The integral Hodge conjecture for threefolds

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Hodge decomposition: For $X$ a smooth complex projective variety we have

$$H^i(X, \mathbb{C}) = \bigoplus_{j=0}^{i} H^{i-j}_j(X),$$

where $H^{i-j}_j(X) := H^{i-j}(X, \Omega^j_X)$.

**Fact:** For any complex subvariety $Y \subset X$ of codimension $i$ the image of $[Y] \in H^{2i}(X, \mathbb{Z})$ in $H^{2i}(X, \mathbb{C})$ lies in $H^{i,i}(X)$.

**Hodge conjecture (HC):** For any $u \in H^{2i}(X, \mathbb{Q})$ whose image in $H^{2i}(X, \mathbb{C})$ is in $H^{i,i}(X)$, $u$ is the class of an algebraic cycle, i.e., a $\mathbb{Q}$-linear combination of codimension $i$ subvarieties of $X$.

**Integral Hodge conjecture (IHC):** same statement for $H^{2i}(X, \mathbb{Z})$ and algebraic cycles with $\mathbb{Z}$ coefficients.

The main evidence is the Lefschetz $(1, 1)$ theorem, i.e., the IHC is true for $i = 1$.

**Corollary 1** The Hodge conjecture (with $\mathbb{Q}$-coefficients) is true for 1-cycles, i.e., for $H^{2n-2}(X, \mathbb{Q})$, where $n = \dim(X)$.

The first open case of the HC: 2-cycles on a 4-fold, even an abelian 4-fold. Weil described explicit Hodge classes in the middle cohomology of a special class of abelian 4-folds which are not known to be algebraic. [Bogomolov: And this is all one needs to prove in the case of abelian 4-folds. Totaro: That’s right. For other classes of abelian 4-folds (not of Weil’s type), there are not many Hodge classes, and the Hodge conjecture is known.]
Remark 1 The Hodge decomposition of $H^{2i}(X, \mathbb{C})$ is not defined over $\mathbb{Q}$.

For a family of smooth projective varieties $X \to B$, the Hodge decompositions on $X_t$ vary continuously for $t \in B$, and filtrations vary holomorphically. So the HC predicts that algebraic cycles ‘jump up’ on special varieties of the family, when $H^{12}(X)$ has larger than usual intersection with $H^{2i}(X, \mathbb{Q})$. The HC is often easy to check for general varieties in a family, as there are few Hodge cycles beyond those generated by divisors.

[Graber: Given the paucity of evidence, why do people believe the Hodge conjecture? Totaro: I cannot really answer this, but it is a key ingredient in a larger framework governing cycles in algebraic geometry.]

Integral Hodge conjecture for threefolds. This is false for very general hypersurfaces in $\mathbb{P}^4$ of degree $d$ for certain $d$ ($d \geq 48$); this is due to Kollár.

For any smooth hypersurface $X$ in $\mathbb{P}^4$, we have

$$H^*(X, \mathbb{Z}) = \left\{ \begin{array}{c}
\mathbb{Z} & 0 & \mathbb{Z}^N & \mathbb{Z} & 0 & \mathbb{Z} \\
0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \right\}$$

and the IHC for $X$ would predict that all of $H^4(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ is generated by curves in $X$. Note that

$$\text{deg} : H^4(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

and thus the IHC is true for $X$ if $X$ contains a line (hence for $\text{deg}(X) \leq 5$) and for some special hypersurfaces of any degree.

Conjecture 1 (Griffiths–Harris) For a very general hypersurface $X$ of degree $d \geq 6$ in $\mathbb{P}^4$, every curve in $X$ has degree $\equiv 0 \pmod{d}$.

Theorem 1 (Kollár) For $X$ a very general hypersurface of degree 48 in $\mathbb{P}^4$, every curve in $X$ has even degree. In particular, IHC is false for $X$.

Proof: Find a singular 3-fold $Y \subset \mathbb{P}^4$ of degree 48 such that every curve on $Y$ has even degree. Then apply a degeneration argument, noting that any specialization of a curve of odd degree has at least one component of odd degree.

To produce the singular hypersurface $Y$, start with $Z$ any smooth projective threefold and $L$ a very ample line bundle with $L^3 = d$ (e.g., 48). Suppose
that every curve \( C \subset Z \) has \( L \cdot C \equiv 0 \pmod{e} \), for \( e \) a positive integer not dividing 6 (e.g., 4). (For example, \( Z = \mathbb{P}^3 \), \( L = O(4) \), \( d = 64 \), \( e = 4 \).

To give an example of degree 48, let \( S \) be a very general quartic surface \( S \), \( Z = S \times \mathbb{P}^1 \), and \( L = \pi_1^*O(1) \otimes \pi_2^*O(4) \) on \( Z \); then \( d = 48 \) and every curve on \( Z \) has degree a multiple of 4 with respect to \( L \).

Embed \( Z \hookrightarrow \mathbb{P}^N \) using \( L \) and take a general linear projection to a hypersurface \( Y \) in \( \mathbb{P}^4 \). We might hope that every curve \( C \subset Y \) has degree \( \equiv 0 \pmod{e} \). However, the finite morphism \( \pi : Z \to Y \) can be 2 : 1 on a surface and 3 : 1 on some curves (and show more complicated behavior over finitely many bad points). Thus every curve \( C \subset Y \) is either

\[
\pi_*(\text{curve}), \quad \frac{1}{2}\pi_*(\text{curve}), \quad \text{or} \quad \frac{1}{3}\pi_*(\text{curve})
\]

as a cycle. So every curve \( C \subset Y \) has degree a multiple of \( e/6 \).

**Hassett-Tschinkel construction** of counterexamples to IHC defined over \( \mathbb{Q} \):

Do the same over \( \text{Spec}(\mathbb{Z}) \), i.e., choose a hypersurface with a reduction modulo \( p \) that is a general projection.

**Positive results on IHC for threefolds:**

**Theorem 2 (Voisin)** Let \( X \) be a smooth projective threefold over \( \mathbb{C} \) which is either uniruled or strongly Calabi-Yau (\( K_X \simeq O_X \) and \( b_1(X) = 0 \)). Then the IHC is true for \( X \), i.e., \( H_2(X,\mathbb{Z}) \) is generated by algebraic curves.

**Question 1** Let \( X \) be a smooth projective threefold, uniruled or simply-connected Calabi-Yau. Is \( H_2(X,\mathbb{Z}) \) generated by rational curves?

**Question 2 (asked by Tschinkel)** Does every simply connected Calabi-Yau threefold \( X \) contain any rational curve?

Question 2 has a positive answer for \( X \) of Picard number at least 14, by R. Heath-Brown and P.M.H. Wilson.

How about IHC for rationally connected varieties of higher dimension?

**Theorem 3 (Colliot-Thélène, Voisin)** There is a rationally connected (RC) 6-fold (fibered over \( \mathbb{P}^3 \) with generic fiber a quadric) for which IHC fails for codimension-two cycles.
This draws on previous work of Ojanguren and Colliot-Thélène on non-vanishing of unramified cohomology in this case. The key recent innovation is the development of links between unramified cohomology and the failure of the integral Hodge conjecture.

[Starr: How about blowing up something for which the IHC fails in projective space? Totaro: That works, but not for codimension two cycles.]

Despite Theorem 3, IHC for 1-cycles on a RC variety of any dimension looks plausible.

**Theorem 4 (Voisin)** If the Tate conjecture (with $\mathbb{Q}_\ell$ coefficients) holds for 1-cycles on all smooth projective surfaces over $\mathbb{F}_q$, then the IHC holds for 1-cycles on RC varieties of any dimension over the complex numbers.

**Proof:** The IHC for 1-cycles on RC varieties is deformation invariant. Indeed, can use very free rational curves to get curves with ample normal bundle, i.e., both

$$C + \text{very free curves} \quad \text{and} \quad \text{very free curves}$$

deform.

Voisin uses Chad Schoen’s theorem: If the Tate conjecture (with $\mathbb{Q}_\ell$-coefficients) holds for all varieties [Colliot-Thélène: suffices for just surfaces] over $\mathbb{F}_q$ then it holds for 1-cycles with $\mathbb{Z}_\ell$ coefficients on all varieties over $\mathbb{F}_q$.

Specialize your RC variety to a separably rationally connected variety over a finite field, then lift curve classes to characteristic zero. \(\square\)

**Theorem 5 (Totaro)** Can omit the assumption that $b_1(X) = 0$ for Calabi Yau 3-folds. In particular, IHC holds for 1-cycles on abelian 3-folds.

How to prove IHC for CY 3-folds $X$ or uniruled 3-folds: Look at the smooth surfaces $S$ in $|dH|$ for $H$ ample on $X$, $d \gg 0$. The Lefschetz hyperplane theorem says that

$$H_2(S, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z})$$

is onto. So let $u \in H_2(X, \mathbb{Z})$ be a Hodge class. If $u$ is the image of a Hodge class on some surface $S_t$ in the family then we win. (This reduces us to a question about Hodge theory.)
Idea: If $K_X$ is “negative” or trivial then

$$h^0(S, N_{S/X}) \geq h^2(S, \mathcal{O}).$$

The former measures the dimension of the deformation space of $S$ in $X$; the latter how complicated the Hodge structure on $H^2(S)$ is. Essentially, $h^2(S, \mathcal{O})$ measures the number of conditions that have to be satisfied for a given integral class to be Hodge.

That inequality suggests the possibility that every homology class in $H_2(X, \mathbb{Z})$ in some open cone near $H^{1,1}(X)$ may become a Hodge class on some surface $S$ in the family near a given one. We justify that by an infinitesimal calculation to show that the Hodge structure on the family of surfaces $S$ is varying “as much as possible”.

[C. Xu: For RC varieties, is the effective cone generated by rational curves? J. Li: How does the simple-connectivity come in? Totaro: For a Calabi-Yau threefold $X$ with first Betti number $X$ not zero, we also have $H^2(X, O_X) \neq 0$. So the IHC becomes a more complicated statement to prove: we have to prove that a specified subgroup of $H_2(X, \mathbb{Z})$ is generated by algebraic curves, rather than showing that all of $H_2(X, \mathbb{Z})$ is generated by algebraic curves. Colliot-Thélène: Do you have a new proof of Voisin’s result? Totaro: No, my goal was to extend her methods to a broader class of 3-folds. McKernan: Does IHC hold for 3-folds of Kodaira dimensions 1 and 2? Totaro: There are counterexamples in those cases too, by Colliot-Thélène and Voisin. So the most one can hope would be that IHC might hold for all 3-folds of Kodaira dimension at most zero.]