

RATIONAL AND NON-RATIONAL FANO 3-FOLDS

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1. SMOOTH FANO 3-FOLDS

Iskovskih, Manin and Mori showed there are 105 families of smooth Fano 3-folds. Most are rational; some are not. In general, there are three invariants one uses to sort out Fano 3-folds:

- the degree $-K_X^3$, which is an even integer, by adjunction;
- the index I , which is the largest positive integer n that divides $-K_X$ in the class group; and
- the rank ρ of the class group of X .

Generally speaking, the larger these invariants are, the likelier it is that X is rational.

1.1. **Fano 3-folds with $\rho = I = 1$.** With these invariants fixed, we have many results on rationality/nonrationality of X :

- (1) Degree 2: here X_2 is a sextic in $\mathbb{P}(1, 1, 1, 1, 2)$, and the projection $X_2 \rightarrow \mathbb{P}^3$ is branched over a sextic curve C_6 . Iskovskih showed that X_2 is nonrational if it is smooth.
- (2) Degree 4: hypersurfaces $X_4 \subset \mathbb{P}^4$. They are nonrational if smooth; this is due to Iskovskih and Manin.
- (3) Degree 6: complete intersections $X_{2,3} \subseteq \mathbb{P}^5$ of a quadric and a cubic. The general such 3-fold is nonrational; this is a result of Iskovskih and Pukhilov.
- (4) Degree 8: complete intersections $X_{2,2,2} \subseteq \mathbb{P}^6$ of three quadrics. Beauville showed they are nonrational if smooth.
- (5) Degree 10: the very general such 3-fold is nonrational (Beauville).
- (6) Degree 14: every smooth such 3-fold is nonrational (it is birational to a smooth cubic threefold).

1.2. **Fano 3-folds with $\rho = 1$ and $I = 2$.** Say $-K_X = 2H$, where H is ample. Then $H^3 \in \{1, 2, 3, 4, 5\}$. We denote such a 3-fold V_i , where $i = H^3$. The ones of type V_4 and V_5 are rational. The other types are not: Clemens-Griffiths showed that a smooth V_3 is nonrational; Voisin showed that V_2 which are double solids ramified over a quartic are nonrational if smooth; Grinenko showed that 3-folds of type V_1 are nonrational if smooth.

1.3. **Fano 3-folds with $\rho \geq 2$.** When $\rho = 2$ there are two new nonrational examples. All other non-rational Fano threefolds with $\rho = 2$ are blow ups of non-rational Fano threefolds with $\rho = 1$. A similar situation holds for $\rho \geq 3$.

2. SINGULAR FANO 3-FOLDS

Throughout, ODP stands for “ordinary double point”.

Theorem 2.1 (Mella). *Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface with ordinary double points, such that $\text{Cl } X_4 \simeq \mathbb{Z}$. Then X_4 is nonrational.*

The situation is more complicated if we remove the hypothesis that $\text{Cl } X_4 \simeq \mathbb{Z}$:

Example 2.2. *If X_4 is a very general quartic threefold containing a plane (so in particular $\text{Cl } X_4 \simeq \mathbb{Z}^2$), then X has 9 ODPs and is nonrational. One uses degeneration methods to prove this result.*

Example 2.3. *If X_4 is a general quartic threefold containing a quartic surface, then it is birational to a Fano 3-fold of the form $X_{2,3} \subset \mathbb{P}^5$ which is singular, having only ODPs, and it is nonrational.*

Example 2.4. *This example is motivated by a question of Prokhorov. Suppose that $X_4 \subset \mathbb{P}^4$ is a quartic threefold with an S_6 -action. We may write such a threefold symmetrically in $\mathbb{P}^5 = \text{Proj } k[x_0, \dots, x_5]$ as the intersection of*

$$\sum_{i=0}^5 x_i = 0, \text{ and}$$

$$\sum_{i=0}^5 x_i^4 = t \left(\sum_{i=0}^5 x_i^2 \right)^2$$

Such threefolds were studied by van der Geer. They are always singular along an S_6 -orbit Σ_{30} of length 30. If $t \neq \frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{7}{10}$, then $\text{Sing}(X_4) = \Sigma_{30}$. If $t \neq \frac{1}{4}$, then X_4 has only ODPs and $\text{Cl } X_4 \not\simeq \mathbb{Z}$. Moreover, we have the following cases:

- $t = \frac{1}{4}$, known as the Igusa quartic. Its singularities are non-isolated. It is rational.
- $t = \frac{1}{2}$, known as the Bukhardt quartic. It has 45 singularities and is rational.
- $t = \frac{1}{6}$, which has 40 ODPs.
- $t = \frac{7}{10}$, which has 36 ODPs.

The last two are rational by work of Cheltsov and Shramov. Beauville showed that X_4 is nonrational if $t \notin \{\frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{7}{10}\}$. Note that $\dim IJ(X_4) = 5$. Question: what are the Mori fibre spaces birational to X_4 in the non-rational case?

3. RATIONALLY CONNECTED 3-FOLDS

If one runs MMP on a rationally connected 3-fold, one gets as outputs

- a conic bundle,
- a del Pezzo fibration over \mathbb{P}^1 ,
- a Fano 3-fold X with terminal singularities and $\text{Cl } X = \mathbb{Z}$ up to torsion. We call such an output a \mathbb{Q} -Fano 3-fold.

The last class is birationally bounded (Kawamata).

The output of MMP need not be unique, and this prompts the following definition, due to Corti:

Definition 3.1. *A \mathbb{Q} -Fano 3-fold X is **birationally rigid** if it is a unique output of MMP.*

Example 3.2. *If $X_4 \subset \mathbb{P}^4$ is a quartic threefold with terminal singularities and $\text{Cl } X \simeq \mathbb{Z}$, then it is \mathbb{Q} -Fano 3-fold.*

Theorem 3.3 (Iskovskikh; Mella). *If $X_4 \subset \mathbb{P}^4$ is a quartic threefold with at worst ODP singularities and $\text{Cl } X_4 = \mathbb{Z}$, then X_4 is birationally rigid.*

Theorem 3.4 (Corti–Mella). *Let $X_4 \subset \mathbb{P}^4$ be a general quartic threefold with terminal singularities, with one singular point locally given by $xy + z^3 + t^3 = 0$ in \mathbb{C}^4 . Then X_4 is nonrational.*

We remark that an X_4 in the above theorem is not birationally rigid: MMP has exactly two outputs.

3.1. Universal birational rigidity. Kollár has written on the behavior of birational rigidity under base extension of the ground field. If $k = \bar{k}$ is an uncountable field of characteristic zero, we say that X/k is **universally birationally rigid** if X_F is birationally rigid for all extensions F/k .

Theorem 3.5 (Kollár). *If $\text{Bir } X$ is countable then*

$$X \text{ is birationally rigid} \iff X \text{ is universally birationally rigid.}$$

There are birationally rigid Fano 3-folds with uncountable group of birational automorphisms. One example to look at is due to Iskovskikh: it is a smooth double cover X'_4 of a quadric 3-fold, with $-K_{X'_4}^3 = 4$; it is birationally rigid, but

$$\text{Bir } X_4 = \langle \tau_C : C \text{ a curve with } -K_{X'_4}^2 C = 1 \rangle$$

which is uncountable.

What if $\text{Bir } X$ is uncountable? Are birational rigidity and universal birational rigidity different notions in this case? We expect that the answer is yes (these are different notions).

3.2. Solid Fano 3-folds. Ahmadinezhad and Okada have introduced the following concept:

Definition 3.6. *A \mathbb{Q} -Fano 3-fold X is **solid** if X is not birational to a conic bundle or a del Pezzo surface fibration.*

They have initiated a program to classify such 3-folds. To get a sense for how complicated this is, recall that there are three basic invariants of Fano 3-folds: its degree, its index and its Picard number. For \mathbb{Q} -Fano 3 folds, the degree satisfies

$$\frac{1}{330} \leq -K_X^3 \leq \begin{cases} 64 & \text{if } X \text{ is Gorenstein,} \\ 125/2 & \text{otherwise.} \end{cases}$$

These bounds are sharp. The index (which one has to be careful in defining since we are dealing with \mathbb{Q} -divisors) satisfies

$$9 \leq I \leq 19.$$

For Fano 3-folds X , one can get a sense for how many families there are by looking at their anticanonical ring $R(X, -K_X)$. We have:

codim	1	2	3	4	\dots	18
$\#$ families	95	85	1 + 69	"145"	\dots	"4709"

In codimension 1 (so that X is a hypersurface in a weighted projective space), there are 95 families discovered by Iano-Fletcher. Every quasismooth 3-fold in these families is birationally rigid (Corti, Pukhlikov, Reid, Cheltsov, Park), so that it is solid. In codimension 2, general elements of 18 families are birationally rigid, and 6 of them are known to be solid. The general elements of remaining $65 - 18 = 47$ families are not birationally rigid (Okada). In codimension 3, 69 families are Pfaffian Fano 3-folds and 1 family is a complete intersection of three quadrics in \mathbb{P}^6 . Only 3 families among the 69 contain birationally rigid (general) members (Ahmadinezhad and Okada). Numbers in quotes signify that all we know is that there are this many different Hilbert Series for the anticanonical ring.