Reflection functors $R_{E_i}, i \in I$ generate a subgroup $Braid_{\mathcal{C}} \subset Aut(\mathcal{C})$, which induces an action on $Stab(\mathcal{C})$. The orbit $D := Braid_{\mathcal{C}}(U) \subset Stab(\mathcal{C})$ is the union of consecutive "chambers" obtained one from another one by reflection functor R_{E_j}. Such consecutive chambers have a common real codimension one boundary singled out by the condition $Im\, Z(E_j) = 0$.

Remark 3.4. *The group $Braid_{\mathcal{C}}$ plays a role of the braid group (or Weyl group) in the theory of Kac-Moody algebras. If we add also the group $\mathbb{Z}$ of shifts $F \mapsto F[n], n \in \mathbb{Z}$ then we obtain an affine version of the braid group $Braid_{\mathcal{C}} \times \mathbb{Z}$. In some examples $\mathbb{Z} \subset Braid_{\mathcal{C}}$.*

3.4 Motivic DT-series for $2CY$ categories

Let $\mathcal{C}$ be an ind-constructible locally regular $2CY$ category over $\mathbf{k}$. Let us fix $R = Mot(Spec(\mathbf{k}))$ as the ground ring for the quantum torus $\mathcal{R}_{\Gamma,R}$. We will denote the latter by $\mathcal{R}_{\Gamma}$. It is a commutative algebra generated by elements $\widehat{e}_{\gamma}, \gamma \in \Gamma$ such that $\widehat{e}_{\gamma_1+\gamma_2} = \widehat{e}_{\gamma_1}\widehat{e}_{\gamma_2}, \widehat{e}_0 = 1$. Let us also fix a stability condition on $\mathcal{C}$ with the central charge $Z : \Gamma \to \mathbb{C}$.

Definition 3.5. *The motivic weight $\omega \in Mot(\mathcal{O}b(\mathcal{C}))$ is defined by $\omega(E) = \mathbb{L}^{\frac{1}{2}(\chi(E,E))}$.*

Then we have the following result.

Proposition 3.6. *The map $\Phi : H(\mathcal{C}) \to \mathcal{R}_{\Gamma}$ given by $\Phi(\nu) = (\nu, \omega)\widehat{e}_{\gamma}, \nu \in H(\mathcal{C})_{\gamma}$ satisfies the condition $\Phi(\nu_1 \cdot \nu_2) = \Phi(\nu_1)\Phi(\nu_2)$ for $Arg(\gamma_1) > Arg(\gamma_2)$, where $\nu_i \in H(\mathcal{C})_{\gamma_i}$. (here $(\bullet, \bullet)$ is the pairing between motivic measures and motivic functions.)*

In other words, Φ can be written as $[\pi : Y \to \mathcal{O}b(\mathcal{C})] \mapsto \int_Y \mathbb{L}^{\frac{1}{2}\chi(\pi(y),\pi(y))}\widehat{e}_{cl(\pi(y))}$.

Proof. It suffices to prove the theorem for $\nu_{E_i} = [\delta_{E_i} : pt \to \mathcal{O}b(\mathcal{C})]$, where $\delta_{E_i}(pt) = E_i \in \mathcal{O}b(\mathcal{C})$. Recall that we denote $\dim Ext^i(E,F)$ by $(E,F)_i, i \in \mathbb{Z}$.

We have $\Phi(\nu_{E_i}) = \mathbb{L}^{\frac{1}{2}\chi(E_i,E_i)}\widehat{e}_{\gamma_i}$, which implies that $\Phi(\nu_{E_1})\Phi(\nu_{E_2}) = \mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))}\widehat{e}_{\gamma_1+\gamma_2}$.

On the other hand, $\nu_{E_1} \cdot \nu_{E_2} = \mathbb{L}^{-(E_2,E_1)_{\leq 0}}[\pi_{21} : Ext^1(E_2,E_1) \to \mathcal{O}b(\mathcal{C})]$. Then

$$\begin{aligned}
&\Phi(\nu_{E_1} \cdot \nu_{E_2}) = \mathbb{L}^{-(E_2,E_1)_{\leq 0}} \int_{\alpha \in Ext^1(E_2,E_1)} \mathbb{L}^{\frac{1}{2}\chi(E_\alpha,E_\alpha)}\widehat{e}_{\gamma_1+\gamma_2} \\
= \;& \mathbb{L}^{-(E_2,E_1)_{\leq 0}}\mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2)+\chi(E_1,E_2)+\chi(E_2,E_1))} \int_{\alpha \in Ext^1(E_2,E_1)} \widehat{e}_{\gamma_1+\gamma_2} \\
= \;& \mathbb{L}^{-(E_2,E_1)_{\leq 0}+\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))+\chi(E_2,E_1)}\mathbb{L}^{(E_2,E_1)_1}\widehat{e}_{\gamma_1+\gamma_2} \\
= \;& \mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))+(E_2,E_1)_2}\widehat{e}_{\gamma_1+\gamma_2}
\end{aligned}$$

If $Arg(\gamma_1) > Arg(\gamma_2)$, then $(E_2,E_1)_2 = (E_1,E_2)_0 = 0$. Thus $\Phi(\nu_{E_1} \cdot \nu_{E_2}) = \Phi(\nu_{E_1})\Phi(\nu_{E_2})$. □

Recall the categories $\mathcal{C}_V$ and set $V = l$ be a ray. For a generic central charge Z let us consider the generating function

$$A_l^{mot} = \sum_{[E], E \in Ob(\mathcal{C}_l)} \frac{\omega(E)\widehat{e}_{cl(E)}}{[Aut(E)]} =$$

$$\sum_{[E], E \in Ob(\mathcal{C}_l)} \mathbb{L}^{\frac{1}{2}(\chi(E,E))} \frac{t^{cl(E)}}{[Aut(E)]},$$

where $t = \widehat{e}_{\gamma_0}$ for a primitive γ_0 such that $Z(\gamma_0) \in l$ generates $Z(\Gamma) \cap l$ and $[Aut(E)]$ denotes the motive of the group of automorphisms of E. More invariantly, $A_l^{mot} = \Phi(A_l^{Hall})$ where $A_l^{Hall} \in H(\mathcal{C}_l)$ corresponds to the characteristic function of the stack of objects of the full subcategory $\mathcal{C}_l \subset \mathcal{C}$ generated by semistables E such that $Z(E) \in l$ (cf. loc.cit.).

Definition 3.7. *We call A_l^{mot} the motivic DT-series of $\mathcal{C}$ corresponding to the ray l.*

Suppose that $\mathcal{C}$ is associated with the preprojective algebra Π_Q. One can show that A_l^{mot} can be obtained from the motivic DT-series for the $3CY$ category associated with $(\widehat{Q}, W)$ by the reduction to $\mathcal{C}$. Similarly to A_l^{mot} we define A_V^{mot} for any strict sector V.

The Proposition 3.6 implies that the series A_V^{mot} is the (clockwise) product of A_l^{mot} over all rays $l \subset V$. This can be also derived from the dimensional reduction and the results of [12].

Corollary 3.8. *The collections of elements $A_V^{mot} = \Phi(A_V^{Hall})$ parametrized by strict sectors $V \subset \mathbf{R}^2$ with the vertex in the origin satisfies the Factorization Property: if a strict sector V is decomposed into a disjoint union $V = V_1 \sqcup V_2$ in the clockwise order, then $A_V^{mot} = A_{V_1}^{mot} A_{V_2}^{mot}$.*

Proposition 3.9. *Motivic DT-series A_V^{mot} is constant on each connected component of the space of stability conditions.*

Proof. Similarly to the case of $3CY$ categories, each element A_V^{mot} does not change when we move in the space of stability conditions on $\mathcal{C}$ in such a way that central charges of semistable object neither enter nor leave the sector V. But in the case of $2CY$ categories the Euler form is symmetric, hence the motivic quantum torus is commutative. It follows that the wall-crossing formulas from [12] are trivial. This implies the result. $\square$

For a $2CY$ category form our class one can construct the corresponding $3CY$ category (see Introduction). We expect that the motivic DT-series arising in this situation are quantum admissible in the sense of [13] and can be described in terms of the corresponding COHA (the latter is expected to exist for quite general $3CY$ categories, see [18]).

Therefore, by analogy with the case of $3CY$ categories, we can define DT-invariants $\Omega(\gamma)$ in $2CY$ case using (quantum) admissibility (see [13], Section 6) of our DT-series by the formula:

$$A_V^{mot} = Sym\left(\sum_{n\geq 0}\mathbb{L}^n \sum_{\gamma\neq 0, Z(\gamma)\in V}\Omega(\gamma)\widehat{e}_\gamma\right) =$$

$$= Sym\left(\frac{\sum_{\gamma\neq 0, Z(\gamma)\in V}\Omega(\gamma)\widehat{e}_\gamma}{1-\mathbb{L}}\right).$$

By Proposition 3.9 our motivic DT-invariants $\Omega(\gamma)$ depend only on the connected component of $Stab(\mathcal{C})$ which contains Z. The Conjecture 3.10 (see next subsection) says that $\Omega(\gamma)$ is (essentially) the same as Kac polynomial $a_\gamma(\mathbb{L})$ (or the motivic DT-invariant of the corresponding $3CY$ category, see Introduction).

Let us fix the connected component in $Stab(\mathcal{C})$ which contains such central charge Z that for each spherical generator E_i of $\mathcal{C}$ we have $Z(E_i) = (0, ..., 1, ...0)$ (the only nontrivial element 1 at the i-th place). We will call the corresponding t-structure *standard*. We denote the corresponding motivic DT-invariants by $\Omega_{\mathcal{C}}^{mot}(\gamma)$.

3.5 Kac polynomial of a $2CY$ category

We can now introduce an analog of the Kac polynomial in the case of a $2CY$ category from our class following the ideas of [16].

Notice that the coefficient ring $Mot(Spec(\mathbf{k}))$ of the quantum torus $\mathcal{R}_\Gamma$ has a $\lambda-$ring structure, which can be lifted to the quantum torus (which is commutative in the case of $2CY$ categories). Recall that for a λ-ring we can introduce the operation of symmetrization by the formula:

$$Sym(r) = \sum_{n\geqslant 0} Sym^n(r) = \sum_{n\geqslant 0}(-1)^n\lambda^n(-r) = (\sum_{n\geqslant 0}(-1)^n\lambda^n(r))^{-1}.$$

For any ray $l \subset \mathbb{H}_+$, where $\mathbb{H}_+$ is the upper half plane, we have the (quantum) admissible element A_l^{mot}.

Let $\mathcal{C}$ be a $2CY$ category from our class. We fix the standard t-structure. Recall the motivic DT-series A_l^{mot}.

Conjecture 3.10. *There exist elements $a_\gamma^{mot}(\mathbb{L}) \in Mot(Spec(\boldsymbol{k}))$ which are polynomials in $\mathbb{L}$ and such that the following formula holds in the (commutative) motivic quantum torus:*

$$A_l^{mot} = Sym\left(\frac{\sum_{\gamma, Z(\gamma)\in l}(-a_\gamma^{mot}(\mathbb{L})\cdot \mathbb{L})\widehat{e}_\gamma}{1-\mathbb{L}}\right).$$

Furthermore, there exists a $3CY$ category $\mathcal{B}$ such that the elements $a^{mot}(\mathbb{L})$ coincide with motivic DT-invariants with respect to some stability condition on $\mathcal{B}$.

Some related results can be found in [3], [6] [8], and especially in [16]. In fact Theorem 5.1 from [16] establishes the Conjecture in the framework of quivers. More precisely, if $\mathcal{C}$ is the $2CY$ category associated with the preprojective algebra of a quiver, then for its standard t-structure the element $a_\gamma^{mot}(\mathbb{L})$ coincides with the Kac polynomial $a_\gamma(\mathbb{L})$ of the Kac-Moody algebra corresponding to the quiver.

We plan to discuss the general case in the future work.

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