

Constructing rational curves on K3 surfaces

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1 Introduction

Let X be a smooth projective variety over \mathbb{C} . At least conjecturally, we have the following behavior of rational curves on X depending on its Kodaira dimension κ .

$\kappa = -\infty$	\Leftarrow	uniruled	\Rightarrow	there exist moving families of rational curves
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$\kappa = \dim(X)$	Lang's conjecture:	there exists a proper closed subset that contains every rational curve of X
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$\kappa = 0$	Abelian variety	\Rightarrow	no rational curves
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Whereas there are no rational curves on Abelian varieties, we expect many rational curves on other types of varieties of $\kappa = 0$, such as K3 surfaces and Calabi–Yau varieties:

2 K3 Surfaces

For K3 surfaces, we have the following well-known conjecture:

Conjecture 1 *Let X be a projective K3 surface. Then it contains infinitely many rational curves.*

Note that rational curves on K3 surfaces over \mathbb{C} cannot move. That is, every curve must have its own “reason of existence”. For K3 surfaces defined over number fields, Bogomolov even asked whether there is a rational curve through every $\overline{\mathbb{Q}}$ -rational point.

We have the following results towards this conjecture:

Theorem 1 (Bogomolov-Mumford) *Let X be a projective K3 surface over an algebraically closed field and \mathcal{L} be a non-trivial and effective invertible sheaf. Then there exists a section*

$$\sum n_i C_i \in |\mathcal{L}|$$

where $n_i \geq 1$ and with each C_i a rational curve.

Corollary 1 *Every projective K3 surface contains at least one rational curve.*

Using degeneration techniques, one shows that the conjecture “almost always” holds:

Theorem 2 (Mori-Mukai) *For every $d \geq 1$, a very general K3 surface in $\mathcal{M}_{2d}/\mathbb{C}$ contains infinitely many rational curves.*

Xi Chen proved that these rational curves are generally nodal, at least in primitive classes. To have explicit examples, we have the following:

Theorem 3 (Bogomolov-Tschinkel) *Elliptic K3 surfaces contain infinitely many rational curves.*

3 Arithmetic approach

We will now discuss the techniques and results found in [BHT] and [LL]. First, we reduce our main conjecture to the case of number fields:

Proposition 1 ([BHT]) *It suffices to establish the conjecture for K3’s over $\overline{\mathbb{Q}}$ to get it over \mathbb{C} .*

The idea of proof is as follows: every K3 surface over \mathbb{C} can be defined over a field that is finitely generated over \mathbb{Q} and if this field is not a number field already, then X can be thought of as the geometric generic fiber of family of K3 surfaces over some number field. There always exists a specialization such that the Picard group does not jump. But then, one can deform all the rational curves on such a specialization back to the general fiber. \square

Thus, we will now assume that X is a K3 surface over $\overline{\mathbb{Q}}$. Then, X can be defined over a number field K , and thus, there exists a smooth projective model

$$\mathcal{X} \rightarrow \operatorname{Spec}(\mathcal{O}_{K,S}),$$

where \mathcal{O}_K denotes the ring of integers of K and S is some finite set of places. Throughout the talk, several arguments will require us to take finite extensions of K and/or to enlarge S . This is one obstacle towards having effective versions of our main result below.

Observation (Artin, Swinnerton-Dyer)

Let Y be a K3 surface over \mathbb{F}_q and assume that it satisfies the Tate conjecture. Then the Picard number $\rho(Y \otimes \overline{\mathbb{F}}_q)$ is even.

Now, for all places $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_{K,S})$ we have an injection

$$\operatorname{Pic}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \operatorname{Pic}(\mathcal{X}_{\mathfrak{p}}).$$

In particular, if $\rho(X_{\overline{\mathbb{Q}}})$ is odd, then the geometric Picard rank jumps for every place of good reduction for which the Tate conjecture holds. Using these “new” lines bundles over finite fields and the Bogomolov–Mumford result above we find rational curves on reductions. Using results of Nygaard–Ogus on the Tate conjecture, and results of Bogomolov–Zarhin and Joshi–Rajan on ordinary reduction of K3 surfaces, we have the following unconditional result:

Proposition 2 *Let (X, H) be a polarized K3 surface over $\overline{\mathbb{Q}}$, with $\rho(X)$ odd. Then there exists a model $\mathcal{X} \rightarrow \operatorname{Spec}(\mathcal{O}_{K,S})$ and a set of primes P of density 1, such that for every $\mathfrak{p} \in P$*

- *there exists a rational curve $C_{\mathfrak{p}} \hookrightarrow \mathcal{X}_{\mathfrak{p}}$;*
- *$\mathcal{O}_{\mathcal{X}_{\mathfrak{p}}}(C_{\mathfrak{p}})$ does not lift to $X_{\overline{\mathbb{Q}}}$;*
- *the height of the formal Brauer group $\widehat{\operatorname{Br}}(\mathcal{X}_{\mathfrak{p}})$ is finite;*

- the set $\{H \cdot C_{\mathfrak{p}} : \mathfrak{p} \in P\} \subseteq \mathbb{Z}_{>0}$ is unbounded.

Upshot: We found infinitely many rational curves on infinitely many reductions modulo positive characteristic, but these curves are not liftable.

Now, for each $\mathfrak{p} \in P$, if $N_{\mathfrak{p}}$ is a sufficiently large integer then

$$|N_{\mathfrak{p}}H - C_{\mathfrak{p}}| \neq \emptyset.$$

Using the Bogomolov–Mumford result, there exists rational curves $C_{\mathfrak{p}}^i$ on $X_{\overline{\mathfrak{p}}}$ and integers $n_i \geq 1$ such that

$$C_{\mathfrak{p}} + \sum n_i C_{\mathfrak{p}}^i \in |N_{\mathfrak{p}}H|,$$

Next, we represent this cycle as the image of a stable map $f \in \overline{\mathcal{M}}_{0,0}(\mathcal{X}_{\mathfrak{p}}, N_{\mathfrak{p}}H)$. If f is a “rigid stable map”, that is

$$\dim_{[f]} \overline{\mathcal{M}}_{0,0}(\mathcal{X}_{\mathfrak{p}}, N_{\mathfrak{p}}H) = 0$$

then we can extend f to $\overline{\mathcal{M}}_{0,0}(\mathcal{X}, N_{\mathfrak{p}}H)$, by dimension estimates from deformation theory. In particular, we can lift the whole cycle of rational curves on $\mathcal{X}_{\overline{\mathfrak{p}}}$ to a cycle of rational curves on $X_{\overline{\mathbb{Q}}}$. However, the n_i could be huge, and so, we might not find a representative f satisfying the dimension condition. (Another problem is that we need non-supersingularity to exclude unirational K3 surfaces with moving rational curves.)

In case there is an involution on the K3 surface, one can control these multiplicities:

Theorem 4 ([BHT]) *If X is a K3 surface over \mathbb{C} with $\text{Pic}(X) = \mathbb{Z}H$, $H^2 = 2$. Then there exist infinitely many rational curves.*

To deal with the general case of odd Picard rank, we introduce the following class of rational curves on a K3 surface.

Definition 1 ([LL]) *Let (X, H) be a polarized K3 surface. A rigidifier R is a curve on X which is*

- an integral rational curve;
- has at worst nodal singularities;

- and $R \in |MH|$ for some $M > 0$.

Theorem 5 ([LL]) *Let (X, H) be a polarized K3 surface over an algebraically closed field k . Assume that there exists a rigidifier R on X . If $\text{char}(k) > 0$, assume moreover that X is not supersingular. Given any cycle $\sum n_i C_i$, where the C_i are rational curves and $n_i \geq 1$, then for some $m > 0$ the cycle*

$$mR + \sum n_i C_i$$

can be represented by some $f \in \overline{\mathcal{M}}_{0,0}(X)$ at which the dimension is zero.

The idea is to “stack” copies of R to construct, combinatorially, the desired stable map. As Ravi Vakil points out, rigidifying is basically the opposite of bend-and-break.

Theorem 6 ([LL]) *Let X be a K3 surface over \mathbb{C} with $\rho(X)$ odd. Then X contains infinitely many rational curves.*

The idea of proof is as follows: we want to lift the cycles $C_{\mathfrak{p}} + \sum_i n_i C_i^{\mathfrak{p}}$ using rigidifiers. However, there might be no rigidifiers on $\mathcal{X}_{\mathfrak{p}}$. But then, we can deform $(\mathcal{X}_{\mathfrak{p}}, C_{\mathfrak{p}})$ until it does have a rigidifier, except for a possibly finite set of \mathfrak{p} . Here, we use Xi Chen’s work on nodal rational curves in linear systems of generic K3 surfaces. \square

So what remains to be done? K3 surfaces with Picard rank $\rho \geq 5$ are elliptic and therefore contain infinitely many rational curves. K3’s with $\rho = 1$ and $\rho = 3$ are covered by our main result. Combining different arguments, one can also do $\rho = 4$ except for a handful of rank 4 Néron–Severi lattices. As Fedor Bogomolov points out, K3 surfaces with Néron–Severi lattice

$$\begin{pmatrix} -2 & n \\ n & -2 \end{pmatrix}$$

and $n > 2$ are still open.

References

- [BHT] F. Bogomolov, B. Hassett, Y. Tschinkel, *Constructing rational curves on K3 surfaces*, Duke Math. J. Volume 157, Number 3 (2011), 535–550.
- [LL] J. Li, C. Liedtke, *Rational Curves on K3 Surfaces*, to appear in Invent. Math.