Plan for the talk:

1. Uniruledness criteria
   1.1 for varieties that are not uniruled
   1.2 for varieties that are uniruled

2. Rational curves of minimal degree

Throughout we work over $\mathbb{C}$; almost all results stated here are open problems in positive characteristic.

1 Uniruledness

Theorem 1 (Miyaoka) Let $X$ be a projective manifold, $C \subset X$ a general complete intersection curve. Then $\Omega^1_X|C$ is nef unless $X$ is uniruled.

Here ‘general’ is in the sense of the Mehta-Ramanathan theorem on restrictions of vector bundles to curves. (Recall this says that the restriction of a stable bundle to a sufficiently high degree complete intersection remains stable.)

A vector bundle $V$ is nef if $\mathcal{O}_{\mathbb{P}(V)}(1)$ is nef.

**Idea of proof:** If $\Omega^1_X|C$ is not nef then it has a negative quotient. Hence $T_X|C$ has a positive subsheaf which extends to $X$. Then do Bend-and-Break, deforming “along that sheaf”.

1
1.1 Applications to non-uniruled spaces

Application to moduli

**Theorem 2** Let $f : X \to Y$ be a smooth projective family of curves of genus $\geq 1$, over a smooth projective $Y$ with $c_1(Y) = 0$. Then the family is isotrivial.

**Idea of proof:** Assume not isotrivial. Then we have a Kodaira-Spencer (KS) map

$$T_Y \to \mathbb{R}^1 f_*T_{X/Y},$$

and dually

$$f_*\omega_{X/Y}^\otimes 2 \to \Omega^1_Y.$$

Restrict to a general complete intersection curve

$$f_*\omega_{X/Y}^\otimes 2|C \to \Omega^1_Y|C,$$

and the former is weakly positive (and weakly stable) in the sense of Viehweg. Thus the latter has a subsheaf of positive degree, hence a quotient of negative degree, and the Miyaoka criterion applies to show $Y$ is uniruled.

This result can be generalized to

- families of higher-dimensional varieties (replace the KS-map with results of Viehweg-Zuo on positivity and higher KS-maps);
- non-compact base manifolds (using high-tech MMP).

**Theorem 3 (K- and Kovács)** Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized manifolds, with $Y^\circ$ a quasi-projective base manifold, and $\dim Y^\circ \leq 3$. If $Y$ is a smooth compactification of $Y^\circ$ with a simple normal crossings (SNC) and complement $D = Y \setminus Y^\circ$, then any run of the MMP gives a commutative diagram

$$
\begin{array}{c}
\xymatrix{ Y \ar[r]_{\text{MMP for } (Y,D)} & Y_{\text{min}} \\
\mathcal{M} \ar[u] & W \ar[l] \\
& \text{Mori/Kodaira fiber space}
\end{array}
$$

In particular,
1. If $\kappa(Y^\circ) = -\infty$ then $\text{Var}(F) < \dim(Y)$.

2. Otherwise $\text{Var}(f) \leq \kappa(Y^\circ)$.

Work with Jabbusch shows the result also holds for the Campana core map.

[Abramovich: Why dimension $\leq 3$? Kebekus: Should hold in all dimensions, conditional on the Abundance Conjecture.]

**Applications to deformations**

**Theorem 4 (Hwang, K-, Peternell)** Let $f : X \to Y$ be surjective between normal projective varieties, with $Y$ not uniruled, and

$$
\text{Hom}_f(X, Y) \subset \text{Hom}(X, Y)
$$

the connected component that contains $f$. Then $\text{Hom}_f(X, Y)$ is Abelian, and in particular, reduced.

**Idea of proof:** For simplicity, assume that $X$ and $Y$ smooth, and that $f$ is finite. Look at the composition

$$
\gamma : \text{Aut}^0(Y) \to \text{Hom}_f(X, Y)
$$

$$
g \mapsto g \circ f
$$

The derivative

$$
T\gamma|_{\text{Id}_Y} : T_{\text{Aut}}|_{\text{Id}} \to T_{\text{Hom}}|_f
$$

can be interpreted as the pull-back

$$
H^0(Y, T_Y) \to H^0(X, f^*T_Y).
$$

Observation 1: If $T\gamma$ is surjective ($\Rightarrow$ isomorphic) then $\gamma$ is isomorphic, and we are done.

Observation 2: Look at

$$
H^0(X, f^*T_Y) = H^0(Y, f_*f^*T_Y)
$$

$$
= H^0(Y, (f_*\mathcal{O}_X) \otimes T_Y)
$$

$$
= H^0(Y, (\mathcal{O}_Y \oplus \mathcal{E}^\vee) \otimes T_Y)
$$

$$
= H^0(Y, T_Y) \oplus \text{Hom}({\mathcal{E}}, T_Y)
$$

3
(here $E$ is a vector bundle, nef on every curve not contained in the branch locus, positive if the curve intersects the branch locus). If $\text{Hom}(E, T_Y) \neq 0$ then either we have some positivity in $T_Y$ and hence rational curves (a problem), or we have triviality in $E$, so that $f$ factors through an étale cover.

[Olsson: Is it clear that the automorphism group of a non-uniruled variety is abelian? Kebekus: Yes, otherwise the linear part of the automorphism group would have rational orbits.]

Detailed analysis with a result of David Liebermann from the 1950’s yields

**Corollary 1**  Keep the assumptions of the theorem and assume $Y$ is smooth. Then

$$\dim \text{Hom}_f(X, Y) \leq \dim Y - \kappa(Y).$$

If $\pi_1(Y)$ is finite then $\text{Hom}_f(X, Y)$ is a reduced point.

[McKernan: Doesn’t the first assertion follow directly from the Iitaka fibration? Kebekus: Maybe, let’s discuss further afterwards.]

Abramovich: What about characteristic $p$? Kebekus: Everything we have presented here fails in positive characteristic.]

### 1.2 Applications to uniruled varieties

**Theorem 5 (Bogomolov-McQuillan, K-Sólá-Toma)**  If $X$ is a projective variety, $\mathcal{F} \subset T_X$ a foliation, $C \subset X$ a curve that stays off the singularities of $X$ and $\mathcal{F}$. If $\mathcal{F}|C$ is ample then any leaf that touches $C$ is algebraic, uniruled, and almost all such are rationally connected.


Now let $X$ be a projective manifold and $C$ a movable curve (as opposed to complete intersection) on $X$, i.e., one passing through the generic point. This is enough to define a Harder-Narasimhan filtration (HNF) of $T_X$

$$0 = \mathcal{F}^0 \subsetneq \mathcal{F}^1 \subsetneq \cdots \mathcal{F}^k = T_X.$$
[Starr: Is this induced from the filtration after restriction of $C$? Kebekus: Not true with only ‘movability’ assumption.]

Set 

$$m = \max\{i : \text{slope } F^i/F^{i+1} > 0\}.$$ 

Then $F^1, \ldots, F^k$ are foliations and uniruledness applies. This yields a commutative diagram of “partial rational quotients”

$$
\begin{array}{c}
X \longrightarrow X \longrightarrow \cdots \longrightarrow X \longrightarrow X \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_m \longrightarrow Z
\end{array}
$$

where the right arrow is the MRC quotient, and the relative tangent bundle of $f_i: X \to Q_i$ is $F_i$.

This means that we can decompose the cone of movable curves into chambers, where two curves are in the same chamber if they induce the same HNF of $T_X$.

**Fact:** (Neumann) Chambers are convex inside movable cone. In the interior, the decomposition is locally finite, and the chambers are locally polyhedral. If the movable cone is polyhedral we have only finitely many chambers.

**Fact:** (Neumann) For low-dimensional Fano varieties, each term in any of the HNF is the relative tangent bundle of a Mori fiber space (perhaps containing a face rather than only a ray).

**Question 1** Does this hold in greater generality?

The first case to consider is $b_2(X) = 1$; here one would ask for $T_X$ to be semistable.

[Bogomolov: A few years before (1978?) published argument that gives effective estimate–this was reproduced in the book of Huybrechts and Lehn. The argument here is quite simple, almost trivial.]
2 Rational curves of Minimal Degrees

Set-up: $X$ is a uniruled projective manifold, $V' \subset \text{Chow}(X)$ a maximal irreducible subvariety, such that

- points of $V'$ correspond to reduced, irreducible rational curves;
- $V'$ is a family of curves dominates $X$;
- degrees are minimal among all families that dominate.

Let $V$ be the normalization of $V'$; given $x \in X$, let $V_x$ be the normalization of $\{ \ell \in V : x \in \ell \}$.

Central idea: The associated curves of “minimal degrees” are in many ways similar to lines:

- The set of lines through a given point is compact. Similarly, for a general $x \in X$, $V_x$ is compact.
- Line are smooth. Similarly, for a general $x \in X$, only finitely many $\ell \in V_x$ are singular at $x$, and singularities are immersed (Kebekus).

**Corollary 2** For a general $x \in X$ there is a map

$$V_x \to \mathbb{P}(T_x|_x)$$

$$\ell \mapsto \mathbb{P}(T_{\ell}|_x)$$

The image of this map is called the “variety of minimal rational tangents,” (VMRT) and it is denoted $C_x$. We can fit these varieties together to get

$$\mathcal{C} := \bigcup_{x \text{ general}} C_x \subset \mathbb{P}(T_X)$$

**Theorem 6 (Cartan-Fubini extension)** Let $X$ and $Y$ be Fano varieties with $G(X) = G(Y) = 1$, and let $U \subset X$ and $V \subset Y$ be open subsets for the Euclidean topology. Suppose $\phi: U \to V$ is a biholomorphic map that respects VMTR’s. If the Gauss map of VRMT is generically finite then $\phi$ extends to a global isomorphism.

This theorem has applications for the study of stability:

**Theorem 7 (Hwong)** Let $X$ be a Fano variety with $\rho(X) = 1$. If $\dim X \leq 5$ (resp. $\dim X \leq 6$) then $T_X$ is stable (resp. semistable).