

EXTENDING THE CÓRDOBA-MARTÍNEZ-ZOROA IPM BLOW-UP TO UNIFORMLY SPACE-TIME SMOOTH FORCING

LEVENT ALPÖGE, TRISTAN BUCKMASTER, AND MATEI P. COICULESCU

ABSTRACT. By adapting the techniques used in the IPM blowup result of Córdoba-Martínez-Zoróa with spatially smooth force, we prove finite-time blow-up for the IPM equation on $\mathbb{T}^2$ with a uniformly spacetime smooth force. In particular, we show there exist a smooth odd initial density, a smooth odd force $F \in C^\infty([0, 1] \times \mathbb{T}^2)$, and a classical solution ρ on $[0, 1)$ whose density gradient and spatial velocity gradient diverge in L^∞ as $t \uparrow 1$. Nevertheless, $\rho(t)$ converges in C^η for every $0 \leq \eta < 1$.

1. INTRODUCTION

The study of incompressible fluid flow, while an active research field for over a hundred years, still presents difficult mathematical challenges due to the nonlinear and nonlocal behavior of its various model equations. Questions about the global-in-time well-posedness of equations such as the incompressible Navier–Stokes and Euler equations remain unsolved, and even one-dimensional model problems such as the surface growth equation or the Córdoba–Córdoba–Fontelos equation are not completely understood. Singularity formation was discovered by Elgindi for $C^{1,\alpha}$ solutions of the three-dimensional Euler equations [5], but smooth blow-up results remain elusive. On the other hand, Córdoba and Martínez-Zoróa proved a breakthrough result [1], establishing finite-time singularity formation for the forced IPM equation from smooth initial data and with a spatially smooth force.

The IPM equation transports a density ρ by the divergence-free Darcy velocity that it generates:

$$\partial_t \rho + u_{\mathbb{T}}(\rho) \cdot \nabla \rho = F, \quad \operatorname{div} u_{\mathbb{T}}(\rho) = 0 \quad \text{on } \mathbb{T}^2,$$

where $u_{\mathbb{T}}$ is the periodic zero-order Fourier multiplier determined by Darcy’s law; its precise normalization is given in the overview below. Córdoba, Gancedo, and Orive established foundational local existence and uniqueness results, global continuation criteria, and blow-up in a special infinite-energy class [2]. Kiselev and Yao constructed smooth data for the unforced equation, including on the torus, for which global smoothness entails unbounded growth of derivatives, and they proved nonlinear instability of a class of stratified steady states [3]. At the critical Sobolev threshold, Bianchini, Córdoba, and Martínez-Zoróa proved nonexistence and strong ill-posedness in H^2 , even near the linearly stable profile $-x_2$ [4]. A more comprehensive literature review is provided in the next section.

The blow-up construction of Córdoba and Martínez-Zoróa relies on a multiscale inductive scheme in which each previous iteration of the solution serves as a background flow that rapidly amplifies an oscillatory wave packet added to the solution. By carefully setting up the induction, they show that the multiscale solution leads to infinite gradient growth in finite time. The theorem of Córdoba and Martínez-Zoróa concerns the forced IPM equation on $\mathbb{R}^2$, with initial density in $C_c^\infty(\mathbb{R}^2)$ and a compactly supported force in $L_t^\infty C_x^\infty$. They show that the density is a classical solution before $t = 1$ and that the gradients of both the density and the velocity field blow up in C^0 as $t \rightarrow 1^-$. We also mention that related blow-up mechanisms have been used by Córdoba, Martínez-Zoróa, and their coauthors for other equations, including the forced hypodissipative Navier–Stokes and generalized SQG equations [7, 8].

The goal of our work is to extend the result of Córdoba and Martínez-Zoroa for the forced IPM equation in the following way. We prove finite-time singularity formation for the forced IPM equation on the torus $\mathbb{T}^2$, with initial density in $C^\infty(\mathbb{T}^2)$ and a space–time smooth force in $C^\infty([0, 1] \times \mathbb{T}^2)$. We also obtain blow-up of the C^0 norms of the gradients of the density and the velocity field as $t \rightarrow 1^-$. We have endeavored to make the proof presented here as clear and readable as possible. Before giving a precise statement of the main theorem and an overview of the proof strategy, we wish to address two possible points of contention. First, we directly compare the present work with that of Córdoba and Martínez-Zoroa and identify what is new and which ideas are attributable to their earlier work. Second, we address the use of language models in the preparation of this work.

The idea of a background flow amplifying a high-frequency perturbation dates back at least to the work of Bourgain and Li, who used it to prove norm inflation for the Euler equations in endpoint critical Sobolev spaces [6]. Córdoba and Martínez-Zoroa successfully implemented this concept for the forced IPM equation, obtaining spatially smooth forced blow-up. Many techniques in the present work originate in their work, including transporting the perturbation’s phase directly by the previous frequency level while maintaining its oscillatory structure and tracking the orientation of the oscillatory wave packet relative to its “parent” field. Of special note is the idea of tying the amplitude of the wave packet to the oscillation parameter in a frequency-dependent way, together with a delicate induction on the number of controlled derivatives. More details are provided below. New features introduced here for forced IPM include Fourier space truncations that avoid a loss of derivatives and a mixed induction in time and space that ultimately shows that the force is smooth in space–time. In addition, we transport the oscillatory packets by the Fourier truncated previous scales, while the work of Córdoba and Martínez-Zoroa transport the next scale by the jet truncated velocity. Another improvement over the work of Córdoba and Martínez-Zoroa is that our solution is given by an exact series and requires no nonconstructive passage through a nonlinear limit. Everything is made as explicit as possible without sacrificing readability.

We now address the use of language models in this work. We used Claude to identify the key elements of the previous proof of Córdoba and Martínez-Zoroa and to reproduce their argument. We used Claude and Codex to write the main body of the text, delegating delicate bookkeeping of inductive orders and constants to the models while implementing our ideas for adapting the forced IPM construction of Córdoba and Martínez-Zoroa to the torus and to a space–time smooth force. We used the models in conjunction with our own ideas to reduce the length of initial versions of this work. Claude originally generated an overly complicated version of the result on the whole space following a proof architecture closer in resemblance to Córdoba and Martínez-Zoroa’s work [1]. In simplifying the proof, we incorporated many of our own ideas, some of which are inspired by analogous ideas from the convex integration literature. This introduction (apart from TeX compatibility with the main body, formatting, and the second-paragraph literature review) was written by the listed authors, but the remainder of the paper was written with the assistance of Claude and Codex, under the strict direction of the authors.

1.1. Blow-up for two-dimensional inviscid Boussinesq and three-dimensional incompressible Euler equations with uniformly space-time smooth forcing. In forthcoming work, the first and second authors prove finite-time singularity formation from smooth, compactly supported initial data for the forced two-dimensional inviscid Boussinesq equations on $\mathbb{R}^2$ and the forced three-dimensional incompressible Euler equations on $\mathbb{R}^3$, in the latter case within the class of axisymmetric solutions with swirl. In both results, the external forcing has fixed compact spatial support and remains C^∞ in space and time up to and including the singular time, with every mixed derivative uniformly bounded.

Under normal circumstances, we would not announce these results before polished, human-readable proofs were available. Those proofs are currently being prepared. For various reasons, a precise preliminary statement has become necessary. Both proofs have been formally verified in

Lean. To identify the precise formal-proof artifacts underlying this announcement, their SHA-256 hashes are

$$[\text{BOUSSINESQ SHA-256}], \quad [\text{EULER SHA-256}].$$

The complete human-readable proofs will be released shortly by the first and second authors, together with an account of the role of artificial intelligence in this work and further commentary on AI in mathematics. We ask the community for patience while these materials are prepared with the care they require.

2. OVERVIEW

Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the two-dimensional torus and normalize Fourier coefficients by

$$\widehat{f}(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x) e^{-ik \cdot x} dx, \quad k \in \mathbb{Z}^2.$$

The periodic incompressible porous media velocity is the zero-order multiplier

$$(1) \quad \widehat{u_{\mathbb{T}}(\rho)}(k) = 2\pi \left(\frac{k_1 k_2}{|k|^2}, -\frac{k_1^2}{|k|^2} \right) \widehat{\rho}(k) \quad (k \neq 0), \quad \widehat{u_{\mathbb{T}}(\rho)}(0) = 0.$$

The vector on the right is orthogonal to k , so $\operatorname{div} u_{\mathbb{T}}(\rho) = 0$. We study

$$(2) \quad \partial_t \rho + u_{\mathbb{T}}(\rho) \cdot \nabla \rho = F \quad \text{on } [0, 1] \times \mathbb{T}^2.$$

The factor 2π in (1) fixes the normalization used below. The goal of this work is to prove:

Theorem 2.1 (Finite-time blow-up). *There exist functions*

$$\rho_{\text{in}} \in C^\infty(\mathbb{T}^2), \quad F \in C^\infty([0, 1] \times \mathbb{T}^2),$$

and a solution ρ of (2) with the following properties. The initial datum ρ_{in} is odd and has zero spatial mean, while $F(t, \cdot)$ is odd and has zero spatial mean for every $t \in [0, 1]$. Moreover,

$$\rho(0) = \rho_{\text{in}}, \quad \rho \in C^\infty([0, T] \times \mathbb{T}^2) \quad \text{for every } T < 1.$$

There is one function

$$\rho_* \in \bigcap_{0 \leq \eta < 1} C^\eta(\mathbb{T}^2)$$

such that, for every $0 \leq \eta < 1$,

$$(3) \quad \rho(t) \longrightarrow \rho_* \quad \text{in } C^\eta(\mathbb{T}^2) \quad \text{as } t \uparrow 1.$$

Nevertheless,

$$(4) \quad \lim_{t \uparrow 1} \|\nabla \rho(t)\|_{L^\infty} = \lim_{t \uparrow 1} \|D_x u_{\mathbb{T}}(\rho(t))\|_{L^\infty} = \infty.$$

Córdoba and Martínez-Zoroa proved the corresponding finite-time C^1 blow-up, including blow-up of the velocity gradient, on $\mathbb{R}^2$ with compactly supported smooth initial density and a force in $L_t^\infty C_x^\infty$ [1, Theorem and Remark 2]. Their construction introduces the multiscale induction, transported phase, a finite asymptotic expansion of the velocity generated by a localized high-frequency oscillation, and the phase-harmonic cancellation mechanism used here [1, Sections 1.2.1–1.2.4]. Joint smoothness in time and space is anticipated there but is not part of that theorem [1, Remark 1]. The present argument implements the mechanism on the torus, with nested Fourier cutoffs and an exact cutoff commutator, and proves that the resulting force belongs to $C^\infty([0, 1] \times \mathbb{T}^2)$.

2.1. Strategy.

Packets and the parent–child step. The elementary model is an oscillatory steady state. For $A \in \mathbb{R}$ and $k \in \mathbb{Z}^2 \setminus \{0\}$, the plane wave $\vartheta(x) = A \sin(k \cdot x)$ has velocity parallel to the multiplier vector in (1). Since this vector is perpendicular to k , one has $u_{\mathbb{T}}(\vartheta) \cdot \nabla \vartheta = 0$. Thus a single plane wave is a stationary solution of the unforced equation on the torus. The following construction localizes and transports this model. A localized oscillation is no longer an exact unforced steady state, so at the initial level we take its nonlinear residual as the force. At every later level the same principle is used: all remainders are estimated, but none are omitted from the equation.

If $(\rho^{(q)}, F^{(q)})$ are the density and force, respectively, at frequency level q , then the *parent* at the next frequency level $q+1$ is the complete preceding density after a smoothing Fourier cutoff, $P_{q+1} = S_{\mu_{q+1}} \rho^{(q)}$. The operator S_{μ} and its commutator are defined in (13) and (14); the child-dependent Fourier scale μ_{q+1} is specified below. The parent therefore contains the contributions of all earlier levels, rather than only the most recent oscillation. The new, much higher-frequency perturbation θ_{q+1} is the *child*. If $\mathcal{C}_{\mu_{q+1}}(\rho^{(q)})$ denotes the exact commutator created by the cutoff and E_{q+1}^{pert} denotes the exact residual of the child relative to P_{q+1} , then the density and force at level $q+1$ are

$$(5) \quad \begin{aligned} \rho^{(q+1)} &= P_{q+1} + \theta_{q+1}, \\ F^{(q+1)} &= S_{\mu_{q+1}} F^{(q)} + \mathcal{C}_{\mu_{q+1}}(\rho^{(q)}) + E_{q+1}^{\text{pert}}. \end{aligned}$$

This is an identity, so every finite stage solves the forced IPM equation exactly.

Let λ_{q+1} be the child frequency and let $0 < \beta_{q+1} \leq 1/8$ be its scale exponent. At leading order the child has the same oscillatory form as the plane-wave model:

$$(6) \quad \theta_{q+1}^{\text{prin}}(x, t) = w_{q+1}(t) b_{q+1}(x, t) \sin(\lambda_{q+1} X_{q+1}(x, t)), \quad b_{q+1}(0, t) = a_{q+1}(t) \lambda_{q+1}^{\beta_{q+1}-1}.$$

Here b_{q+1} is a compactly supported, slowly varying coefficient propagated along the parent flow, X_{q+1} is the transported phase, a_{q+1} controls the leading amplitude, and w_{q+1} switches the packet on smoothly. More precisely, $w_{q+1} = 0$ up to the activation time s_{q+1} , vanishes there to infinite order, rises smoothly from zero to one, and then remains one. Thus the packet is not switched off after its activation interval. Its local frequency vector is $\lambda_{q+1} \nabla X_{q+1}$.

The principal packet alone does not have a sufficiently small residual. We therefore construct the full child $\theta_{q+1} = \Theta_M := \zeta_1 + \dots + \zeta_M$, where $M = \lceil \beta_{q+1}^{-2} \rceil$, as a finite nonlinear *parametrix*: an explicit approximate solution whose exact residual is quantitatively controlled. The term ζ_1 is the principal first harmonic, and each later ζ_j corrects the residual left by the preceding partial sum. For a pure plane wave, the translation-invariant velocity multiplier acts exactly by multiplication by a fixed vector. For a localized packet with transported phase, it admits a finite inverse-frequency expansion $u^{[M]}$ about the local frequency vector. Its zeroth-order term is obtained by evaluating the multiplier symbol at that vector, while the higher-order terms contain derivatives of the envelope and of the non-affine part of the phase. For an individual harmonic z , the exact remainder $(u_{\mathbb{T}} - u^{[M]})(z)$ is retained intact and included in E_{q+1}^{pert} . This multiplier expansion is one ingredient of the nonlinear parametrix.

The density ansatz has a finite phase-harmonic structure. For $1 \leq j \leq M$,

$$\zeta_j = \sum_{\ell=1}^{2^j} (g_{\ell,j}^s \sin(\ell \lambda_{q+1} X_{q+1}) + g_{\ell,j}^c \cos(\ell \lambda_{q+1} X_{q+1})),$$

where $g_{\ell,j}^s$ and $g_{\ell,j}^c$ are slowly varying coefficient functions. The principal term ζ_1 is carried by the phase harmonic $\ell = 1$ and is distinct from the leading frozen-frequency velocity response. The true zero harmonic $\ell = 0$ is a nonoscillatory coefficient created when equal harmonics interact; it is not the Fourier zero mode and need not be spatially constant. Products of two harmonics generate their sums and differences, so the structured residual after level j contains indices up to 2^{j+1} . For $1 \leq j < M$, a weighted Duhamel formula chooses ζ_{j+1} to cancel the complete correctable

nonzero-harmonic defect $\mathcal{Q}_j$. The nonlinear interactions are then recomputed, producing a new and smaller defect $\mathcal{Q}_{j+1}$. At coefficient order zero, each successive correction is smaller by one factor of $\lambda_{q+1}^{-\beta_{q+1}}$, up to logarithmic factors. Taking $M = \lceil \beta_{q+1}^{-2} \rceil$ correction levels supplies enough decay to absorb the losses in the prescribed mixed-derivative range.

The correction procedure reduces the residual to the prescribed size; it does not make it vanish. At the next level, the activation error, the zero-harmonic terms due to nonlinear interaction, the exact multiplier remainders, and the final nonzero-harmonic defect all remain in E_{q+1}^{pert} . This residual and the cutoff commutator are placed in the force through (5). The higher corrections are lower order in the origin-gradient geometry, so the principal packet continues to carry the instability.

The amplification is geometric. The activation time is the unique time at which the child phase-gradient direction is perpendicular to the leading phase-gradient direction of the q -th packet, which dominates ∇P_{q+1} near the origin. The leading child velocity is transverse to its own phase gradient and is therefore aligned with the leading parent-density gradient. Both principal trigonometric factors are then close to one, giving a favorable amplitude rate of size $\lambda_q^{\beta_q}$. The child itself vanishes at the precise activation time because $w_{q+1} = 0$, but it is switched on immediately afterward with an exponentially small coefficient. During the favorable part of the evolution, the parent shear produces an exponentially large amplitude gain; the normalization $a_{q+1}(1) = 1$ then gives a seed small enough to insert the child at the next frequency.

The same parent shear transports the child phase gradient from the perpendicular configuration toward the parent direction. By terminal time the two phase gradients are nearly parallel, so the favorable amplitude rate has faded. The remaining angular motion is slow. This is the precise sense in which the leading origin geometry of the terminal child is almost stationary: the phase is still transported, but $a_{q+1}(t) = 1 + o(1)$ and $\nabla X_{q+1}(0, t)$ is $o(1)$ -close to its terminal value uniformly on the child's terminal interval of time. The former child then has the geometry needed to become the dominant newest part of the next parent.

After choosing $\beta_{q+1} \in [\beta_q/2, \beta_q]$, a deterministic frequency law selects λ_{q+1} exponentially large in $\lambda_q^{\beta_q/8}$, and the Fourier cutoff scale is $\mu_{q+1} = \lambda_{q+1}^{\beta_{q+1}/2}$. These choices separate the child oscillation, its cutoff, and its terminal interval of time from the corresponding parent scales.

The inductive step is closed by truncating the complete stage, not the parent and child separately. Namely, the following level begins from $P_{q+2} = S_{\mu_{q+2}} \rho^{(q+1)}$. The nested cutoffs preserve all older cutoff components and truncate only the newest child. The Fourier cutoff supplies the higher spatial derivatives needed at the following step through Bernstein estimates, thereby preventing a loss of derivatives from propagating through the iteration. The cutoff does introduce an error, but its exact commutator is again retained in the force. This completes one parent-child cycle.

The global induction and the scale exponent. The iteration has two simultaneous goals. First, the new force increment $G_{q+1} := \mathcal{C}_{\mu_{q+1}}(\rho^{(q)}) + E_{q+1}^{\text{pert}}$ must decay at the child frequency. Once $\beta_{q+1} \leq 1/40$, the estimates from the inductive step described above give the following estimate uniformly in time for all integers $m, k \geq 0$:

$$(7) \quad \|\partial_t^m G_{q+1}(t)\|_{C_x^k} \leq \lambda_{q+1}^{-1/4} \quad \text{whenever} \quad m + k \leq \left\lfloor \frac{1}{4\beta_{q+1}} \right\rfloor - 10.$$

The second goal is that the terminal density gradient and velocity gradient must keep growing through the inductive process. Their principal scales are respectively $\lambda_q^{\beta_q}$ and $\lambda_q^{7\beta_q/8}$.

The scale exponent β_q therefore has two roles. It enters the size and spatial concentration of the packets, and it controls the finite range of mixed derivatives in (7). Roughly $1/\beta_q$ derivatives of the force are controlled at level q . Smaller β_q gives a larger range, but it also increases both the correction depth $M_q \simeq \beta_q^{-2}$ and the number of parent derivatives required to construct the next child. Although every finite stage is smooth, the recursion records only a finite mixed-derivative range for the density $\rho^{(q)}$ at level q , so β_q cannot simply be halved at every step.

The recursion instead follows a retain-or-halve rule. It retains β_q while the verified derivative range of the density $\rho^{(q)}$ at level q catches up with the larger finite requirement at exponent $\beta_q/2$. Once that range and the associated finite frequency thresholds are available, it sets $\beta_{q+1} = \beta_q/2$. While β_q is retained, frequency separation increases the available derivative reserve until the halving criterion is satisfied; hence a positive exponent cannot be retained forever. Consequently $\beta_q \downarrow 0$, and every fixed pair (m, k) therefore eventually lies in the range of (7).

The frequency growth dominates the decrease of the scale exponent: the scale separation gives $\beta_q \log \lambda_q \rightarrow \infty$. Thus $\lambda_q^{\beta_q}$ and $\lambda_q^{7\beta_q/8}$ still diverge. The same recursion can therefore make the force increments small in successively more derivatives while preserving the density-gradient and velocity-gradient amplification.

The mixed space–time induction. The force must be smooth jointly in space and time, whereas at a fixed stage the decay estimate (7) covers only finitely many mixed derivatives. Let $D = \partial_t + u_{\mathbb{T}}(P_{q+1}) \cdot \nabla$ be the parent material derivative. After the exact scalar amplitude factor is divided out, a normalized Duhamel coefficient J satisfies $(D + \tilde{c}_P)J = \varphi$ with $J(s_{q+1}) = 0$, where c_P is the leading parent-interaction coefficient, $\tilde{c}_P = c_P - c_P(0, t)$ is its centered form, and φ is the already constructed Duhamel source. For $p \geq 1$, differentiating this identity $p - 1$ times gives

$$(8) \quad D^p J = D^{p-1} \varphi - \sum_{r=0}^{p-1} \binom{p-1}{r} (D^r \tilde{c}_P) D^{p-1-r} J.$$

An order- p correction therefore requires positive material derivatives of its Duhamel source only through order $p-1$, at the cost of additional spatial derivatives. This one-order gain is the mechanism by which the time integral closes the induction.

The construction first closes the complete correction cascade at time order zero. For each $p \geq 1$, it next proves the order- p estimate for the seeded principal coefficient and then, successively, those for the higher-correction coefficients, using only the fixed seed and already known lower time orders of the parent inputs and preceding defects. Only afterward does it estimate the order- p structured defects, perturbation residual, cutoff commutator, and force increment. Thus no order- p defect estimate is used to prove an order- p child-coefficient estimate. Since the phase satisfies $DX_{q+1} = 0$, conversion from material to ordinary time derivatives uses only the additional spatial derivatives already built into the finite range.

The activation cutoff and all its derivatives vanish at s_{q+1} , and this infinite-order vanishing propagates through the Duhamel corrections. Each child and its residual therefore extend smoothly by zero to earlier times. Because $\beta_q \rightarrow 0$, every fixed mixed derivative eventually satisfies (7); the summability of $\lambda_q^{-1/4}$ then yields $F \in C^\infty([0, 1] \times \mathbb{T}^2)$, including at the endpoint.

Finally, the activation times tend to one and the frequencies are lacunary. On every compact interval $[0, T]$, $T < 1$, all sufficiently high-frequency children vanish, so the finite stages stabilize there and give a classical solution without a separate nonlinear compactness argument. The density increments are summable in C^η for every $\eta < 1$, while the terminal density gradients and velocity gradients diverge. Support separation and Fourier-cutoff tail estimates ensure that later children do not interfere with the dominant geometry on the preceding terminal intervals. This proves both limits in (4) and the convergence in (3).

Section 3 develops the localized oscillation calculus using the periodic multiplier, cutoff, and kernel estimates proved in the appendix. Section 4 states the finite-stage invariant and proves the angle–amplitude mechanism. Section 5 constructs one corrected child, and Section 6 gives the global recursion and proves Theorem 2.1. The appendix also supplies the all-order mixed estimates used to obtain joint smoothness of the force.

2.2. Relevant Literature.

Classical IPM and stable regimes. The behavior of the incompressible porous media equation (IPM) depends strongly on the solution class and the background stratification. Córdoba, Gancedo, and Orive established local existence and uniqueness for classical solutions and several continuation criteria; they also exhibited blow-up in a special infinite-energy class [2]. Around stable stratifications, Elgindi proved nonlinear asymptotic stability [9], Castro, Córdoba, and Lear obtained global quasi-stratified solutions in a confined domain [10], and Bianchini, Crin-Barat, and Paicu established asymptotic stability near a linearly stratified state [11].

The stability theory coexists with mechanisms that create small scales. Kiselev and Yao constructed solutions whose derivatives must grow without bound if they remain smooth globally, and deduced nonlinear instability for a class of stratified steady states [3]. At the critical Sobolev index, Bianchini, Córdoba, and Martínez-Zoroa proved nonexistence and strong ill-posedness in H^2 for small perturbations of a linearly stable profile [4]. Park recently sharpened stability analysis in a periodic channel by exploiting the energy structure shared by IPM and Stokes transport [12]. None of these stability or small-scale-growth results supplies a smooth, finite-energy, unforced IPM singularity.

Weak nonuniqueness and mixing. At lower regularity, the picture changes sharply. Córdoba, Faraco, and Gancedo constructed nontrivial bounded weak solutions compactly supported in time, yielding nonuniqueness from zero initial data [13]. Székelyhidi then computed an explicit relaxation of IPM and used convex integration to construct weak mixing solutions for the unstable Muskat problem [14]. Isett and Vicol proved the existence of Hölder-continuous weak solutions and an h -principle for active-scalar multipliers that are not odd, a class containing IPM [15].

Subsequent work extended these constructions substantially. Förster and Székelyhidi treated broad classes of Muskat interfaces [16]; Castro, Córdoba, and Faraco constructed mixing solutions for fully unstable H^5 data [17]; Castro, Faraco, and Mengual quantified degraded mixing [18] and later localized the mixing zone for bubbles and turned interfaces [19]; and Mengual incorporated viscosity jumps [20]. Castro, Faraco, and Gebhard developed a macroscopic entropy formulation intended to select a coarse-scale evolution within this nonunique regime [21]. These are rigorous results about low-regularity weak solutions and must be kept logically separate from loss of regularity along a classical solution.

Rigorous IPM singularity results. The known rigorous singularity results occupy several distinct settings. Castro, Córdoba, Gancedo, and Orive identified an explicit blow-up mechanism in a special infinite-energy reduction [22]. Collot, Prange, and Tan recently proved nonlinear stability of the resulting self-similar blow-up, again within that infinite-energy reduction on an infinite periodic strip [23]. Kiselev and Sarsam proved finite-time blow-up for a one-dimensional boundary-layer model of IPM, rather than the full two-dimensional system [24].

Two recent works are closer to the full equation but retain essential qualifications. Dembski's preprint proves singularity formation for the unforced IPM equation in a wedge, starting from Lipschitz data whose density vanishes on the boundary [25]. Córdoba and Martínez-Zoroa construct smooth, finite-energy solutions of two-dimensional IPM that become singular under a compactly supported spatially smooth force [1]. The latter provides the closest precedent for the present periodic construction. Its introductory theorem states only $F \in L_t^\infty C_x^\infty$, while Remark 1 treats higher time regularity as requiring additional work.

The $C^{1,\alpha}$ Euler comparison. The $C^{1,\alpha}$ incompressible Euler literature supplies an important comparison. Earlier strong-solution constructions of Elgindi and Jeong used conical or corner domains [26]. In the whole space, Elgindi constructed finite-time singularities for unforced, axisymmetric Euler flows without swirl when α is sufficiently small [5]. Elgindi, Ghoul, and Masmoudi subsequently obtained finite-energy, locally self-similar examples and stability in the same regularity class [27].

A separate boundary mechanism originated in the numerical scenario of Luo and Hou [28, 29]. For axisymmetric Euler with swirl in a cylindrical domain, the boundary stagnation structure is related to a two-dimensional Boussinesq system. Chen and Hou proved blow-up for $C^{1,\alpha}$ data in

this boundary setting [30] and later analyzed stability and instability of the associated singular solutions [31].

Among boundary-free constructions, Córdoba, Martínez-Zorúa, and Zheng produced an infinitely layered, unforced solution that is globally $C^{1,\alpha}$, smooth away from one point, and singular in finite time [32]. Elgindi and Pasqualotto announced a different mechanism driven by Taylor–Couette instability [33]. Chen showed that the earlier Elgindi and Chen–Hou constructions can be modified so that the velocity is nonsmooth only at the singular point [34]. For every $\varepsilon > 0$, Córdoba and Martínez-Zorúa also constructed a forced, nonaxisymmetric $C^{1,\alpha}$ singularity with a force uniformly bounded in $C^{1,1/2-\varepsilon} \cap L^2$ [35].

Two 2026 developments reach the sharp range below the classical threshold. For every $0 < \alpha < 1/3$, Shkoller announced Type-I blow-up for unforced, finite-energy, axisymmetric no-swirl flow with $C^{1,\alpha}$ initial velocity [36]. Independently, Chen’s companion preprints construct exact C^α self-similar vorticity profiles and asymptotically self-similar blow-up with locally C^α vorticity and locally $C^{1,\alpha}$ velocity for the same range [37, 38]. These results approach $C^{1,1/3}$ from below; neither claims the endpoint.

Mollification as a methodological precedent. The use of an intermediate smoothing operator has a direct methodological precedent in the admissible-Euler convex-integration construction of Buckmaster, De Lellis, Székelyhidi, and Vicol [39, Section 2.4]. In that paper, an Euler–Reynolds pair is mollified at a scale ℓ before the gluing and perturbation stages to control derivative loss typical of convex-integration schemes. In the present IPM argument, nested Fourier cutoffs play an analogous bookkeeping role, but the exact cutoff commutators are retained in the force. The analogy is methodological only; it does not invoke convex integration or a weak-nonuniqueness conclusion.

PINN discovery of candidate blow-up profiles. Physics-informed neural networks have recently been used as targeted discovery tools for self-similar singular profiles. Wang, Lai, Gómez-Serrano, and Buckmaster found numerical profiles for the two-dimensional Boussinesq and three-dimensional axisymmetric Euler equations, as well as an unstable profile for the CCF model [40]. A subsequent large-scale search reported stable and unstable self-similar profiles for IPM with boundary and for Boussinesq, with the latter linked to axisymmetric Euler with boundary [41]. Wang, Léger, Lai, and Buckmaster later introduced a gradient-normalized residual to resolve sharp profiles more accurately and numerically identified a fourth unstable IPM candidate [42].

3. PERIODIC MULTIPLIERS, NESTED CUTOFFS, AND TRANSPORTED OSCILLATIONS

For $\alpha \in \mathbb{R}$, set

$$e(\alpha) = (\cos \alpha, \sin \alpha), \quad e^\perp(\alpha) = (-\sin \alpha, \cos \alpha).$$

To avoid overloading the notation later, we write $C_0 = 2\pi$.

3.1. The periodic operator and parity. The Fourier multiplier symbol in (1) is real, even, homogeneous of degree zero away from the origin, and orthogonal to the frequency. Consequently $u_{\mathbb{T}}$ commutes with translations and spatial derivatives and maps real scalars to real divergence-free vector fields.

Define the scalar operator $V_{\mathbb{T}}$ by

$$\widehat{V_{\mathbb{T}}f}(k) = \frac{2\pi i k_1}{|k|^2} \widehat{f}(k) \quad (k \neq 0), \quad \widehat{V_{\mathbb{T}}f}(0) = 0.$$

Writing $u_{\mathbb{T}}(\rho) = (u_{\mathbb{T},1}(\rho), u_{\mathbb{T},2}(\rho))$ for its two vector components, we have

$$(9) \quad u_{\mathbb{T},1}(\rho) = -V_{\mathbb{T}}(\partial_2 \rho), \quad u_{\mathbb{T},2}(\rho) = V_{\mathbb{T}}(\partial_1 \rho).$$

Write $A \Subset B$ when $\overline{A}$ is a compact subset of B . Fix symmetric coordinate balls

$$(10) \quad U_0 \Subset U \Subset (-\pi, \pi)^2,$$

centered at the origin. Every function supported in U_0 below is identified with its local lift; in Euclidean expressions, this lift is extended by zero to $\mathbb{R}^2$. Fix an even function $\eta_{\text{lin}} \in C_c^\infty(U)$ that equals one on a neighborhood of $\overline{U_0}$. For $p \in \mathbb{R}^2$, let $\psi_p \in C^\infty(\mathbb{T}^2)$ be the periodic function obtained by extending

$$x \mapsto \eta_{\text{lin}}(x)(p \cdot x)$$

by zero from the chart. Thus ψ_p is odd and agrees with the linear phase $p \cdot x$ near $\overline{U_0}$. Extend the periodic Fourier multiplier symbol homogeneously to

$$(11) \quad m(\xi) = 2\pi \left(\frac{\xi_1 \xi_2}{|\xi|^2}, -\frac{\xi_1^2}{|\xi|^2} \right), \quad \xi \in \mathbb{R}^2 \setminus \{0\}.$$

For $\lambda > 0$, $\alpha, \vartheta \in \mathbb{R}$, and a smooth amplitude $b = b(x)$ compactly supported in U_0 , set

$$z(x) = b(x) \sin(\lambda e(\alpha) \cdot x + \vartheta).$$

Define the zeroth-order approximation to the velocity by

$$(12) \quad u^{[0]}[z] = -C_0 \cos \alpha e^\perp(\alpha) z.$$

Indeed, evaluating the homogeneous Fourier multiplier symbol (11) at $e(\alpha)$ gives

$$2\pi(\cos \alpha \sin \alpha, -\cos^2 \alpha) = -C_0 \cos \alpha e^\perp(\alpha).$$

The high-order localized version of this calculation is proved below.

3.2. Nested Fourier cutoffs. Fix an even radial function $\chi \in C_c^\infty(\mathbb{R}^2)$ such that

$$\chi(\xi) = 1 \quad (|\xi| \leq 1), \quad \chi(\xi) = 0 \quad (|\xi| \geq 2).$$

For $\mu \geq 1$, define

$$(13) \quad \widehat{S_\mu f}(k) = \chi(k/\mu) \widehat{f}(k), \quad k \in \mathbb{Z}^2.$$

For later use, define the cutoff commutator by

$$(14) \quad \mathcal{C}_\mu(h) := u_{\mathbb{T}}(S_\mu h) \cdot \nabla S_\mu h - S_\mu(u_{\mathbb{T}}(h) \cdot \nabla h).$$

If f is odd, then $u_{\mathbb{T}}(f) \cdot \nabla f$ is odd and has zero mean. The same is true after applying any cutoff. These observations account both for the global zero mode and for the exact nonlinear Fourier support used later in the force recursion. Proofs of the Bernstein-type estimates for S_μ and $\mathcal{C}_\mu$ used throughout this work may be found in the appendix.

3.3. The local periodic kernel. For $f \in C^\infty(\mathbb{T}^2)$ supported in U_0 , define

$$(15) \quad V_{\mathbb{R}} f(x) = \int_{\mathbb{R}^2} \frac{h_1}{|h|^2} f(x+h) dh,$$

acting on the extension by zero to $\mathbb{R}^2$ of the local lift of f .

Remark. For f supported in U_0 , we decompose $V_{\mathbb{T}} f$ on U into $V_{\mathbb{R}} f$ plus an integral of f against a smooth kernel; this decomposition is proved in the appendix in Lemma B.3.

3.4. Affine high-frequency velocity expansion. The results below isolate the oscillatory inputs used in the layer construction. An affine expansion identifies the local multiplier response; transport keeps the nonlinear phase nearly affine on the perturbation support; the periodic expansion separates finitely many correctable terms from a small global remainder; and a common transported phase yields the anisotropic gain in quadratic interactions.

Choose $\kappa_0 \in C_c^\infty([0, \infty))$, equal to one near zero, so that $h \mapsto \kappa_0(|h|)$ is smooth on $\mathbb{R}^2$. For a unit vector $\omega = (\omega_1, \omega_2)$ and a multi-index γ , set

$$\mathcal{K}_\gamma(h) = \frac{h_1 h^\gamma}{|h|^{2\gamma!}},$$

and define

$$(16) \quad c_\gamma^s(\omega) = \lim_{\sigma \downarrow 0} \int_{\mathbb{R}^2} \mathcal{K}_\gamma(h) \cos(\omega \cdot h) \kappa_0(\sigma|h|) dh,$$

$$(17) \quad c_\gamma^e(\omega) = \lim_{\sigma \downarrow 0} \int_{\mathbb{R}^2} \mathcal{K}_\gamma(h) \sin(\omega \cdot h) \kappa_0(\sigma|h|) dh.$$

For a scalar-valued profile f , an affine phase $\Phi = \nu\omega \cdot x + \vartheta$, and an integer $N \geq 0$, put

$$\mathcal{V}_\Phi^N[f] = \sum_{|\gamma| \leq N} \nu^{-|\gamma|-1} (\partial^\gamma f) \cdot (c_\gamma^s(\omega) \sin \Phi + c_\gamma^e(\omega) \cos \Phi), \quad \mathcal{V}_\Phi^{-1} = 0.$$

Define the Euclidean singular velocity by

$$u_{\mathbb{R}}(h) = (-V_{\mathbb{R}}(\partial_2 h), V_{\mathbb{R}}(\partial_1 h)).$$

The subscript pw below indicates that the expansion is built from the affine plane wave $f \sin \Phi$. The order- N truncated high-frequency velocity expansion is

$$(18) \quad u_{\text{pw},1}^{[N]}[f; \Phi] = -\mathcal{V}_{\Phi+\pi/2}^N[\nu\omega_2 f] - \mathcal{V}_\Phi^{N-1}[\partial_2 f],$$

$$(19) \quad u_{\text{pw},2}^{[N]}[f; \Phi] = \mathcal{V}_{\Phi+\pi/2}^N[\nu\omega_1 f] + \mathcal{V}_\Phi^{N-1}[\partial_1 f].$$

Lemma 3.1 (Affine high-frequency velocity expansion). *The limits in (16)–(17) exist uniformly in $\omega \in \mathbb{S}^1$ and define smooth functions on $\mathbb{S}^1$. They satisfy*

$$(20) \quad c_\gamma^s = 0 \quad (|\gamma| \text{ even}), \quad c_\gamma^e = 0 \quad (|\gamma| \text{ odd}), \quad c_0^s = 0, \quad c_0^e(\omega) = 2\pi\omega_1.$$

Let $N, J \geq 0$, $0 < \varepsilon < 1$, $\nu \geq 2$, $\omega \in \mathbb{S}^1$, $\vartheta \in \mathbb{R}$, and a scalar-valued profile $f \in C_c^\infty(U_0)$. For $\Phi = \nu\omega \cdot x + \vartheta$,

$$(21) \quad \|V_{\mathbb{R}}(f \sin \Phi) - \mathcal{V}_\Phi^N[f]\|_{C^J(\mathbb{R}^2)} \leq C_{N,J,\varepsilon} \left(\sum_{d=0}^N \nu^{d\varepsilon+J-N} \|f\|_{C^{N-d+J}} + \nu^{J-(N+2)\varepsilon} \|f\|_{C^{N+J+1}} \right).$$

For $N \geq 0$, define

$$\mathcal{R}_\Phi^N[f] := V_{\mathbb{R}}(f \sin \Phi) - \mathcal{V}_\Phi^N[f].$$

If $N \geq 1$, then

$$u_{\mathbb{R}}(f \sin \Phi) - u_{\text{pw}}^{[N]}[f; \Phi] = (-\mathcal{R}_{\Phi+\pi/2}^N[(\nu\omega_2)f] - \mathcal{R}_\Phi^{N-1}[\partial_2 f], \\ \mathcal{R}_{\Phi+\pi/2}^N[(\nu\omega_1)f] + \mathcal{R}_\Phi^{N-1}[\partial_1 f]).$$

Thus $u_{\text{pw}}^{[N]}[f; \Phi]$ approximates the Euclidean singular velocity of the localized oscillation $f \sin \Phi$, with each scalar remainder controlled by (21) at order N or $N-1$. It does not approximate the oscillatory scalar itself.

Proof. Subtract two radial cutoffs in (16)–(17) and integrate in a coordinate for which $|\omega_a| \geq 1/2$. For every $R > 0$, more than $|\gamma| + 2 + R$ integrations by parts make the difference $O(\sigma^R)$, uniformly on each of two direction charts. Differentiating in ω proves smooth uniform convergence. Reflection $h \mapsto -h$ gives the first two identities in (20). The Fourier multiplier of (15) is $2\pi i \xi_1 |\xi|^{-2}$; applying it to $\sin(\omega \cdot x)$ gives $2\pi\omega_1 \cos(\omega \cdot x)$, proving the last two identities.

Set $\kappa_\nu(h) = \kappa_0(\nu^\varepsilon|h|)$ and split (15) smoothly into the terms with factors κ_ν and $1 - \kappa_\nu$. In the near-field term, Taylor-expand $f(x+h)$ through order N . Scaling h by ν gives $\mathcal{V}_\Phi^N[f]$; after J output derivatives, the Taylor remainder is bounded by

$$C\nu^{J-(N+2)\varepsilon} \|f\|_{C^{N+J+1}}.$$

In the far-field term, choose a with $|\omega_a| \geq 1/2$ and use, for its complex form,

$$\int (1 - \kappa_\nu) \frac{h_1}{|h|^2} f(x+h) e^{i\nu\omega \cdot h} dh = \frac{(-1)^N}{(i\nu\omega_a)^N} \int e^{i\nu\omega \cdot h} \partial_{h_a}^N \left[(1 - \kappa_\nu) \frac{h_1}{|h|^2} f(x+h) \right] dh.$$

There are no boundary terms because the split is smooth and the profile is compactly supported. When d derivatives hit the cutoff or kernel, the cutoff cost is $\nu^{d\varepsilon}$, and the contribution is bounded by

$$C\nu^{d\varepsilon+J-N}\|f\|_{C^{N-d+J}}, \quad 0 \leq d \leq N.$$

After scaling the near-field term, its cutoff is $\kappa_0(\nu^{\varepsilon-1}|h|)$; hence the finite-cutoff moments differ from (16)–(17) by an arbitrarily negative power of ν . This proves (21). Finally,

$$\partial_j(f \sin \Phi) = (\nu\omega_j)f \sin(\Phi + \pi/2) + (\partial_j f) \sin \Phi.$$

Apply the scalar identity at orders N and $N-1$ to obtain the displayed velocity remainder. $\square$

At a fixed time, let $N \geq 0$, $\lambda > 0$, $\ell \in \mathbb{N}$, $\vartheta \in \mathbb{R}$, $X \in C^\infty(U; \mathbb{R})$ with $A := \nabla X(0) \neq 0$, and $b \in C_c^\infty(U_0)$. Set

$$Y_\ell = \ell(X - A \cdot x), \quad \nu_\ell = \lambda\ell|A|, \quad \omega = A/|A|, \quad \Phi_{\ell,\vartheta} = \nu_\ell\omega \cdot x + \vartheta.$$

For the chosen sine-band presentation (b, ℓ, ϑ) , define

$$(22) \quad \begin{aligned} u^{[N]}[b \sin(\lambda\ell X + \vartheta)] &= u_{\text{pw}}^{[N]}[b \cos(\lambda Y_\ell); \Phi_{\ell,\vartheta}] \\ &\quad + u_{\text{pw}}^{[N]}[b \sin(\lambda Y_\ell); \Phi_{\ell,\vartheta} + \pi/2]. \end{aligned}$$

Cosine bands are defined by shifting ϑ by $\pi/2$, and finite presentations are treated band by band. Thus $u^{[N]}$ here denotes a quantity defined from the chosen presentation, rather than an operator determined only by the resulting scalar function. Formula (22) is the exact sine-addition split; no affine approximation of the density is made. When the data depend smoothly on time, this definition is applied at each fixed time.

3.5. Transported phase and oscillatory products. For $a \in \mathbb{R}$, write $a_+ = \max\{a, 0\}$.

The next lemma assumes an odd parent with compact Fourier support and with scale-local derivative bounds and an odd transported phase that is nearly affine on B_r . It controls the nonlinear phase remainder Y_ℓ and the time dependence of the induced frequency ν_ℓ and direction α ; the following corollary then converts these bounds into estimates for localized oscillatory products.

Lemma 3.2 (Transported-phase estimates). *Let $0 < \beta \leq 1/8$, $\lambda \geq e$, and set*

$$L = \log \lambda, \quad \mu = \lambda^{\beta/2}, \quad M = \lceil \beta^{-2} \rceil, \quad r = \lambda^{-2\beta} L^4.$$

Let $I = [s, 1]$, where $0 \leq s \leq 1$, and let B_r denote the Euclidean ball of radius r centered at the origin. Assume $\overline{B_r} \subset U_0$. Fix integers $T, K \geq 0$, and define

$$(23) \quad N_{\text{prof}} = K + M + T + 2, \quad N_{\text{phase}} = N_{\text{prof}} + 1.$$

The parent estimates use derivatives through N_{prof} , whereas the phase estimates use one additional derivative through N_{phase} . All constants in the hypotheses below are independent of λ and the harmonic index ℓ . Let $P \in C^\infty(\mathbb{T}^2 \times I)$ be odd in space and satisfy $\text{supp } \widehat{P(\cdot, t)} \subset \{|k| \leq 2\mu\}$ for every $t \in I$. Put

$$v = u_{\mathbb{T}}(P), \quad D = \partial_t + v \cdot \nabla, \quad \mathcal{G} = \nabla P,$$

and suppose on $B_r \times I$ that

$$(24) \quad \begin{aligned} \|D^p v\|_{C^0(B_r)} &\leq C_p L^{C_p} r, \\ \|D^p v\|_{C^{j+1}(B_r)} &\leq C_{p,j} L^{C_{p,j}} \mu^j, \end{aligned}$$

for $0 \leq p \leq (T-1)_+$, $0 \leq j \leq N_{\text{prof}}$, and

$$(25) \quad \|D^p \mathcal{G}\|_{C^j(B_r)} \leq C_{p,j} L^{C_{p,j}} \mu^j \quad (0 \leq p \leq T, \quad p+j \leq N_{\text{prof}}).$$

Let $X \in C^\infty(U \times I; \mathbb{R})$ be odd in space and satisfy on $B_r \times I$

$$DX = 0, \quad L^{-C} \leq |\nabla X| \leq L^C,$$

and

$$\begin{aligned}
& \|\nabla^j X\|_{C^0(B_r)} \leq C_j L^{C_j} \mu^{(j-1)+} & (1 \leq j \leq N_{\text{phase}}), \\
& \|X - \nabla X(0, t) \cdot x\|_{C^0(B_r)} \leq L^C \lambda^{-4\beta}, \\
(26) \quad & \|\nabla X - \nabla X(0, t)\|_{C^0(B_r)} \leq L^C \lambda^{-2\beta}.
\end{aligned}$$

The Fourier-support condition on P records how (24)–(25) are obtained later; the arguments from this lemma through Corollary 3.6 use those displayed bounds and oddness, not that support condition itself. The last two lines of (26) also follow from its first line and oddness by Taylor's theorem, but are recorded in the form used below. Set

$$A(t) = \nabla X(0, t), \quad Y_\ell = \ell(X - A(t) \cdot x), \quad \nu_\ell = \lambda \ell |A(t)|.$$

Assume that $\nu_\ell \geq 2$ on I for every integer $1 \leq \ell \leq 2^{M+1}$.

Choose the continuous angle α by

$$\frac{\nabla X(0, t)}{|\nabla X(0, t)|} = e(\alpha(t)),$$

with $\alpha(1) \in [-\pi, \pi)$. One has

$$\begin{aligned}
(27) \quad & \max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p Y_\ell\|_{C^0(B_r)} \leq C_p L^{C_p} \lambda^{-4\beta}, \\
& \max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p \nabla Y_\ell\|_{C^j(B_r)} \leq C_{p,j} L^{C_{p,j}} \lambda^{-2\beta+2j\beta},
\end{aligned}$$

$$(28) \quad \nu_\ell^{-1} |\partial_t^p \nu_\ell| + |\partial_t^p \alpha| \leq C_p L^{C_p}.$$

Here $0 \leq p \leq T$, and $p + j \leq N_{\text{prof}}$ in the second line. Also $L^{-C} \lambda \leq \nu_\ell \leq C_\beta L^C \lambda$. The constants depend only on β, T, K , the fixed chart and cutoff data, and the finitely many constants in the hypotheses. They are independent of λ and ℓ . In the iteration, $\beta \in \{2^{-3-k} : k \geq 0\}$; for $\beta \geq \beta_0 > 0$, all constants and logarithmic exponents here and in the results below may therefore be replaced by their maximum over this finite set.

Proof. Oddness of X gives zero constant and quadratic jets, and $Y_\ell = \ell(X - A \cdot x)$ has zero linear jet. The spatial conclusions at $p = 0$ follow from (26), Taylor's theorem, and $\mu = \lambda^{\beta/2}$. Differentiating $DX = 0$ at the origin gives

$$\dot{A} = -D_x v(0, t)^\top A,$$

and hence

$$(29) \quad DY_\ell = -\ell A(t) \cdot (v(x, t) - D_x v(0, t)x).$$

Every material derivative of the right side is odd with zero linear jet. For the material estimates, induct on $n = p + j$, where j is the C^j -order of ∇Y_ℓ . For $p > 0$, differentiate (29) $p - 1$ times and commute the spatial derivatives using

$$[D, \partial_a] = -(\partial_a v) \cdot \nabla$$

together with (24). Every parent factor has material order at most $p - 1$. Every phase factor has material order at most $p - 1$ and spatial order at most j , hence total order at most $n - 1$. Thus no output pair is used to prove itself. The largest parent demand occurs when all spatial derivatives land on the velocity remainder in (29); it is $D^{p-1}v$ in C^{j+1} , which lies in (24). At low spatial order,

$$\|D^p Y_\ell\|_{C^0(B_r)} \leq L^C \mu^2 r^3 \leq L^C \lambda^{-4\beta}, \quad \|\nabla D^p Y_\ell\|_{C^0(B_r)} \leq L^C \mu^2 r^2 \leq L^C \lambda^{-2\beta}.$$

The Hessian is odd and therefore vanishes at the origin, so

$$\|\nabla^2 D^p Y_\ell\|_{C^0(B_r)} \leq L^C \mu^2 r \leq L^C.$$

For $j \geq 2$, $\mu^j \leq \lambda^{-2\beta+2j\beta}$. At the top spatial endpoint $(p, j) = (0, N_{\text{prof}})$, all spatial derivatives may land on the phase, requiring $\nabla^{N_{\text{prof}}+1}X$. This is exactly the one additional derivative in $N_{\text{phase}} = N_{\text{prof}} + 1$. At the opposite endpoint with $T \geq 1$, the material identity uses only $D^{T-1}v$; when $T = 0$, the positive-material-order induction is vacuous.

To obtain ordinary-time estimates, iterate $\partial_t = D - v \cdot \nabla$. The largest phase demand occurs when all transport factors land on the phase block, reaching spatial order $j + p$. Each accompanying undifferentiated velocity costs only $r\lambda^{2\beta} = L^4$; if a derivative instead lands on a velocity factor, the phase order decreases. Hence the diagonal restriction $p + j \leq N_{\text{prof}}$ is sufficient. The origin equation gives (28). This proves (27). $\square$

Corollary 3.3 (Oscillatory-product estimates). *Assume the hypotheses of Lemma 3.2. Let $\mathcal{A} > 0$ and $b \in C^\infty(U \times I)$, with*

$$\bigcup_{t \in I} \text{supp } b(\cdot, t) \subseteq B_r,$$

and suppose that

$$(30) \quad \|D^p b\|_{C^j(B_r)} \leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^{2j\beta} \quad (0 \leq p \leq T, \ p + j \leq N_{\text{prof}}).$$

The constants in this hypothesis are independent of λ, ℓ, ϑ , and $\mathcal{A}$. For $\mathbf{t} \in \{\sin, \cos\}$, $\vartheta \in \mathbb{R}$, and an integer $1 \leq \ell \leq 2^{M+1}$, let

$$z = b\mathbf{t}(\lambda\ell X + \vartheta),$$

extended by zero from the coordinate chart. For $0 \leq p \leq T$ and $0 \leq j \leq K$,

$$(31) \quad \begin{aligned} \|D^p z\|_{C^j(B_r)} &\leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^j, \\ \|\partial_t^p z\|_{C^j(B_r)} &\leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^{j+p(1-2\beta)}. \end{aligned}$$

The conclusion constants depend only on β, T, K , the fixed chart and cutoff data, and the finitely many constants in the hypotheses. They are independent of λ, ℓ, ϑ , and $\mathcal{A}$.

Proof. Leibniz' rule and $DX = 0$ give the material derivative estimate in (31). Converting to ordinary time derivatives costs $r\lambda = L^4 \lambda^{1-2\beta}$ whenever a derivative hits the oscillatory factor. $\square$

3.6. Localized periodic velocity expansion. For the data of Corollary 3.3, a *localized oscillatory packet* is a scalar

$$z = b\mathbf{t}(\lambda\ell X + \vartheta), \quad \mathbf{t} \in \{\sin, \cos\}, \quad \ell \in \mathbb{Z}, \quad 1 \leq \ell \leq 2^{M+1},$$

with b supported in U_0 and satisfying the coefficient bounds in that corollary.

Proposition 3.4 (Localized periodic velocity expansion). *Assume the hypotheses of Corollary 3.3, and use its notation. For a localized oscillatory packet z , the truncated expansion in (22) has a finite-order decomposition at the same phase. The index n is the net power of ν_ℓ^{-1} , namely $n = |\gamma|$ in the branch containing $\nu_\ell \omega_a f$ and $n = |\gamma| + 1$ in the branch containing $\partial_a f$.*

$$(32) \quad u^{[M]}[z] = \sum_{n=0}^M (B_n^s \sin(\lambda\ell X + \vartheta) + B_n^c \cos(\lambda\ell X + \vartheta)),$$

where the coefficients are supported in $\text{supp } b$, extended by zero from the coordinate chart. All unqualified C^j norms of these coefficients are periodic norms. For $0 \leq p \leq T$, $0 \leq j \leq K$, and $0 \leq n \leq M$,

$$(33) \quad \max_{D \in \{D, \partial_t\}} (\|D^p B_n^s\|_{C^j} + \|D^p B_n^c\|_{C^j}) \leq C_{p,j,n} L^{C_{p,j,n}} \mathcal{A} \lambda^{2(j-n)\beta}.$$

For the same p, j , the exact periodic velocity satisfies

$$(34) \quad \left\| \partial_t^p (u_{\mathbb{T}} - u^{[M]}[z]) \right\|_{C^j(\mathbb{T}^2)} \leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^{p+2(j+1)(1-\beta)-M\beta}.$$

No material derivative estimate or local phase-harmonic presentation is asserted for this grouped exact remainder. The constants have the dependence stated in Corollary 3.3.

Proof. Apply Lemma 3.1 with $\varepsilon = 1 - \beta$ to the exact sine-addition decomposition (22). For each affine branch, choose the coefficients recursively, with $a \in \{1, 2\}$ and e_a the corresponding coordinate multi-index, by

$$\begin{aligned} C_{\gamma+e_a}^s &= \partial_a C_\gamma^s - \lambda(\partial_a Y_\ell) C_\gamma^c, \\ C_{\gamma+e_a}^c &= \partial_a C_\gamma^c + \lambda(\partial_a Y_\ell) C_\gamma^s, \end{aligned}$$

starting from $(C_0^s, C_0^c) = (b, 0)$ for a sine input and $(0, b)$ for a cosine input. Order the pairs $(|\gamma|, j)$ lexicographically by $(|\gamma| + j, |\gamma|)$. The derivative term at $(|\gamma| + 1, j)$ uses the preceding coefficient in C^{j+1} , on the same diagonal but with smaller $|\gamma|$, so this recursion is noncircular. The largest regularity demand occurs when these derivative terms put all $|\gamma| + j$ derivatives on b ; it is exactly the range in (30). At derivative order d , the carrier term has the largest frequency exponent, since $\|\lambda \nabla Y_\ell\|_{C^d} \leq L^C \lambda^{1-2\beta+2d\beta}$, whereas the derivative term costs only $\lambda^{2\beta} \leq \lambda^{1-2\beta}$. Leibniz' rule therefore gives, for $|\gamma| \leq M$ and $|\gamma| + j \leq N_{\text{prof}}$,

$$\partial^\gamma [b \mathfrak{t}(\lambda Y_\ell)] = C_\gamma^s \sin(\lambda Y_\ell) + C_\gamma^c \cos(\lambda Y_\ell), \quad \|C_\gamma^{s,c}\|_{C^j} \leq L^C \mathcal{A} \lambda^{|\gamma|(1-2\beta)+2j\beta}.$$

An order- n term in the affine expansion contains ν_ℓ^{-n} , so it is at most $L^C \mathcal{A} \lambda^{2(j-n)\beta}$. To see the exact recombination, put $g = b e^{i\lambda Y_\ell}$ and $\Phi = \Phi_{\ell,\vartheta}$. For each scalar branch,

$$(35) \quad \mathcal{V}_\Phi^N[\text{Re } g] + \mathcal{V}_{\Phi+\pi/2}^N[\text{Im } g] = \sum_{|\gamma| \leq N} \nu_\ell^{-|\gamma|-1} \text{Im}[(c_\gamma^s + i c_\gamma^c) \partial^\gamma g e^{i\Phi}].$$

Thus the terms with phase $\lambda Y_\ell - \Phi$ cancel index by index, and the remaining oscillatory factor is $\mathfrak{t}(\lambda \ell X + \vartheta)$. Since $DX = 0$, its material derivative vanishes, so material derivatives fall only on the recombined coefficients. Equivalently, these coefficients are finite linear combinations of products of smooth functions of ν_ℓ and α from the affine moments with the real and imaginary parts of

$$e^{-i\lambda Y_\ell} \partial^\gamma (b e^{i\lambda Y_\ell}).$$

When D hits this expression, the term from the first exponential cancels the term in which D hits the second exponential without a spatial derivative falling on DY_ℓ . Every remaining term contains a material derivative of b or one of those moment factors, a commutator derivative of v , or a spatial derivative of DY_ℓ . The coefficient input, (29), Lemma 3.2, and (28) therefore close a triangular induction in material order. At the endpoint it uses $p + |\gamma| + j \leq T + M + K = N_{\text{prof}} - 2$. Writing $\partial_t = D - v \cdot \nabla$, and using the C^0 bound for v , the spatial coefficient bounds, and (28), gives the same estimate for ordinary-time derivatives. This conversion uses only coefficient pairs on the diagonal $p' + j' \leq T + K$, which are supplied by the preceding induction. Lemma 3.2 and Corollary 3.3 therefore give (32) and (33) under material and ordinary derivatives.

For $p \geq 1$, we differentiate the three-part proof of (21) term by term, rather than differentiating its fixed-parameter statement. After p ordinary-time and j spatial derivatives, every singular remainder is a far integration-by-parts term, a near Taylor remainder, or a differentiated moment tail. On $\text{supp } b \subset B_r$, a derivative of the affine phase costs at most $L^C \lambda r$, derivatives of its cutoff and moments cost at most L^C , and derivatives of its profile cost at most $L^C \lambda^{1-2\beta}$. The largest regularity demand is the order- M Taylor remainder in the differentiated velocity branch, with all $p + j$ output derivatives on that branch. It uses $M + p + j + 2$ coefficient derivatives, exactly N_{prof} at $(p, j) = (T, K)$, and has at most p angular or radial parameter derivatives. Each time derivative costs at most $L^C \lambda$. If E_{far} , E_{Tay} , and E_{mom} denote the three corresponding contributions to the singular velocity remainder, substitution in (21) gives, for each $\star \in \{\text{far}, \text{Tay}, \text{mom}\}$,

$$\|\partial_t^p E_\star\|_{C^j} \leq C_{p,j} L^C \mathcal{A} \lambda^{p+2(j+1)(1-\beta)-M\beta}.$$

For the smooth periodic part, Lemma B.3 reduces the estimate to finitely many operators

$$\mathcal{H}z(x, t) = \int_{U_0} H(x, y) b(y, t) e^{i\lambda \ell X(y, t)} dy$$

with smooth H . On a neighborhood of $\text{supp } b(\cdot, t) \Subset B_r$, the lower gradient bound in Lemma 3.2 makes

$$\mathcal{L} = \frac{\nabla_y X}{i\lambda \ell |\nabla_y X|^2} \cdot \nabla_y$$

well defined. Direct differentiation gives

$$\mathcal{L}(e^{i\lambda \ell X}) = \frac{\nabla_y X \cdot (i\lambda \ell \nabla_y X)}{i\lambda \ell |\nabla_y X|^2} e^{i\lambda \ell X} = e^{i\lambda \ell X}.$$

The compact support of b removes the boundary term. Integrate exactly M times with the formal transpose of $\mathcal{L}$. Every integration gains λ^{-1} and costs at most $L^C \lambda^{2\beta}$; every ordinary-time derivative costs at most $L^C \lambda^{1-2\beta}$. Hence

$$\|\partial_t^p \mathcal{H}z\|_{C^j(\mathbb{T}^2)} \leq C_{p,j} L^C \mathcal{A} \lambda^{(p-M)(1-2\beta)}.$$

Only $M + p \leq M + T$ coefficient derivatives are used. Since

$$(p - M)(1 - 2\beta) \leq p + 2(j + 1)(1 - \beta) - M\beta$$

for $\beta \leq 1/8$, the smooth and singular contributions prove (34) within the block (23). $\square$

Corollary 3.5 (Leading response at the origin). *Under the hypotheses of Proposition 3.4, let z be a sine input. Then*

$$(36) \quad B_0^s = -C_0 \cos \alpha e^\perp(\alpha) b, \quad B_0^c = 0,$$

and

$$(37) \quad D_x u^{[0]}[z](0, t) = -C_0 \cos \alpha(t) e^\perp(\alpha(t)) (\nabla z(0, t))^\top.$$

Proof. At order zero, (20) gives (36); differentiating that response gives (37). $\square$

3.7. Common-phase interactions.

Corollary 3.6 (Common-phase interaction bounds). *Assume the parent and phase hypotheses of Lemma 3.2 for a single shared $P, D, X, \lambda, \beta, T, K$ and the associated scales. For $\iota = 1, 2$, let $\mathcal{A}_\iota > 0$ and $b_\iota \in C^\infty(U \times I)$, and set*

$$z_\iota = b_\iota \mathbf{t}_\iota (\lambda \ell_\iota X + \vartheta_\iota),$$

where $\ell_\iota \in \mathbb{Z}$, $1 \leq \ell_\iota \leq 2^{M+1}$, $\vartheta_\iota \in \mathbb{R}$, $\mathbf{t}_\iota \in \{\sin, \cos\}$, $\bigcup_{t \in I} \text{supp } b_\iota(\cdot, t) \Subset B_r$, and b_ι satisfies (30) with size $\mathcal{A}_\iota$. Thus both oscillations share the Fourier-truncated parent, material derivative, transported phase, base oscillation parameter, and scale exponent; their harmonic indices and phase shifts may differ.

Every coefficient obtained from $u^{[M]}[z_1] \cdot \nabla z_2$ by product-to-sum reduction satisfies

$$(38) \quad \max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p c\|_{C^j} \leq C_{p,j} L^{C_{p,j}} \mathcal{A}_1 \mathcal{A}_2 \lambda^{1-2\beta+2j\beta}.$$

For either $z = z_\iota$, every coefficient in $(u^{[M]} - u^{[0]})[z] \cdot \mathcal{G}$ satisfies

$$(39) \quad \max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p c\|_{C^j} \leq C_{p,j} L^{C_{p,j}} \mathcal{A}_\iota \lambda^{-2\beta+2j\beta}.$$

Both estimates hold for $0 \leq p \leq T$ and $0 \leq j \leq K$. The constants have the dependence stated in Corollary 3.3.

Proof. Put $A = \nabla X(0, t) = |A|e(\alpha(t))$. For every permitted harmonic index ℓ , the definition of Y_ℓ gives

$$\mathcal{W}_X := e^\perp(\alpha) \cdot \nabla X = e^\perp(\alpha) \cdot (\nabla X - A) = \ell^{-1} e^\perp(\alpha) \cdot \nabla Y_\ell.$$

This function and all its material or ordinary time derivatives vanish at the origin. The mean-value theorem, together with (27) and (28), gives, for $\mathcal{D} \in \{D, \partial_t\}$,

$$\|\mathcal{D}^p \mathcal{W}_X\|_{C^0(B_r)} \leq r \|\nabla \mathcal{D}^p \mathcal{W}_X\|_{C^0(B_r)} \leq L^C r \leq L^C \lambda^{-2\beta},$$

and

$$\|\mathcal{D}^p \mathcal{W}_X\|_{C^j(B_r)} \leq L^C \lambda^{2(j-1)+\beta}.$$

Thus the leading oscillatory derivative costs only $\lambda^{1-2\beta}$, and every higher velocity term gains $\lambda^{-2\beta}$. This proves (38). Product-to-sum reduction produces only indices $|\ell_1 \pm \ell_2| \leq 2^{M+2}$; no velocity expansion is applied to these products.

For ordinary-time derivatives of b_2 , ∇b_2 , and $\mathcal{G}$, use $\partial_t = D - v \cdot \nabla$ at one additional spatial order. The respective transport costs are $r\lambda^{2\beta} = L^4$ and $r\mu \leq L^4$, within the diagonal ranges of (30) and (25). Finally, every term in $(u^{[M]} - u^{[0]})[z] \cdot \mathcal{G}$ has $n \geq 1$, so (33) and (25) give (39). $\square$

4. FINITE-STAGE GEOMETRY AND AMPLIFICATION

This section states the finite-stage invariant, selects the next spatial oscillation parameter, and proves the phase-direction and amplitude mechanism for one adjacent pair of perturbations. An overdot denotes differentiation in time. With $C_0 = 2\pi$, define

$$\mathcal{M}(\alpha) := -e^\perp(\alpha)e(\alpha)^\top, \quad \alpha \in \mathbb{R}.$$

Fix

$$\tilde{C} = \frac{16\pi^3}{C_0} = 8\pi^2.$$

For every layer, set

$$(40) \quad \underline{t}_q = 1 - \tilde{C}\lambda_q^{-3\beta_q/4}, \quad I_q = [\underline{t}_q, 1].$$

For parameters $\lambda_i > 2$ and $\beta_i > 0$, the associated scales are defined by

$$(41) \quad L_i = \log \lambda_i, \quad \mu_i = \lambda_i^{\beta_i/2}, \quad M_i = \lceil \beta_i^{-2} \rceil, \quad r_i = \lambda_i^{-2\beta_i} L_i^4.$$

Thus λ_i is the base oscillation parameter, μ_i is the Fourier cutoff scale, M_i is both the velocity-expansion order and the number of correction levels, and r_i is the support radius.

4.1. The periodic inductive invariant. The induction preserves four kinds of information: the exact stage equation, separation of adjacent scales, support and derivative bounds, and terminal geometry at the origin.

Definition 4.1 (Admissible layered family). An admissible layered family through level q consists of functions that are odd in the spatial variable,

$$P_i, \theta_i, \rho^{(i)}, F^{(i)} \in C^\infty([0, 1] \times \mathbb{T}^2), \quad 0 \leq i \leq q,$$

and parameters

$$\lambda_i > 2, \quad 0 < \beta_i \leq \frac{1}{8}, \quad k_i, K_i \in \mathbb{N},$$

with the following properties.

(i) *Stage decomposition and equation.*

At level zero,

$$P_0 = 0, \quad \rho^{(0)} = \theta_0.$$

For $1 \leq i \leq q$,

$$(42) \quad P_i = S_{\mu_i} \rho^{(i-1)}, \quad \rho^{(i)} = P_i + \theta_i, \quad \text{supp } \widehat{P}_i \subset \{k : |k| \leq 2\mu_i\}.$$

Every finite stage solves

$$(43) \quad \partial_t \rho^{(i)} + u_{\mathbb{T}}(\rho^{(i)}) \cdot \nabla \rho^{(i)} = F^{(i)} \quad \text{on } [0, 1] \times \mathbb{T}^2.$$

Oddness makes every displayed scalar mean zero. Here $F^{(i)}$ denotes the cumulative finite-stage force.

(ii) *Scale ranges and separation.*

The integer k_i specifies the finite range $j + 2m \leq k_i$ for the perturbation θ_i , whereas K_i specifies the propagated range $1 \leq j + 2m \leq K_i$ for the complete stage $\rho^{(i)}$. They satisfy

$$(44) \quad k_i, K_i \geq d(\beta_i) := \left\lceil \frac{4}{\beta_i^2} \right\rceil + 1.$$

The terminal window is required to lie in the time domain:

$$(45) \quad \widetilde{C} \lambda_i^{-3\beta_i/4} \leq 1.$$

Thus $0 \leq t_i < 1$ and $I_i \subset [0, 1]$. In addition,

$$(46) \quad K_i \geq d(\beta_{i+1}) \quad (0 \leq i < q).$$

The scales L_i, μ_i, M_i, r_i are those in (41). When both indices exist,

$$(47) \quad \frac{\beta_i}{2} \leq \beta_{i+1} \leq \beta_i,$$

$$(48) \quad \lambda_{i+1} \geq 4\lambda_i, \quad \lambda_{i+1}^{\beta_{i+1}/8} \geq 4\lambda_i,$$

$$(49) \quad 2\lambda_{i+1}^{-3\beta_{i+1}/4} \leq \lambda_i^{-3\beta_i/4},$$

$$(50) \quad \mu_{i+1} \geq \lambda_i^4, \quad \mu_{i+1} \geq 4\mu_i.$$

(iii) *Support, activation, and derivative estimates.*

Each perturbation is supported in the fixed coordinate chart:

$$(51) \quad \text{supp } \theta_i(t) \subset B_{r_i} \Subset U_0.$$

For $i \geq 1$, there is an activation time $s_i \in I_{i-1}$ such that $\theta_i(t) = 0$ for $t \leq s_i$ (and hence vanishes to infinite order in time at $t = s_i$), and

$$(52) \quad s_i + L_i^{-8} < \underline{t}_i.$$

The finite spatial and mixed-derivative estimates are

$$(53) \quad \|\partial_t^m \theta_i(t)\|_{C^j} \leq \lambda_i^{j+2m} \quad (j + 2m \leq k_i),$$

$$(54) \quad \|\partial_t^m \rho^{(i)}(t)\|_{C^j} \leq 2\lambda_i^{j+2m} \quad (1 \leq j + 2m \leq K_i).$$

For every fixed $(j, m) \in \mathbb{N}_0^2$, the perturbation also satisfies

$$(55) \quad \|\partial_t^m \theta_i(t)\|_{C^j} \leq C_{j,m}(\beta_i) L_i^{N_{j,m}(\beta_i)} \lambda_i^{j+m(1-2\beta_i)-1+\beta_i}.$$

The families $C_{j,m}$ and $N_{j,m}$ are fixed before the stage induction. Induct on m : once every spatial order at time orders below m has been fixed, choose the order- (j, m) values simultaneously for all j . In the order- m density pass, obtain every threshold-relevant cutoff-parent input from the constant-free $(j, m) = (2, 0)$ case of (54) and the Fourier-cutoff Bernstein inequality. The remaining sharp inputs have time order strictly below m , so

their constants have already been fixed. Same-time parent estimates are used only in the subsequent defect and force estimates and do not enter this choice. For $\beta \leq b \leq \min\{2\beta, 1/8\}$, choose finite $\widehat{C}_{j,m}(b, \beta) \geq 1$ and $\widehat{N}_{j,m}(b, \beta) \in \mathbb{N}_0$ that dominate the resulting perturbation estimate. At $(b, \beta) = (1/8, 1/8)$, include the level-zero estimate. Finally, set

$$(56) \quad C_{j,m}(\beta) := \sup_{\substack{\beta \leq \gamma \leq 1/8 \\ \gamma \leq b \leq \min\{2\gamma, 1/8\}}} \widehat{C}_{j,m}(b, \gamma), \quad N_{j,m}(\beta) := \left\lceil \sup_{\substack{\beta \leq \gamma \leq 1/8 \\ \gamma \leq b \leq \min\{2\gamma, 1/8\}}} \widehat{N}_{j,m}(b, \gamma) \right\rceil.$$

Local uniformity of the estimates makes these suprema finite. The resulting families are independent of the stage index and both base oscillation parameters, and (56) is the envelope used when the earlier exponents lie in $[\beta, 1/8]$.

(iv) *Terminal origin geometry and angle evolution.*

On I_i , the inherited cutoff density is small relative to the newest perturbation:

$$(57) \quad \|P_i(t)\|_{C^1} + \|u_{\mathbb{T}}(P_i)(t)\|_{C^1} \leq \lambda_i^{\beta_i/8}.$$

The two amplitudes have different roles: A_i measures the perturbation density gradient, while B_i is the coefficient in its velocity-Jacobian expansion. There are positive functions

$$A_i, B_i \in C\left(I_i; [\lambda_i^{\beta_i}/2, 2\lambda_i^{\beta_i}]\right),$$

an angle $\alpha_i \in C^1(I_i)$, and remainders

$$E_i^\rho \in C(I_i; \mathbb{R}^2), \quad E_i^u \in C(I_i; \mathbb{R}^{2 \times 2})$$

such that

$$(58) \quad \nabla \theta_i(0, t) = A_i(t)e(\alpha_i(t)) + E_i^\rho(t), \quad |E_i^\rho(t)| \leq \sqrt{2} \lambda_i^{\beta_i/8},$$

$$(59) \quad D_x u_{\mathbb{T}}(\theta_i)(0, t) = C_0 B_i(t) \cos \alpha_i(t) \mathcal{M}(\alpha_i(t)) + E_i^u(t), \quad |(E_i^u)_{ab}(t)| \leq \lambda_i^{\beta_i/8}.$$

The reference angle is transported by P_i :

$$(60) \quad \dot{\alpha}_i = - \left\langle D_x u_{\mathbb{T}}(P_i)(0, t)^\top e(\alpha_i), e^\perp(\alpha_i) \right\rangle, \quad \alpha_i(1) = \frac{\pi}{2} + \lambda_i^{-\beta_i/8}.$$

For $i = 0$, the right side is zero.

The stage decomposition and cutoff nesting give an exact finite-stage formula before any global parameter recursion is imposed. For every $1 \leq Q \leq q$,

$$(61) \quad \rho^{(Q)} = S_{\mu_1} \theta_0 + \sum_{i=1}^{Q-1} S_{\mu_{i+1}} \theta_i + \theta_Q.$$

Indeed, $S_{\mu_j} S_{\mu_i} = S_{\mu_i}$ for $j > i$, by (50) and Lemma B.2; induction in Q proves the formula.

4.2. Selection of the next scales. Set

$$k_- := \frac{1}{192\pi}.$$

Proposition 4.2 (Deterministic scale selection). *Let an admissible family be given through level q , and choose β_{q+1} subject to*

$$(62) \quad \frac{\beta_q}{2} \leq \beta_{q+1} \leq \beta_q, \quad K_q \geq d(\beta_{q+1}).$$

Define

$$(63) \quad \boxed{\lambda_{q+1} = \exp\left(\frac{\beta_{q+1}}{192\pi} \lambda_q^{\beta_q/8}\right)}.$$

There is a finite threshold

$$\Lambda_{\text{sep}} = \Lambda_{\text{sep}}(\beta_q)$$

such that, if $\lambda_q \geq \Lambda_{\text{sep}}$, the adjacent frequency, terminal-time, and frequency separations (48)–(50) hold for the pair $(q, q+1)$. In particular,

$$(64) \quad \lambda_{q+1}^{\beta_{q+1}/8} \geq 4\lambda_q, \quad \lambda_{q+1}^{-\beta_{q+1}/8} \leq \lambda_q^{-1}, \quad \mu_{q+1} \geq \lambda_q^4, \quad \mu_{q+1} \geq 4\mu_q.$$

The exact transfer identity is

$$(65) \quad \lambda_q^A = \left(\frac{192\pi L_{q+1}}{\beta_{q+1}} \right)^{8A/\beta_q} \quad (A \in \mathbb{R}).$$

Proof. Put $b = \beta_q$, $\beta = \beta_{q+1}$, and $\Lambda = \lambda_q$. Then

$$(66) \quad L_{q+1} = k_- \beta \Lambda^{b/8}.$$

Since $\beta \in [b/2, b]$, the quantities

$$\frac{\beta L_{q+1}}{8} - \log(4\Lambda), \quad \frac{3\beta L_{q+1}}{4} - \frac{3b}{4} \log \Lambda, \quad \frac{\beta L_{q+1}}{2} - 4 \log \Lambda$$

tend to infinity with Λ , uniformly for $\beta \in [b/2, b]$. These three limits give respectively (48), (49), and $\mu_{q+1} \geq \lambda_q^4$. They also imply $\lambda_{q+1} \geq 4\lambda_q$, and $\lambda_{q+1}^{-\beta/8} \leq \Lambda^{-1}$, above one threshold depending only on b . Moreover, $\mu_{q+1} \geq \Lambda^4 \geq 4\Lambda^{b/2} = 4\mu_q$ for $\Lambda \geq 2$ and $b \leq 1/8$, so this separation requires no further threshold. Solving (66) for Λ^A proves (65). $\square$

4.3. Cutoff-parent geometry, flow, and transported phase.

Definition 4.3 (Adjacent-layer setup). Fix an admissible family through level q , and set the parent aliases

$$\Lambda := \lambda_q, \quad b := \beta_q.$$

Choose the child exponent β_{q+1} subject to

$$\beta := \beta_{q+1} \in [b/2, b], \quad K_q \geq d(\beta).$$

Define the child frequency λ_{q+1} from (Λ, b, β) by (63), write $\lambda := \lambda_{q+1}$, and assume

$$\Lambda \geq \Lambda_{\text{sep}}(b).$$

Proposition 4.2 supplies all adjacent-scale separations. Set

$$\begin{aligned} L &:= L_{q+1} = \log \lambda, & M &:= M_{q+1} = \lceil \beta^{-2} \rceil, & \mu &:= \mu_{q+1} = \lambda^{\beta/2}, \\ R &:= \lambda^{-2\beta}, & r &:= r_{q+1} = RL^4, \end{aligned}$$

where B_R is the initial child core and B_r is its buffered support region. Define the cutoff parent

$$(67) \quad P_{q+1} = S_\mu \rho^{(q)}, \quad v_q = u_{\mathbb{T}}(P_{q+1}).$$

Constants in the remainder of this section may depend on the adjacent exponents, displayed finite derivative orders, the cutoff χ , the constants in Definition 4.1, but not on Λ or λ .

Cutting off the complete stage at the child Fourier scale leaves all older cutoff multipliers unchanged and removes only a rapidly decaying tail of θ_q ; the next lemma transfers the terminal origin geometry to this cutoff parent.

Lemma 4.4 (Cutoff preservation of the newest geometry). *Assume Definition 4.3. There is a finite threshold*

$$\Lambda_{\text{geom}} = \Lambda_{\text{geom}}(b, \beta)$$

such that, if $\Lambda \geq \Lambda_{\text{geom}}$, the following conclusions hold for $t \in I_q$:

$$(68) \quad |\alpha_q(t) - \alpha_q(1)| \leq \Lambda^{-b/2},$$

$$(69) \quad -2\Lambda^{-b/8} \leq \cos \alpha_q(t) \leq -\frac{1}{2}\Lambda^{-b/8},$$

$$(70) \quad |\cos \alpha_q(t)| \leq (1 + \Lambda^{-3b/8})\Lambda^{-b/8}.$$

The nested Fourier cutoffs give, for $t \in [0, 1]$, the exact identity

$$(71) \quad P_{q+1} = P_q + S_\mu \theta_q.$$

For $\tau_q = (I - S_\mu)\theta_q$,

$$(72) \quad \|\tau_q(t)\|_{C^2} + \|D_x u_{\mathbb{T}}(\tau_q)(t)\|_{L^\infty} \leq C\Lambda^{-4},$$

for every $t \in [0, 1]$, where C is absolute and in particular independent of b, β . Consequently, for $t \in I_q$,

$$(73) \quad \nabla S_\mu \theta_q(0, t) = A_q e(\alpha_q) + \tilde{E}_q^\rho, \quad |\tilde{E}_q^\rho| \leq 4\Lambda^{b/8},$$

$$(74) \quad D_x u_{\mathbb{T}}(S_\mu \theta_q)(0, t) = C_0 B_q \cos \alpha_q \mathcal{M}(\alpha_q) + \tilde{E}_q^u, \quad |(\tilde{E}_q^u)_{ab}| \leq 4\Lambda^{b/8}.$$

Proof. The exact angle law and (57) give

$$|\dot{\alpha}_q| \leq C\Lambda^{b/8}.$$

Integration over I_q proves (68). Write $\alpha_q = \pi/2 + \Lambda^{-b/8} + h$, where $|h| \leq \Lambda^{-b/2}$. The bounds $2|z|/\pi \leq |\sin z| \leq |z|$ for $|z| \leq \pi/2$ give (69) and (70).

The Fourier support of P_q lies in $B_{2\mu_q}$, and $\mu \geq 2\mu_q$. Lemma B.2 gives $S_\mu P_q = P_q$, proving (71). Taking $j = 2, s = 3$ in (197), using $\|\theta_q\|_{C^s} \leq \Lambda^8$ and $\mu \geq \Lambda^4$, yields

$$\|\tau_q\|_{C^2} \leq C\mu^{-3}\Lambda^8 \leq C\Lambda^{-4}.$$

The Jacobian estimate in Lemma B.1 proves the second part of (72). Combining this tail with (58) and (59) proves the last two conclusions. $\square$

The next three results prepare the hypotheses of the transported-phase calculus in Section 3. The cutoff-parent flow first keeps the child core inside the coordinate chart and controls its derivatives. Pulling a terminal linear phase back by this flow then constructs the child phase; the final corollary records the resulting finite-order inputs.

Fix once and for all a radius $\rho_{\text{ch}} > 0$ such that

$$(75) \quad \overline{B_{2\rho_{\text{ch}}}} \subset U_0.$$

The subscript ch is shorthand for child.

Lemma 4.5 (Cutoff-parent flow). *Assume the hypotheses of Lemma 4.4, and fix an integer $J \geq 1$. There is a finite threshold*

$$\Lambda_{\text{flow}} = \Lambda_{\text{flow}}(b, \beta)$$

such that the following conclusions hold when $\Lambda \geq \Lambda_{\text{flow}}$, where the threshold is chosen at least as large as Λ_{geom} .

Let $\Gamma_q(t; \sigma)$ solve

$$\partial_t \Gamma_q(t; \sigma) = D_x v_q(0, t) \Gamma_q(t; \sigma), \quad \Gamma_q(\sigma; \sigma) = \text{Id}.$$

All the following conclusions hold for arbitrary $\sigma, t \in I_q$, in either time order. First,

$$(76) \quad \|\Gamma_q(t; \sigma)\|_{\text{op}} + \|\Gamma_q(\sigma; t)\|_{\text{op}} \leq C\Lambda^{b/4}.$$

Let $\Phi_q(t; \sigma, \cdot)$ be the global flow on $\mathbb{T}^2$ of v_q :

$$\partial_t \Phi_q(t; \sigma, x) = v_q(\Phi_q(t; \sigma, x), t), \quad \Phi_q(\sigma; \sigma, x) = x.$$

For $x \in B_{2r}$, let $\tilde{\Phi}_q(t; \sigma, x)$ denote the lift of this trajectory whose value at $t = \sigma$ is the coordinate representative x . Then

$$(77) \quad \sup_{x \in B_{2r}} |\tilde{\Phi}_q(t; \sigma, x)| \leq C_{b,\beta} L^2 r < \rho_{\text{ch}},$$

$$(78) \quad \Phi_q(t; \sigma, \overline{B_R}) \subset B_{3r/4} \Subset B_r,$$

$$(79) \quad \|D_x^j \Phi_q(t; \sigma, \cdot)\|_{C^0(B_r)} \leq C_j L^{C_j} \mu^{(j-1)+} \quad (1 \leq j \leq J).$$

Moreover, for $x \in B_r$,

$$(80) \quad \det D_x \Phi_q(t; \sigma, x) = 1, \quad \left\| (D_x \Phi_q(t; \sigma, x))^{-1} \right\|_{\text{op}} \leq CL^C.$$

Thus every trajectory used below to define the transported phase stays in the fixed coordinate chart.

Proof. We estimate the inverse transpose in the frame frozen at $\alpha_q(1)$, where the leading system is triangular. We first control its second component, then its first; incompressibility transfers the resulting bounds to Γ_q and its inverse. Let $Z = \Gamma_q^{-\top}$. By (71), (74), and (57), each column $z = (z_1, z_2)$ of Z satisfies

$$\begin{aligned} \dot{z}_1 &= C_0 B_q \cos \alpha_q z_2 + \varepsilon_{11} z_1 + \varepsilon_{12} z_2, \\ \dot{z}_2 &= \varepsilon_{21} z_1 + \varepsilon_{22} z_2, \end{aligned}$$

where

$$\sum_{a,b} |\varepsilon_{ab}| \leq C\Lambda^{b/2}.$$

The last bound includes the angular change of the principal matrix: $\Lambda^{7b/8} \Lambda^{-b/2} = \Lambda^{3b/8} \leq \Lambda^{b/2}$. It suffices to consider forward time; the same estimates apply after reversing time. For $\sigma \leq u \leq t$, set

$$m_a(u) = \sup_{\sigma \leq \tau \leq u} |z_a(\tau)|, \quad a \in \{1, 2\}.$$

Integration of the second equation and absorption of $C\Lambda^{-b/4} m_2(u)$ give

$$m_2(u) \leq C(1 + \Lambda^{-b/4} m_1(u)).$$

Since $|C_0 B_q \cos \alpha_q| \leq C\Lambda^{7b/8}$, the coefficient of m_1 after this substitution is bounded by $C\Lambda^{7b/8-b/4} = C\Lambda^{5b/8}$, while the source is bounded by $C\Lambda^{7b/8}$. Hence

$$1 + m_1(u) \leq C + \int_{\sigma}^u \left(C\Lambda^{5b/8} m_1(\tau) + C\Lambda^{7b/8} \right) d\tau.$$

The interval has length at most $\tilde{C}\Lambda^{-3b/4}$. Integral Grönwall therefore gives $m_1(u) \leq C\Lambda^{b/8}$, and hence $\|Z\| \leq C\Lambda^{b/4}$. The same argument with reversed time controls the inverse. Since v_q is divergence free, $\det \Gamma_q = 1$, and in two dimensions these bounds give (76) for both Γ_q and its inverse.

Cutoff boundedness, the $j = 2, m = 0$ case of (54), the Fourier support of P_{q+1} , Bernstein's inequality, the periodic Schauder estimate, and (65) give, for every $k \geq 1$, constants C_k, N_k , depending only on b, β, k and the fixed cutoff, such that

$$(81) \quad \|\nabla^k v_q\|_{L^\infty(\mathbb{T}^2)} \leq C_k L^{N_k} \mu^{k-1}.$$

In particular, fix $C_* = \max\{C_1, C_2\}$ and $N_* = \max\{N_1, N_2\}$. These two constants, and not the requested order J , determine the threshold below. Put

$$\mathbf{A}(t) = D_x v_q(0, t), \quad \mathcal{N}(x, t) = v_q(x, t) - \mathbf{A}(t)x.$$

Oddness gives $v_q(0, t) = 0$. Taylor's theorem and (81) therefore imply, for $x \in B_{\rho_{\text{ch}}}$,

$$(82) \quad |\mathcal{N}(x, t)| \leq C_* L^{N_*} \mu |x|^2, \quad \|D_x \mathcal{N}(x, t)\| \leq C_* L^{N_*} \mu |x|.$$

Let

$$H = \tilde{C}\Lambda^{-3b/4}, \quad G = \max\{1, C_{b,\beta}\Lambda^{b/4}\},$$

where G dominates both linear-flow norms in (76). The frequency law gives

$$(83) \quad H = \tilde{C}(k_- \beta)^6 L^{-6}, \quad G \leq C_{b,\beta} L^2, \quad \mu r = \lambda^{-3\beta/2} L^4.$$

After increasing Λ_{flow} , we may require simultaneously

$$(84) \quad 4Gr < \rho_{\text{ch}}, \quad 2GR < r, \quad 8C_* G^2 H L^{N^*} \mu r \leq \frac{1}{2}.$$

These conditions are compatible because $\Lambda^{b/4} = (L/(k_- \beta))^2$, $GR/r = G/L^4 \rightarrow 0$, and $L^N \mu r \rightarrow 0$ for every fixed N .

Fix $\sigma, t \in I_q$, put $h = |t - \sigma| \leq H$, and write

$$y(u) = \tilde{\Phi}_q(u; \sigma, x).$$

On the interval between σ and t , variation of constants gives

$$(85) \quad y(u) = \Gamma_q(u; \sigma)x + \int_{\sigma}^u \Gamma_q(u; \tau) \mathcal{N}(y(\tau), \tau) d\tau.$$

For $x \in B_{2r}$, bootstrap $|y| \leq 4Gr$. Equations (82) and (84) give

$$|y(u)| \leq 2Gr + 16C_* G^3 h L^{N^*} \mu r^2 < 4Gr.$$

The inequality is strict, so the continuation argument proves (77), in both time directions.

For $x \in \overline{B_R}$, bootstrap instead $|y| \leq r$. Then

$$|y(u)| \leq GR + C_* G h L^{N^*} \mu r^2 < \frac{r}{2} + \frac{r}{16} < \frac{3r}{4}$$

by (84). This proves (78).

For $x \in B_r$, set

$$J_1(u) = D_x \tilde{\Phi}_q(u; \sigma, x), \quad E(u) = D_x v_q(y(u), u) - \mathbf{A}(u).$$

The buffered-flow estimate and (82) give

$$\|E(u)\| \leq 4C_* G L^{N^*} \mu r.$$

Variation of constants around Γ_q yields

$$J_1(u) = \Gamma_q(u; \sigma) + \int_{\sigma}^u \Gamma_q(u; \tau) E(\tau) J_1(\tau) d\tau.$$

Hence

$$\sup_u \|J_1(u)\| \leq G + 4C_* G^2 H L^{N^*} \mu r \sup_u \|J_1(u)\| \leq G + \frac{1}{4} \sup_u \|J_1(u)\|,$$

and therefore $\sup_u \|J_1(u)\| \leq 2G \leq CL^2$.

For $m \geq 2$, let $J_m = D_x^m \tilde{\Phi}_q$. Differentiating the flow equation gives

$$\partial_u J_m = \mathbf{A} J_m + E J_m + \mathcal{F}_m, \quad J_m(\sigma) = 0,$$

where $\mathcal{F}_m$ is the sum over partitions $m_1 + \dots + m_a = m$ with $a \geq 2$. Assuming the result below orders m , each such term satisfies

$$\|\nabla^a v_q\| \prod_{\nu=1}^a \|J_{m_\nu}\| \leq C_m L^{C_m} \mu^{a-1} \prod_{\nu=1}^a \mu^{m_\nu-1} = C_m L^{C_m} \mu^{m-1}.$$

Variation of constants, followed by the same absorption used for J_1 , proves

$$\|J_m\|_{C^0(B_r)} \leq C_m L^{C_m} \mu^{m-1} \quad (2 \leq m \leq J).$$

This proves (79).

Finally, incompressibility gives

$$\partial_u \det J_1(u) = (\operatorname{div} v_q)(y(u), u) \det J_1(u) = 0.$$

Thus $\det J_1 = 1$. In two dimensions a determinant-one matrix and its inverse have the same Euclidean operator norm. The $m = 1$ estimate proves (80). $\square$

Lemma 4.6 (Transported child phase). *Assume Definition 4.3, fix an integer $J \geq 3$, and take the flows from Lemma 4.5 with $\Lambda \geq \Lambda_{\text{flow}}$. Put*

$$p_{\text{term}} = e\left(\frac{\pi}{2} + \lambda^{-\beta/8}\right).$$

Define globally on $\mathbb{T}^2 \times I_q$

$$(86) \quad X_{q+1}(x, t) = \psi_{p_{\text{term}}}(\Phi_q(1; t, x)).$$

Then X_{q+1} is odd,

$$(\partial_t + v_q \cdot \nabla)X_{q+1} = 0, \quad \nabla X_{q+1}(0, t) = \Gamma_q(1; t)^\top p_{\text{term}},$$

and the following estimates hold on $B_r \times I_q$:

$$(87) \quad \begin{aligned} L^{-C} &\leq |\nabla X_{q+1}| \leq L^C, \\ \|\nabla^j X_{q+1}\|_{C^0(B_r)} &\leq C_j L^{C_j} \mu^{(j-1)+}, \quad 1 \leq j \leq J, \end{aligned}$$

$$(88) \quad \begin{aligned} \|X_{q+1} - \nabla X_{q+1}(0, t) \cdot x\|_{C^0(B_r)} &\leq L^C \lambda^{-4\beta}, \\ \|\nabla X_{q+1} - \nabla X_{q+1}(0, t)\|_{C^0(B_r)} &\leq L^C \lambda^{-2\beta}. \end{aligned}$$

The continuous lift α_{q+1} determined by

$$(89) \quad e(\alpha_{q+1}(t)) = \frac{\nabla X_{q+1}(0, t)}{|\nabla X_{q+1}(0, t)|}, \quad \alpha_{q+1}(1) = \frac{\pi}{2} + \lambda^{-\beta/8},$$

satisfies

$$(90) \quad \dot{\alpha}_{q+1} = -\left\langle D_x u_{\mathbb{T}}(P_{q+1})(0, t)^\top e(\alpha_{q+1}), e^\perp(\alpha_{q+1}) \right\rangle.$$

Proof. Write $X := X_{q+1}$ in this fixed-step proof. The flow property gives

$$(\partial_t + v_q \cdot \nabla)X = 0 \quad \text{on } \mathbb{T}^2 \times I_q.$$

Both v_q and $\psi_{p_{\text{term}}}$ are odd, so uniqueness for the flow gives that X is odd. By (77), if $x \in B_r$, then $\tilde{\Phi}_q(1; t, x) \in B_{\rho_{\text{ch}}} \subset U_0$. Consequently, on B_r ,

$$X(x, t) = p_{\text{term}} \cdot \tilde{\Phi}_q(1; t, x), \quad \nabla X(0, t) = \Gamma_q(1; t)^\top p_{\text{term}}.$$

Equations (79) and (80) give

$$L^{-C} \leq |\nabla X(x, t)| \leq L^C \quad (x \in B_r),$$

together with all the asserted higher spatial derivative bounds.

Since X is odd, its quadratic jet vanishes at the origin. Taylor's theorem and $\|D_x^3 X\| \leq L^C \mu^2$ give

$$\|X - \nabla X(0, t) \cdot x\|_{C^0(B_r)} \leq L^C \mu^2 r^3, \quad \|\nabla X - \nabla X(0, t)\|_{C^0(B_r)} \leq L^C \mu^2 r^2.$$

Substitution of $\mu = \lambda^{\beta/2}$ and $r = \lambda^{-2\beta} L^4$ proves (88). Differentiating the transport equation at the origin gives

$$\partial_t \nabla X(0, t) = -D_x v_q(0, t)^\top \nabla X(0, t).$$

Taking its angular component proves (90). $\square$

The following corollary packages the hypotheses of Lemma 3.2 in the derivative ranges authorized by K_q .

Corollary 4.7 (Finite-order transported-phase inputs). *Assume Definition 4.3. Fix integers $T, K \geq 0$, let $N_{\text{prof}}, N_{\text{phase}}$ be defined by (23), and suppose*

$$(91) \quad N_{\text{phase}} + 2T + 2 \leq K_q.$$

Set $I = I_q$, let X_{q+1} be the child phase supplied by Lemma 4.6, and set

$$D_q = \partial_t + v_q \cdot \nabla.$$

There is a finite threshold

$$\Lambda_{\text{pkt}} = \Lambda_{\text{pkt}}(b, \beta)$$

such that, when $\Lambda \geq \Lambda_{\text{pkt}}$, the fields $P_{q+1}, v_q, D_q, X_{q+1}$ satisfy, on $B_r \times I_q$, (24)–(26) with $P = P_{q+1}$, $v = v_q$, $D = D_q$, and $X = X_{q+1}$, $\mathcal{G} = \nabla P_{q+1}$. More precisely, the velocity input holds for

$$0 \leq p \leq (T-1)_+, \quad 0 \leq j \leq N_{\text{prof}},$$

the density-gradient input holds for

$$0 \leq p \leq T, \quad p + j \leq N_{\text{prof}},$$

and the spatial phase estimates hold through order N_{phase} . Moreover,

$$\lambda \ell |\nabla X_{q+1}(0, t)| \geq 2 \quad (\ell \in \mathbb{Z}, 1 \leq \ell \leq 2^{M+1}).$$

Proof of Corollary 4.7. Apply Lemma 4.5 and Lemma 4.6 with $J = N_{\text{phase}}$. Their common threshold is independent of this requested order; only the constants in the resulting derivative estimates depend on it. Fourier support limitation, the mixed stage estimate (54), and (66) give, in the displayed finite ranges,

$$\|D_q^p v_q\|_{C^{j+1}} + \|D_q^p \nabla P_{q+1}\|_{C^j} \leq C_{p,j} L^{C_{p,j}} \mu^j.$$

Indeed, expansion of D_q^p uses ordinary stage derivatives of weighted order at most $j + 2p + 2$, all of which are supplied by (91). Oddness improves the $C^0(B_r)$ velocity bound to $C_p L^{C_{p,r}}$. The phase conclusions are those of Lemma 4.6. Its lower gradient bound also gives the stated lower frequency bound after increasing a threshold depending only on b, β , since that bound uses only the first flow derivative. $\square$

The identity $S_\mu P_q = P_q$ has a second consequence. The angle α_q in (60) continues to be driven by exactly P_q when the cutoff parent is formed. Thus there is no cutoff error in the reference-angle equation; only the spectral tail created when θ_q is cut off, estimated in (72), enters the child equation.

4.4. Perpendicularity time. The child is activated when its phase-gradient direction is perpendicular to the newest parent direction. The parent shear makes the relative angle satisfy a Riccati comparison, which yields one such time and locates it inside I_q .

Define the positive strain coefficient of the newest perturbation by

$$(92) \quad \kappa_q(t) = -C_0 B_q(t) \cos \alpha_q(t).$$

When $\Lambda \geq \Lambda_{\text{geom}}(b, \beta)$, Lemma 4.4 gives on I_q

$$(93) \quad \frac{C_0}{4} \Lambda^{7b/8} \leq \kappa_q(t) \leq 2C_0(1 + \Lambda^{-3b/8}) \Lambda^{7b/8}.$$

Lemma 4.8 (Perpendicularity crossing). *Assume Definition 4.3, and take the child phase X_{q+1} and angle α_{q+1} from Lemma 4.6 with $J = 3$. Thus, on I_q ,*

$$\begin{aligned} e(\alpha_{q+1}) &= \frac{\nabla X_{q+1}(0, t)}{|\nabla X_{q+1}(0, t)|}, & \alpha_{q+1}(1) &= \frac{\pi}{2} + \lambda^{-\beta/8}, \\ \dot{\alpha}_{q+1} &= - \left\langle D_x u_{\text{T}}(P_{q+1})(0, t)^\top e(\alpha_{q+1}), e^\perp(\alpha_{q+1}) \right\rangle. \end{aligned}$$

Choose a finite threshold

$$\Lambda_{\perp} = \Lambda_{\perp}(b, \beta)$$

with

$$\Lambda_{\perp} \geq \max \{ \Lambda_{\text{geom}}(b, \beta), \Lambda_{\text{flow}}(b, \beta) \}.$$

If $\Lambda \geq \Lambda_{\perp}$, there is a unique $s_{q+1} \in I_q$ such that

$$(94) \quad \alpha_{q+1}(s_{q+1}) - \alpha_q(s_{q+1}) = -\frac{\pi}{2}.$$

Writing $\delta = \alpha_{q+1} - \alpha_q$, one has on $[s_{q+1}, 1]$

$$(95) \quad -\frac{\pi}{2} \leq \delta(t) \leq -\frac{3}{4}\Lambda^{-b/8},$$

$$(96) \quad \frac{C_0}{8\pi^3}\Lambda^{7b/8}\delta(t)^2 \leq \dot{\delta}(t) \leq 3C_0\Lambda^{7b/8}\delta(t)^2.$$

Moreover,

$$(97) \quad \frac{1}{6C_0}\Lambda^{-3b/4} \leq 1 - s_{q+1} \leq \frac{32\pi^3}{3C_0}\Lambda^{-3b/4} < \tilde{C}\Lambda^{-3b/4},$$

and the crossing is transverse:

$$(98) \quad \dot{\delta}(s_{q+1}) \geq \frac{C_0}{32\pi}\Lambda^{7b/8}.$$

Proof. Subtract (60) from (90). For a matrix B , set

$$\mathbf{a}_B(\vartheta) := -\langle B^{\top}e(\vartheta), e^{\perp}(\vartheta) \rangle.$$

Then

$$\mathbf{a}'_B(\vartheta) = \langle B^{\top}e(\vartheta), e(\vartheta) \rangle - \langle B^{\top}e^{\perp}(\vartheta), e^{\perp}(\vartheta) \rangle,$$

so $|\mathbf{a}'_B| \leq 2\|B\|_{\text{op}}$. By (71), the contribution of P_q is $\mathbf{a}_{D_x u_{\mathbb{T}}(P_q)(0,t)}(\alpha_{q+1}) - \mathbf{a}_{D_x u_{\mathbb{T}}(P_q)(0,t)}(\alpha_q)$. The mean-value theorem and (57) bound it by $C\Lambda^{b/8}|\delta|$. The principal part of (74) gives exactly $\kappa_q \sin^2 \delta$. Hence

$$(99) \quad \dot{\delta} = \kappa_q \sin^2 \delta + \mathcal{E}_q^{\alpha}, \quad |\mathcal{E}_q^{\alpha}| \leq 16\Lambda^{b/8}|\delta| + 8\Lambda^{b/8}.$$

There is no term involving $(I - S_{\mu})P_q$, because that function is identically zero.

At $t = 1$,

$$-\Lambda^{-b/8} \leq \delta(1) \leq -\frac{3}{4}\Lambda^{-b/8}.$$

On the strip

$$-\frac{\pi}{2} \leq \delta \leq -\frac{1}{4}\Lambda^{-b/8},$$

the remainder in (99) is absorbed by the lower bound in (93). Using $2|z|/\pi \leq |\sin z| \leq |z|$ for $|z| \leq \pi/2$, and the sharp upper bound in (93), gives (96). In particular, the gap is strictly increasing in forward time throughout the strip.

At every $t \in I_q$ for which $\delta(t) = -\pi/2$, (99) and (93) give

$$\dot{\delta}(t) \geq \frac{C_0}{4}\Lambda^{7b/8} - (8\pi + 8)\Lambda^{b/8} > 0.$$

Thus every crossing of $-\pi/2$ is upward, so there can be at most one such crossing in I_q .

Integrating the inequalities for $-1/\delta$ from the terminal gap backward to $-\pi/2$ gives (97). The upper endpoint lies strictly inside I_q because $32/3 < 16$, which gives existence. The preceding estimate at an arbitrary crossing gives uniqueness on all of I_q . Evaluating the lower inequality in (96) at $\delta = -\pi/2$ proves (98). $\square$

4.5. **Amplitude gain.** Set

$$k_g := \frac{1}{96\pi}.$$

The cutoff-parent gradient decomposes on I_q as

$$(100) \quad \nabla P_{q+1}(0, t) = A_q(t)e(\alpha_q(t)) + E_q^{\text{par}}(t), \quad |E_q^{\text{par}}(t)| \leq C\Lambda^{b/8}.$$

Indeed, combine (71), (57), and (73).

The principal perturbation constructed in the next section, with amplitude a_{q+1} from Lemma 4.9 and activation cutoff w from (130), has origin coefficient $w(t)a_{q+1}(t)\lambda^{\beta-1}$. Since $X_{q+1}(0, t) = 0$, its origin gradient is

$$w(t)a_{q+1}(t)\lambda^\beta \nabla X_{q+1}(0, t).$$

For $t \geq s_{q+1} + L_{q+1}^{-8}$, the activation cutoff satisfies $w(t) = 1$. Its leading velocity response is multiplication by

$$-C_0 \cos \alpha_{q+1} e^\perp(\alpha_{q+1}).$$

The following equation cancels the resulting leading interaction with ∇P_{q+1} .

Lemma 4.9 (Amplitude gain). *Under the hypotheses and notation of Lemma 4.8, let s_{q+1} be the unique crossing furnished there. Choose a finite threshold*

$$\Lambda_{\text{gain}} = \Lambda_{\text{gain}}(b, \beta) \geq \Lambda_\perp(b, \beta).$$

If $\Lambda \geq \Lambda_{\text{gain}}$, let a_{q+1} be the unique positive solution on I_q of

$$(101) \quad \frac{\dot{a}_{q+1}}{a_{q+1}} = C_0 \cos \alpha_{q+1} e^\perp(\alpha_{q+1}) \cdot \nabla P_{q+1}(0, t), \quad a_{q+1}(1) = 1.$$

Then

$$(102) \quad \frac{a_{q+1}(1)}{a_{q+1}(s_{q+1})} \geq \exp(k_g \Lambda^{b/8}).$$

On every subinterval,

$$(103) \quad \frac{a_{q+1}(t_2)}{a_{q+1}(t_1)} \geq e^{-1} \quad (s_{q+1} \leq t_1 \leq t_2 \leq 1).$$

Proof. Write $a := a_{q+1}$. We isolate a short interval of positive gain and then bound every later negative contribution. By (100), the amplitude equation is equivalently

$$(104) \quad \begin{aligned} \frac{\dot{a}}{a} &= C_0 A_q \cos \alpha_{q+1} \sin(\alpha_q - \alpha_{q+1}) + g_q, \\ g_q &= C_0 \cos \alpha_{q+1} e^\perp(\alpha_{q+1}) \cdot E_q^{\text{par}}, \quad |g_q| \leq C\Lambda^{b/8}. \end{aligned}$$

For the lower bound, let t_- be the unique time at which $\delta(t_-) = -2\pi/5$. The upper Riccati inequality gives

$$(105) \quad t_- - s_{q+1} \geq \frac{1}{6\pi C_0} \Lambda^{-7b/8}.$$

Throughout this interval, by (68),

$$\alpha_q(t) = \frac{\pi}{2} + \Lambda^{-b/8} + O(\Lambda^{-b/2}), \quad -\frac{\pi}{2} \leq \delta(t) \leq -\frac{2\pi}{5}.$$

Consequently, after increasing the threshold, $0 < \alpha_{q+1}(t) < \pi/6$ and $2\pi/5 \leq \alpha_q - \alpha_{q+1} \leq \pi/2$. Thus $\cos \alpha_{q+1} \geq 1/2$ and $\sin(\alpha_q - \alpha_{q+1}) \geq 1/2$. Since $A_q \geq \Lambda^b/2$, the principal contribution to $\log(a(t_-)/a(s_{q+1}))$ is at least

$$\frac{1}{48\pi} \Lambda^{b/8}.$$

The angle α_{q+1} is strictly increasing on $[s_{q+1}, 1]$: the lower Riccati bound dominates $|\dot{\alpha}_q|$ even at the terminal gap. Let t_0 be its unique crossing of $\pi/2$. The principal amplitude term is nonnegative on $[t_-, t_0]$. By (64), on $[t_0, 1]$,

$$0 \leq \alpha_{q+1} - \frac{\pi}{2} \leq \lambda^{-\beta/8} \leq \Lambda^{-1},$$

and (97) gives

$$\left| \int_{t_0}^1 C_0 A_q \cos \alpha_{q+1} \sin(\alpha_q - \alpha_{q+1}) dt \right| \leq C \Lambda^{b/4-1},$$

$$\int_{s_{q+1}}^1 |g_q| dt \leq C \Lambda^{-5b/8}.$$

Both powers tend to zero because $0 < b \leq 1/8$. For a sufficiently large threshold, the total logarithmic gain is therefore at least $(96\pi)^{-1} \Lambda^{b/8}$.

The same displayed bounds on an arbitrary subinterval show, after increasing the threshold, that its negative principal contribution and its g_q -contribution have total absolute value at most 1. This proves (103). $\square$

Corollary 4.10 (Deterministic seed smallness). *Assume the hypotheses and notation of Lemma 4.9. There is a finite threshold*

$$\Lambda_{\text{seed}} = \Lambda_{\text{seed}}(b, \beta) \geq \Lambda_{\text{gain}}(b, \beta)$$

such that, if $\Lambda \geq \Lambda_{\text{seed}}$, then

$$(106) \quad a_{q+1}(s_{q+1}) \leq \lambda^{-2/\beta} \leq \lambda^{-1/\beta}.$$

Proof. By (66), $k_g = 2k_-$, and $a_{q+1}(1) = 1$, the lower bound in (102) gives

$$a_{q+1}(s_{q+1}) \leq e^{-2k_- \Lambda^{b/8}} = \lambda^{-2/\beta} \leq \lambda^{-1/\beta}.$$

$\square$

5. CONSTRUCTION OF ONE NEW LAYER

We first form the cutoff parent and its leading transported oscillation. The leading linearized operator can be inverted along the parent flow; repeating this operation finitely many times cancels the successive nonzero phase-harmonic errors. The final uncanceled term, together with the exact velocity remainders and the cutoff commutator, is placed in the force.

Fix the adjacent-layer setup of Definition 4.3. We retain the one-step aliases

$$\Lambda = \lambda_q, \quad b = \beta_q, \quad \beta = \beta_{q+1}, \quad \lambda = \lambda_{q+1}, \quad L = L_{q+1}, \quad \mu = \mu_{q+1}, \quad r = r_{q+1},$$

together with its local core radius R , and set

$$M := M_{q+1} = \lceil \beta^{-2} \rceil.$$

Proposition 4.2 supplies all scale separations used below. The adjacent-layer setup includes $K_q \geq d(\beta)$ and defines λ by (63).

All constants in this section may depend on the two adjacent exponents, the displayed derivative orders, the fixed perturbation and activation profiles, and the cutoff in (13), but not on either base oscillation parameter. They may also depend on the finite subset of the a priori family $C_{j,m}(\eta), N_{j,m}(\eta)$ fixed in Definition 4.1 that is explicitly used.

Mixed time derivatives close in two passes. Within one child, at each order p , all its perturbation coefficients are estimated first, using structured-defect estimates only below order p . Only then are the order- p residual and force-increment terms formed. The closure order is stated and explained after the correction hierarchy. Appendix A proves Lemmas 5.1, 5.2, and 5.4, and supplies the technical endpoint and derivative-range details for Lemma 5.8.

5.1. The cutoff parent and its weighted propagator. The parent for the new perturbation is the cutoff of the complete preceding stage. Write

$$(107) \quad P := P_{q+1} = S_\mu \rho^{(q)}.$$

Lemma 4.4 gives

$$(108) \quad S_\mu P_q = P_q, \quad P = P_q + S_\mu \theta_q,$$

and

$$(109) \quad \|(I - S_\mu)\theta_q\|_{C^2} + \|D_x u_{\mathbb{T}}((I - S_\mu)\theta_q)\|_{L^\infty} \leq C\Lambda^{-4}.$$

Thus the previously filtered terms are unchanged and only the newest perturbation is truncated.

Put

$$v := u_{\mathbb{T}}(P), \quad D := \partial_t + v \cdot \nabla.$$

Lemma 5.1 (All-orders cutoff-parent inputs). *Fix $T, S \in \mathbb{N}_0$, and assume the adjacent-layer setup above. Use the constant-free $(j, m) = (2, 0)$ case of (54) at time order zero. For $1 \leq p \leq T$, use the sharp perturbation estimate (55) only at spatial order two, with the a priori constants fixed in Definition 4.1. Assume also*

$$\lambda = \exp\left(\frac{\beta}{192\pi}\Lambda^{b/8}\right),$$

and hence the exact transfer identity (65). Then there are constants depending on b, β, T, S and the fixed profile and cutoff data, but not on the base oscillation parameters, such that

$$(110) \quad \max_{0 \leq p \leq T} \left(\|\partial_t^p \rho^{(q)}\|_{C^2} + \|\partial_t^p u_{\mathbb{T}}(\rho^{(q)})\|_{C^1} \right) \leq C_T L^{C_T}.$$

Consequently,

$$(111) \quad \|\partial_t^p v\|_{C^{j+1}} \leq C_{p,j} L^{C_{p,j}} \mu^j,$$

$$(112) \quad \|\partial_t^p \nabla P\|_{C^j} \leq C_{p,j} L^{C_{p,j}} \mu^j,$$

for $0 \leq p \leq T$ and $0 \leq j \leq S$. On B_r ,

$$(113) \quad \|\partial_t^p v\|_{C^0(B_r)} \leq C_p L^{C_p} r.$$

For $0 \leq p \leq T$ and $0 \leq j \leq S$, the local material estimates are

$$(114) \quad \begin{aligned} \|D^p v\|_{C^{j+1}(B_r)} + \|D^p \nabla P\|_{C^j(B_r)} &\leq C_{p,j} L^{C_{p,j}} \mu^j, \\ \|D^p v\|_{C^0(B_r)} &\leq C_p L^{C_p} r. \end{aligned}$$

At material order p , only ordinary parent time orders at most p occur. The high spatial orders in (111)–(114) follow from the Fourier cutoff and Bernstein's inequality; they consume no higher-order member of the a priori family. In the local material estimates, the extra derivative from a transport factor is absorbed by $r\mu = L^4 \lambda^{-3\beta/2} \leq 1$.

Hereafter, the cutoff-parent inputs mean the local estimates (114) on B_r , supplied by Lemma 5.1 under admissibility through level q . At coefficient time order zero only the constant-free stage bound is used. For $p \geq 1$, the coefficient pass invokes this lemma only through parent time order $p - 1$; the sharp order- p parent estimate is used afterwards in the defect and force pass.

Let $\alpha = \alpha_{q+1}$ be the phase angle transported by $u_{\mathbb{T}}(P)$, with terminal value

$$\alpha(1) = \frac{\pi}{2} + \lambda^{-\beta/8}.$$

The admissible geometry and scale separation permit the application of Lemmas 4.5 and 4.6. Lemma 4.8 gives the unique crossing time $s = s_{q+1}$. Lemma 4.9 supplies the positive amplitude $a = a_{q+1}$, normalized by $a(1) = 1$, and Corollary 4.10 gives

$$(115) \quad a(s) \leq \lambda^{-2/\beta}.$$

Let $\Phi := \Phi_q$, $X := X_{q+1}$, and $p_{\text{term}} = e(\pi/2 + \lambda^{-\beta/8})$ be the flow, transported phase, and terminal direction supplied by Lemmas 4.5 and 4.6. Thus $DX = 0$, $X(\cdot, 1) = p_{\text{term}} \cdot x$ on B_r , X is odd, and

$$(116) \quad \Phi(t; \sigma, \overline{B_R}) \subset B_{3r/4} \Subset B_r \Subset U_0.$$

The leading periodic packet response in Corollary 3.5 is

$$u^{[0]}[z] = m_0(t)z, \quad m_0(t) := -C_0 \cos \alpha(t) e^\perp(\alpha(t)).$$

Put $c_P(x, t) := m_0(t) \cdot \nabla P(x, t)$. On a shared phase, the leading linearized operator acting on the coefficient is $D + c_P$. For a coefficient φ prescribed at time σ , its characteristic solution operator is

$$(117) \quad H_{\sigma, t}[\varphi](x) := \varphi(\Phi(\sigma; t, x)) \exp\left(-\int_\sigma^t c_P(\Phi(\tau; t, x), \tau) d\tau\right).$$

It satisfies $(D + c_P)H_{\sigma, t}[\varphi] = 0$ and $H_{\sigma, \sigma}[\varphi] = \varphi$. Write

$$(118) \quad \mathcal{S}_{\sigma, t}[\varphi; \ell, \vartheta] := H_{\sigma, t}[\varphi] \sin(\lambda \ell X + \vartheta).$$

The cosine slot is $\vartheta = \pi/2$. Since the phase is transported and the leading velocity is multiplication by m_0 ,

$$(119) \quad D\mathcal{S}_{\sigma, t}[\varphi; \ell, \vartheta] + u^{[0]}[\mathcal{S}_{\sigma, t}[\varphi; \ell, \vartheta]] \cdot \nabla P = 0.$$

Oddness fixes the origin. The amplitude equation is $\dot{a}/a = -c_P(0, t)$; hence

$$(120) \quad H_{\sigma, t}[\varphi](0) = \frac{a(t)}{a(\sigma)} \varphi(0).$$

The normalization below factors out the exact origin growth $a(t)/a(\sigma)$. Define

$$(121) \quad \tilde{H}_{\sigma, t}[\varphi] := \frac{a(\sigma)}{a(t)} H_{\sigma, t}[\varphi].$$

It solves

$$(D + c_P - c_P(0, t)) \tilde{H}_{\sigma, t}[\varphi] = 0, \quad \tilde{H}_{\sigma, \sigma}[\varphi] = \varphi.$$

Put

$$\Omega_t := \Phi(t; s, B_R) \Subset B_r.$$

Lemma 5.2 (Normalized weighted-propagator estimates). *Fix $T, S \in \mathbb{N}_0$ and $\mathcal{A} > 0$, and assume the cutoff-parent estimates of Lemma 5.1 through ordinary time orders $0 \leq m \leq (T-1)_+$ and spatial order $\max\{S+T, 2\}$. Assume $\Lambda \geq \Lambda_{\text{flow}}(b, \beta)$, whose threshold is independent of derivative order, and take the flow and phase from Lemmas 4.5 and 4.6 with $J \geq \max\{S+T+1, 3\}$. Thus (79)–(80) and (87)–(88) hold through every order used below. Let $\varphi \in C^\infty([s, 1] \times \mathbb{T}^2)$, with $\text{supp } \varphi(t) \Subset \Omega_t$ for every t , and assume*

$$(122) \quad \|D^m \varphi\|_{C^k(B_r)} \leq C_{m, k} L^{C_{m, k}} \mathcal{A} \lambda^{2k\beta} \quad (0 \leq m \leq T, \ m + k \leq S + T).$$

Then, for $s \leq \sigma \leq t \leq 1$,

$$(123) \quad \left\| D_t^p \partial_\sigma^n \tilde{H}_{\sigma, t}[\varphi(\sigma)] \right\|_{C^j(B_r)} \leq C_{p, n, j} L^{C_{p, n, j}} \mathcal{A} \lambda^{2j\beta},$$

whenever $p + n \leq T$ and $p + n + j \leq S + T$. Here D_t is the material derivative in the output variables (x, t) . For a fixed admissible triple (p, n, j) , the estimate uses datum bounds (122) only for $0 \leq m \leq n$ and $0 \leq k \leq j$. At positive total endpoint order $p + n$, it uses ordinary parent time derivatives only through order $p + n - 1$; at endpoint order zero, only order-zero parent coefficients enter. The propagated coefficient has support compactly contained in Ω_t . The conclusion constants may depend on the finite collection of datum constants that occurs in the stated ranges.

Corollary 5.3 (Ordinary-time oscillatory estimates). *Under the hypotheses of Lemma 5.2, one has*

$$(124) \quad \|\partial_t^p \varphi\|_{C^j(B_r)} \leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^{2j\beta},$$

for $0 \leq p \leq T$ and $0 \leq j \leq S$. Suppose in addition that the phase hypotheses of Lemma 3.2 and, taking the coefficient there to be φ , the coefficient hypotheses of Corollary 3.3 hold with output orders T, S . Then, for the same ranges, $\mathfrak{t} \in \{\sin, \cos\}$, $\vartheta \in \mathbb{R}$, and $\ell \in \mathbb{Z}$, $1 \leq \ell \leq 2^{M+1}$,

$$(125) \quad \|\partial_t^p \partial_x^\kappa [\varphi \mathfrak{t}(\lambda \ell X + \vartheta)]\|_{C^0} \leq C_{p,j} L^{C_{p,j}} \mathcal{A} \lambda^{j+p(1-2\beta)}, \quad |\kappa| = j.$$

Lemma 5.4 (Weighted Duhamel estimate). *Fix $T, S \in \mathbb{N}_0$ and $\mathcal{A} > 0$. Assume the cutoff-parent estimates of Lemma 5.1 through ordinary time order $(T-1)_+$ and spatial order $\max\{S+T, 2\}$. Assume $\Lambda \geq \Lambda_{\text{flow}}(b, \beta)$, independently of T, S , and take the flow and phase in Lemmas 4.5 and 4.6 with $J \geq \max\{S+T+1, 3\}$. Let $\varphi \in C^\infty([s, 1] \times \mathbb{T}^2)$, with $\text{supp } \varphi(t) \Subset \Omega_t$ for every t , and assume that its pulled-back supports are uniformly compact:*

$$(126) \quad K_\varphi := \overline{\bigcup_{\sigma \in [s, 1]} \Phi(s; \sigma, \text{supp } \varphi(\sigma))} \Subset B_R.$$

Assume also

$$\begin{aligned} \|\varphi\|_{C^k(B_r)} &\leq C_k L^{C_k} \mathcal{A} \lambda^{2k\beta} & (0 \leq k \leq S+T), \\ \|D^m \varphi\|_{C^k(B_r)} &\leq C_{m,k} L^{C_{m,k}} \mathcal{A} \lambda^{2k\beta} & (1 \leq m \leq T-1, m+k \leq S+T-1). \end{aligned}$$

The second range is empty when $T = 0$ or $T = 1$. Set

$$h(\sigma) = a(\sigma) \varphi(\sigma), \quad g(t) = \int_s^t H_{\sigma,t}[h(\sigma)] d\sigma.$$

Then

$$(127) \quad \max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p g(t)\|_{C^j(B_r)} \leq C_{p,j} a(t) L^{C_{p,j}} \mathcal{A} \lambda^{2j\beta},$$

for $0 \leq p \leq T$ and $0 \leq j \leq S$. For a fixed output pair (p, j) , only the undifferentiated Duhamel source φ through spatial order $j+p$ and the Duhamel-source derivative pairs

$$(m, j+p-1-m), \quad 0 \leq m \leq p-1,$$

enter. In particular, for $p \geq 1$, no order- p material or ordinary time derivative of φ is assumed; for $p = 0$, only the undifferentiated source is used. The function $g(t)$ has support compactly contained in Ω_t .

5.2. The finite correction cascade. Products of corrections with the common phase X generate finitely many harmonics of λX . We encode their coefficients in arrays so that the next Duhamel correction can cancel the nonzero harmonics, while the zero harmonic and the exact velocity remainders remain in the perturbation residual.

For $H \geq 0$, a fixed phase-harmonic coefficient array is

$$\mathbf{g} = (g_0, (g_h^s, g_h^c)_{1 \leq h \leq H}),$$

with evaluation

$$(128) \quad \text{Ev}_{\lambda, X} \mathbf{g} := g_0 + \sum_{h=1}^H (g_h^s \sin(\lambda h X) + g_h^c \cos(\lambda h X)).$$

This is a presentation rather than an intrinsic decomposition: evaluation need not be injective. When X is odd, we call the array odd if g_0 and every cosine coefficient g_h^c are odd while every sine coefficient g_h^s is even, componentwise for vector-valued arrays. A band with a fixed phase shift is first put into this convention by the angle-addition identities. Arrays of different lengths are identified by appending zero coefficients.

Their product $\mathbf{g} \star \mathbf{h}$ is defined formally: multiply the displayed sine and cosine slots, apply the product-to-sum identities, and collect coefficients with the same nonnegative harmonic index. Thus

$$\text{Ev}_{\lambda,X}(\mathbf{g} \star \mathbf{h}) = \text{Ev}_{\lambda,X} \mathbf{g} \text{Ev}_{\lambda,X} \mathbf{h}.$$

For each coordinate d , define formal spatial differentiation by

$$\begin{aligned} (\partial_d^{\text{arr}} \mathbf{g})_0 &= \partial_d g_0, \\ (\partial_d^{\text{arr}} \mathbf{g})_\ell^s &= \partial_d g_\ell^s - \lambda \ell (\partial_d X) g_\ell^c, \\ (\partial_d^{\text{arr}} \mathbf{g})_\ell^c &= \partial_d g_\ell^c + \lambda \ell (\partial_d X) g_\ell^s. \end{aligned}$$

Then

$$\text{Ev}_{\lambda,X}(\partial_d^{\text{arr}} \mathbf{g}) = \partial_d \text{Ev}_{\lambda,X} \mathbf{g}.$$

All array operations act on the chosen presentations, not on equivalence classes modulo evaluation. We write $\nabla \mathbf{g} = (\partial_1^{\text{arr}} \mathbf{g}, \partial_2^{\text{arr}} \mathbf{g})$. If $\mathbf{a}$ and $\mathbf{b}$ are vector valued, then $\mathbf{a} \cdot \mathbf{b}$ denotes the scalar array obtained by applying $\star$ componentwise and summing.

Define operations on arrays by

$$(129) \quad \Pi_0 \mathbf{g} = (g_0, (0, 0)_{1 \leq \ell \leq H}), \quad \Pi_{>0} \mathbf{g} = \mathbf{g} - \Pi_0 \mathbf{g}.$$

These operations are not defined on arbitrary functions. In particular, $\text{Ev}_{\lambda,X}(\Pi_0 \mathbf{g}) = g_0$ is a spatially varying coefficient, not a Fourier zero mode. When $u^{[M]}$ is applied band by band, we retain the vector-valued array produced by the formulas of Proposition 3.4 and denote it by $\mathbf{u}^{[M]}[\mathbf{g}]$.

Fix an even $f \in C_c^\infty(B_1)$, equal to one on $B_{1/2}$, and a nondecreasing $W \in C^\infty(\mathbb{R})$ such that $W = 0$ on $(-\infty, 0]$ and $W = 1$ on $[1, \infty)$. Set

$$(130) \quad w(t) := W(L^8(t-s)), \quad \varphi_s(x) := a(s)\lambda^{\beta-1}f(\lambda^{2\beta}x).$$

The principal correction and its activation error are

$$(131) \quad \zeta_1(t) := w(t)\mathcal{S}_{s,t}[\varphi_s; 1, 0], \quad \mathcal{A}_0(t) := w'(t)\mathcal{S}_{s,t}[\varphi_s; 1, 0].$$

They are smooth after extension by zero to $[0, s]$. Equation (120) gives $H_{s,t}[\varphi_s](0) = a(t)\lambda^{\beta-1}$. Hence the coefficient of the activated correction ζ_1 at the origin is exactly $w(t)a(t)\lambda^{\beta-1}$.

Suppose $\zeta_1, \dots, \zeta_j$ have been defined. Let $\mathbf{z}_j$ be the fixed coefficient array whose evaluation is

$$(132) \quad \zeta_j = \text{Ev}_{\lambda,X} \mathbf{z}_j = \sum_{\ell=1}^{2^j} (g_{\ell,j}^s \sin(\lambda \ell X) + g_{\ell,j}^c \cos(\lambda \ell X)).$$

For $j = 1$, the $\ell = 2$ slot is zero by the padding convention above. Put

$$\mathbf{T}_j := \sum_{i=1}^j \mathbf{z}_i, \quad \Theta_j := \text{Ev}_{\lambda,X} \mathbf{T}_j, \quad \mathbf{T}_0 = 0, \quad \Theta_0 = 0.$$

Let $u^{[M]}$ be the truncated periodic velocity expansion in Proposition 3.4. Define the quadratic-increment array and its evaluation by

$$(133) \quad \mathbf{N}_j := \mathbf{u}^{[M]}[\mathbf{z}_j] \cdot \nabla \mathbf{T}_j + \mathbf{u}^{[M]}[\mathbf{T}_{j-1}] \cdot \nabla \mathbf{z}_j, \quad \mathcal{N}_j := \text{Ev}_{\lambda,X} \mathbf{N}_j.$$

Since $u^{[0]}[z] = m_0(t)z$, define its bandwise coefficient array on $\mathbf{z}_j$ by

$$\mathbf{u}^{[0]}[\mathbf{z}_j] := \left(0, (m_0(t)g_{\ell,j}^s, m_0(t)g_{\ell,j}^c)_{1 \leq \ell \leq 2^j}\right).$$

Regard ∇P as a vector-valued array with only its zero-harmonic coefficient, and set

$$\mathbf{R}_j^P := (\mathbf{u}^{[M]}[\mathbf{z}_j] - \mathbf{u}^{[0]}[\mathbf{z}_j]) \cdot \nabla P.$$

Define the structured-defect array and function by

$$(134) \quad \mathbf{Q}_j := \Pi_{>0} \mathbf{N}_j + \mathbf{R}_j^P, \quad \mathcal{Q}_j := \text{Ev}_{\lambda,X} \mathbf{Q}_j.$$

Thus $\mathcal{Q}_j$ is the correctable nonzero-harmonic defect. The part deposited directly into the perturbation residual is

$$\begin{aligned}
 \mathcal{E}_j &:= \text{Ev}_{\lambda, X}(\Pi_0 \mathbf{N}_j) \\
 &\quad + (u_{\mathbb{T}} - u^{[M]})(\zeta_j) \cdot \nabla P \\
 &\quad + (u_{\mathbb{T}} - u^{[M]})(\zeta_j) \cdot \nabla \Theta_j \\
 &\quad + (u_{\mathbb{T}} - u^{[M]})(\Theta_{j-1}) \cdot \nabla \zeta_j.
 \end{aligned}
 \tag{135}$$

The same field P supplies both the transport velocity and the gradient in (119).

The array $\mathbf{Q}_j$ has nonzero harmonics at most 2^{j+1} ; write

$$\mathbf{Q}_j = (0, (h_{\ell, j}^s, h_{\ell, j}^c)_{1 \leq \ell \leq 2^{j+1}}).$$

For $1 \leq j < M$, define

$$\zeta_{j+1}(t) := - \sum_{\ell=1}^{2^{j+1}} \int_s^t (\mathcal{S}_{\sigma, t}[h_{\ell, j}^s(\sigma); \ell, 0] + \mathcal{S}_{\sigma, t}[h_{\ell, j}^c(\sigma); \ell, \pi/2]) d\sigma.
 \tag{136}$$

Apply $D + u^{[0]}[\cdot] \cdot \nabla P$ to each integral. Identity (119) annihilates the integrand, while the upper endpoint is respectively $h_{\ell, j}^s \sin(\lambda \ell X)$ or $h_{\ell, j}^c \cos(\lambda \ell X)$. Summing the fixed array coefficients therefore gives

$$D\zeta_{j+1} + u^{[0]}[\zeta_{j+1}] \cdot \nabla P = -\mathcal{Q}_j.
 \tag{137}$$

This is an explicit recursion: $\mathcal{Q}_j$ depends only on $\zeta_1, \dots, \zeta_j$, which are known before ζ_{j+1} is defined. It is also triangular after differentiation. For an order- p , C^k output, the weighted Duhamel estimate uses the undifferentiated defect through C^{k+p} and positive defect time orders only through $p-1$; the exact pairs are recorded below in Lemma 5.8.

Lemma 5.5 (Parity of the array operations). *Suppose that X and P are odd. Applying Proposition 3.4 harmonic by harmonic to an odd array produces an odd vector-valued array. The grouped exact velocity remainder is odd. Moreover, the propagator $H_{\sigma, t}$ preserves parity, and, when initialized by an odd array, the recursion (131)–(136) produces odd $\zeta_j, \mathcal{Q}_j, \mathcal{E}_j$.*

Proof. For a sine input, the coefficient recursion in the proof of Proposition 3.4 gives

$$C_{\gamma}^s(-x) = (-1)^{|\gamma|} C_{\gamma}^s(x), \quad C_{\gamma}^c(-x) = (-1)^{|\gamma|+1} C_{\gamma}^c(x);$$

the cosine input has the reversed parities. This follows inductively because $\partial_d X$ is even. The vanishing identities (20) then leave even sine coefficients and odd cosine coefficients in the velocity array. The Fourier multiplier symbol of the exact velocity is even, so the exact velocity and its grouped remainder preserve oddness.

The flow of the odd field $u_{\mathbb{T}}(P)$ is odd, while c_P is even; hence (117) preserves parity. Finally, products with gradients, the harmonic projections, and the Duhamel step preserve the stated parities. Induction in j proves the last claim. $\square$

Lemma 5.6 (Exact cascade identity). *For $1 \leq J \leq M$,*

$$\begin{aligned}
 &\partial_t \Theta_J + u_{\mathbb{T}}(P) \cdot \nabla \Theta_J + u_{\mathbb{T}}(\Theta_J) \cdot \nabla P + u_{\mathbb{T}}(\Theta_J) \cdot \nabla \Theta_J \\
 &= \mathcal{A}_0 + \sum_{j=1}^J \mathcal{E}_j + \mathcal{Q}_J.
 \end{aligned}
 \tag{138}$$

Proof. For $J = 1$, one has $\Theta_1 = \zeta_1$, and (119) together with (131) gives

$$D\zeta_1 + u^{[0]}[\zeta_1] \cdot \nabla P = \mathcal{A}_0.$$

The remaining parent interaction splits exactly as

$$(u_{\mathbb{T}} - u^{[0]})[\zeta_1] \cdot \nabla P = (u^{[M]} - u^{[0]})[\zeta_1] \cdot \nabla P + (u_{\mathbb{T}} - u^{[M]})[\zeta_1] \cdot \nabla P.$$

Since $\mathbf{T}_0 = 0$, the finite part of $u_{\mathbb{T}}(\zeta_1) \cdot \nabla \zeta_1$ is $\mathcal{N}_1$, and its grouped exact remainder is the third term in (135). Splitting $\mathbf{N}_1$ by $\Pi_{>0} + \Pi_0$ and using (134)–(135) gives

$$\mathcal{A}_0 + \mathcal{Q}_1 + \mathcal{E}_1,$$

which proves the base case.

Assume the identity at level $J < M$. The exact nonlinear increment after adding ζ_{J+1} is

$$\partial_t \zeta_{J+1} + u_{\mathbb{T}}(P) \cdot \nabla \zeta_{J+1} + u_{\mathbb{T}}(\zeta_{J+1}) \cdot \nabla P + u_{\mathbb{T}}(\zeta_{J+1}) \cdot \nabla \Theta_{J+1} + u_{\mathbb{T}}(\Theta_J) \cdot \nabla \zeta_{J+1}.$$

By (137), the first two terms split as

$$\begin{aligned} D\zeta_{J+1} + u_{\mathbb{T}}(\zeta_{J+1}) \cdot \nabla P \\ = -\mathcal{Q}_J + (u^{[M]} - u^{[0]})[\zeta_{J+1}] \cdot \nabla P + (u_{\mathbb{T}} - u^{[M]})[\zeta_{J+1}] \cdot \nabla P. \end{aligned}$$

The last two terms in the increment are

$$\begin{aligned} \mathcal{N}_{J+1} + (u_{\mathbb{T}} - u^{[M]})[\zeta_{J+1}] \cdot \nabla \Theta_{J+1} \\ + (u_{\mathbb{T}} - u^{[M]})[\Theta_J] \cdot \nabla \zeta_{J+1}. \end{aligned}$$

The positive-harmonic part of $\mathcal{N}_{J+1}$, together with the nonleading finite parent interaction, is $\mathcal{Q}_{J+1}$. Its zero-harmonic part and the three grouped exact remainders are $\mathcal{E}_{J+1}$. Thus the increment is exactly

$$-\mathcal{Q}_J + \mathcal{Q}_{J+1} + \mathcal{E}_{J+1}.$$

It cancels the terminal $+\mathcal{Q}_J$ in the induction hypothesis and proves the claim. In particular, the two ordered quadratic cross terms are both contained in $\mathcal{N}_{J+1}$. $\square$

Define the new perturbation and its exact residual by

$$(139) \quad \theta_{q+1} := \Theta_M, \quad E_{q+1}^{\text{pert}} := \mathcal{A}_0 + \sum_{j=1}^M \mathcal{E}_j + \mathcal{Q}_M.$$

Equivalently, Lemma 5.6 states the intrinsic identity

$$(140) \quad \begin{aligned} E_{q+1}^{\text{pert}} &= \partial_t \theta_{q+1} + u_{\mathbb{T}}(P_{q+1}) \cdot \nabla \theta_{q+1} + u_{\mathbb{T}}(\theta_{q+1}) \cdot \nabla P_{q+1} \\ &\quad + u_{\mathbb{T}}(\theta_{q+1}) \cdot \nabla \theta_{q+1}. \end{aligned}$$

Both θ_{q+1} and E_{q+1}^{pert} vanish to infinite order in time at $t = s_{q+1}$. Indeed, since W vanishes to infinite order at zero, we have $|\zeta_1(t)| \leq C_N(t-s)^N$ for every N , and (136) gives inductively

$$|\zeta_{j+1}(t)| \leq C_N(t-s)^{N+1}.$$

The same argument after differentiation, followed by (139), proves the assertion.

5.3. Descending derivative orders and residual estimates. Appendix A proves the all-order parent and weighted propagator estimates used in this subsection. Put $\gamma = 1 - 2\beta$. One correction step uses $M + T + 2$ additional coefficient derivatives and one additional phase derivative; the following descending derivative orders reserve that loss. For requested output orders T, K , define

$$(141) \quad \mathcal{K}_{M+1} := K, \quad \mathcal{K}_j := K + (M - j + 1)(M + T + 2) \quad (1 \leq j \leq M).$$

The identity $\mathcal{K}_j = \mathcal{K}_{j+1} + M + T + 2$ supplies the additional coefficient orders required by Proposition 3.4 at the next correction. Lemma 5.1 supplies every cutoff-parent order; Lemma 4.6, via Lemma 4.5, supplies the additional phase derivative. At the passage from ζ_j to $\mathcal{Q}_j$, the exact orders used by that proposition are

$$(142) \quad N_{\text{prof}} = \mathcal{K}_{j+1} + M + T + 2 = \mathcal{K}_j, \quad N_{\text{phase}} = \mathcal{K}_j + 1.$$

Set

$$(143) \quad \Lambda_{\text{cas}}(b, \beta) := \max \{ \Lambda_{\text{seed}}(b, \beta), \Lambda_{\text{geom}}(b, \beta), \Lambda_{\text{flow}}(b, \beta), \Lambda_{\text{pkt}}(b, \beta) \}.$$

This threshold is independent of T, K ; derivative orders affect only the constants in the estimates below.

Lemma 5.7 (Correction estimates). *Assume the adjacent-layer setup in Definition 4.3, fix $T, K \in \mathbb{N}_0$, and let the derivative orders $\mathcal{K}_j$ be defined by (141). Assume $\Lambda \geq \Lambda_{\text{cas}}(b, \beta)$. Take the local parent estimates (114) in the ranges*

$$\begin{aligned} D^p v : \quad & 0 \leq p \leq (T-1)_+, \quad d \in \mathbb{N}_0, \quad p+d \leq \mathcal{K}_1 + T, \\ D^p \nabla P : \quad & 0 \leq p \leq T, \quad d \in \mathbb{N}_0, \quad p+d \leq \mathcal{K}_1 + T. \end{aligned}$$

and take (79)–(88) with $J \geq \mathcal{K}_1 + T + 1$. These inputs are supplied, respectively, by Lemma 5.1 and by Lemmas 4.5–4.6. Consequently (24)–(26) hold for every $1 \leq j \leq M$ with $N_{\text{prof}} = \mathcal{K}_j$ and $N_{\text{phase}} = \mathcal{K}_j + 1$. Let $P, D, X, H_{\sigma,t}$, the seed, and the correction cascade be defined by (107)–(136), and use (103). For $\mathcal{D} \in \{D, \partial_t\}$, $p, k \in \mathbb{N}_0$, $0 \leq p \leq T$, $1 \leq j \leq M$, $1 \leq \ell \leq 2^j$, and $k+p \leq \mathcal{K}_j + T$, the coefficients in (132) satisfy

$$(144) \quad \|\mathcal{D}^p g_{\ell,j}^s\|_{C^k} + \|\mathcal{D}^p g_{\ell,j}^c\|_{C^k} \leq C_{p,k,j} a(t) L^{C_{p,k,j}} \lambda^{-1+(2k+2-j)\beta}.$$

If

$$\mathcal{Q}_j = \sum_{\ell=1}^{2^{j+1}} (h_{\ell,j}^s \sin(\lambda \ell X) + h_{\ell,j}^c \cos(\lambda \ell X)),$$

then

$$(145) \quad \|\mathcal{D}^p h_{\ell,j}^s\|_{C^k} + \|\mathcal{D}^p h_{\ell,j}^c\|_{C^k} \leq C_{p,k,j} a(t) L^{C_{p,k,j}} \lambda^{-1+(2k+1-j)\beta},$$

for $\mathcal{D} \in \{D, \partial_t\}$, $p, k \in \mathbb{N}_0$, $0 \leq p \leq T$, $1 \leq j \leq M$, $1 \leq \ell \leq 2^{j+1}$, and $k+p \leq \mathcal{K}_{j+1} + T$. All presented coefficients are supported in the transported perturbation region. Every correction ζ_j is odd, as are $\mathcal{Q}_j$ and $\mathcal{E}_j$.

Proof. The seed coefficient has size $a(t)\lambda^{-1+\beta+2k\beta}$, which is (144) for $j=1$. The propagator identity shows inductively that every correction coefficient has the form

$$g_{\ell,i} = a \tilde{g}_{\ell,i}.$$

The amplitude equation gives $|\partial_t^m a^{\pm 1}| \leq C_m L^{C_m} a^{\pm 1}$; the same estimate applies under material differentiation because $Da = \dot{a}$. Thus the normalized coefficients satisfy the constant-size hypotheses of Proposition 3.4 and Corollary 3.6, with size $\lambda^{-1+(2-i)\beta}$. After applying those results band by band, we restore one factor a in a linear or background term and two factors in a quadratic term. The no-collapse estimate and $a(1) = 1$ give $a(t)^2 \leq Ca(t)$.

For the induction step, assume that every correction with $1 \leq i \leq j$ has the preceding coefficient size. The corrections share the parent, transported phase, and compact transported coefficient support; their harmonics are at most $2^{j+1} \leq 2^{M+1}$. Every pair in $\mathcal{N}_j$ has indices $i_1 + i_2 \geq j+1$. The common-phase estimate (38) gives

$$-2 + (4 - i_1 - i_2)\beta + 1 - 2\beta + 2k\beta \leq -1 + (2k+1-j)\beta.$$

The background estimate (39) gives another factor $\lambda^{-\beta}$ in the interaction with ∇P . Fix an individual output pair (p, k) with $k+p \leq \mathcal{K}_{j+1} + T$. The localized velocity expansion at this pair is applied with output parameters $(T, K) = (p, k)$, as are Corollaries 3.3 and 3.6. Their constants may depend on (p, k) , but not on λ . This application requires coefficient order

$$k + M + p + 2 \leq \mathcal{K}_{j+1} + T + M + 2 = \mathcal{K}_j$$

and phase order one larger. Both are supplied by the parent and phase hypotheses of the lemma. The preceding exponent calculation therefore proves (145) throughout its diagonal range.

If $j < M$, the Duhamel step constructs the next correction. Fix (p, k) with $k + p \leq \mathcal{K}_{j+1} + T$, and let h be one coefficient of $\mathcal{Q}_j$. Denote the corresponding Duhamel coefficient in ζ_{j+1} by

$$g_{j+1}(t) = - \int_s^t H_{\sigma,t}[h(\sigma)] d\sigma.$$

Write

$$h = a\varphi, \quad \varphi = \frac{h}{a}, \quad \mathbf{a}_j = \lambda^{-1+(1-j)\beta}.$$

The amplitude equation gives $|\partial_t^m(a^{-1})| \leq C_m L^{C_m} a^{-1}$. Here is the support induction used in this step. Put

$$K_0 := \text{supp}(f(\lambda^{2\beta} \cdot)) \Subset B_R, \quad K_t := \Phi(t; s, K_0).$$

The seed coefficient is supported in K_t . Products, spatial derivatives, and the array projections Π_0 and $\Pi_{>0}$ do not enlarge coefficient support, so the local truncated-expansion formulas show that every coefficient of $\mathcal{Q}_j(t)$, including $h(t)$, is supported in K_t . Division by the time-dependent scalar $a(t)$ does not change this support. Consequently

$$\Phi(s; \sigma, \text{supp } \varphi(\sigma)) \subset K_0 \quad (\sigma \in [s, 1]),$$

which is (126). Moreover, flow composition shows that each integrand in the Duhamel formula is supported in

$$\Phi(t; \sigma, K_\sigma) = \Phi(t; s, K_0) = K_t.$$

Thus the new coefficient has the same transported support, completing the support induction. The order-zero defect bound through spatial order $k + p$, and the lower-time defect bounds at

$$(m, k + p - 1 - m), \quad 0 \leq m \leq p - 1,$$

imply precisely the Duhamel-source hypotheses of Lemma 5.4, with its parameters $(T, S, \mathcal{A}) = (p, k, \mathbf{a}_j)$. The undifferentiated pair belongs to the defect range because $k + p \leq \mathcal{K}_{j+1} + T$, and every endpoint pair belongs to it because

$$m + (k + p - 1 - m) = k + p - 1 < \mathcal{K}_{j+1} + T.$$

The local parent estimates from Lemma 5.1, applied through ordinary time order $(p - 1)_+$ and spatial order $k + p$, supply exactly the cutoff-parent hypotheses of Lemma 5.4. Consequently,

$$\max_{\mathcal{D} \in \{D, \partial_t\}} \|\mathcal{D}^p g_{j+1}\|_{C^k} \leq C_{p,k,j} a(t) L^{C_{p,k,j}} \lambda^{-1+(2k+1-j)\beta},$$

which is (144) at correction index $j + 1$. For $p = 0$, this step uses the already formed order-zero defect; for $p \geq 1$, it uses only defect orders at most $p - 1$.

Thus, for $1 \leq j < M$, the time-order-zero proof alternates the coefficient estimate at correction index j , the corresponding structured-defect estimate, and the coefficient estimate at index $j + 1$; after the last coefficient, it forms the final uncanceled structured defect at index M . At each $p \geq 1$, after all lower-time coefficient and defect estimates are known, it first obtains the order- p coefficient estimates from the seed datum prescribed at the fixed initial time s and the preceding Duhamel step, and only then forms the order- p structured-defect estimates by the interaction calculation. Parity and support follow from the odd phase, even weight, harmonic projections, and the transported coefficient support. The exact periodic multiplier preserves oddness; no support property is assigned to its grouped remainder. $\square$

For each correction index j , let $\mathbf{C}_{p,j}$ denote the order- p coefficient estimate (144) in the diagonal range $k + p \leq \mathcal{K}_j + T$, and let $\mathbf{R}_{p,j}$ denote the order- p structured-defect estimate (145) in the range $k + p \leq \mathcal{K}_{j+1} + T$.

Lemma 5.8 (Within-child mixed closure). *Assume precisely the hypotheses and order ranges of Lemma 5.7, including $\Lambda \geq \Lambda_{\text{cas}}(b, \beta)$. Within the single child construction, the estimates close in the order*

$$\mathbf{C}_{0,1}, \mathbf{R}_{0,1}, \dots, \mathbf{C}_{0,M}, \mathbf{R}_{0,M},$$

followed, for each $1 \leq p \leq T$, by

$$\mathbf{C}_{p,1}, \dots, \mathbf{C}_{p,M}, \quad \mathbf{R}_{p,1}, \dots, \mathbf{R}_{p,M}.$$

For $1 \leq j < M$, the order- p , C^k coefficient at index $j+1$ uses $\mathbf{R}_{0,j}$ through C^{k+p} and, if $p \geq 2$, $\mathbf{R}_{m,j}$ through $C^{k+p-1-m}$ for $1 \leq m \leq p-1$. Thus each positive-time coefficient pass uses only defect time orders below p ; the sharp order- p parent estimate enters only afterwards, when the order- p defects and force-increment terms are formed.

Proof outline. For the seed datum at time s , set

$$(146) \quad \widehat{\varphi}_s := \frac{\varphi_s}{a(s)} = \lambda^{\beta-1} f(\lambda^{2\beta} x), \quad \widehat{\varphi}(\sigma, x) := \widehat{\varphi}_s(\Phi(s; \sigma, x)).$$

Then $D\widehat{\varphi} = 0$, its support is compactly contained in Ω_σ , and its pulled-back support is the fixed compact set $\text{supp } \widehat{\varphi}_s \Subset B_R$. It therefore supplies $\mathbf{C}_{0,1}$ and every fixed- s endpoint derivative. At order zero, each newly obtained coefficient forms its defect, which is then the Duhamel datum for the next coefficient.

For $p \geq 1$, assume all lower time orders. The precise source pairs in Lemma 5.4 are

$$(0, k+p), \quad (m, k+p-1-m) \quad (1 \leq m \leq p-1).$$

They are available from the order-zero enlarged spatial range and the strictly lower time orders. Hence every $\mathbf{C}_{p,j}$ is proved before any $\mathbf{R}_{p,j}$. The interaction estimates then give all $\mathbf{R}_{p,j}$ and the residual terms. The complete derivative-range verification is given in Appendix A.

Thus the input chain is explicit: Lemma 5.1 supplies all cutoff-parent spatial orders, Lemmas 4.5 and 4.6 supply the phase orders, and $K_q \geq d(\beta)$ is used only for the final unsmoothed residual and commutator estimates. For the next layer, the logical order is the order-zero cascade, the origin geometry of θ_{q+1} , and only then the crossing and thresholds for level $q+2$.

Lemma 5.9 (Packet residual estimate). *Assume the adjacent-layer setup in Definition 4.3 and $\beta \leq 1/40$. Set*

$$m_{q+1} := \left\lfloor \frac{1}{4\beta} \right\rfloor - 10.$$

For each $m, k \in \mathbb{N}_0$ with $m+k \leq m_{q+1}$, assume the hypotheses of Proposition 3.4 with output orders $T = m$, $K = k+1$; the extra spatial order is reserved for the gradients in (135). Use Lemma 5.7 with

$$T = m, \quad K = k + M + 3.$$

These additional $M+2$ orders supply the coefficient range needed when (125) is applied to $\mathcal{Q}_M$. The cutoff-parent and phase inputs for this enlarged cascade range come from Lemma 5.1 and Lemmas 4.5–4.6; only the final grouped remainder uses the finite unsmoothed range authorized by $K_q \geq d(\beta)$. The exact velocity remainder in (34) is kept as one grouped global remainder and is differentiated only by ordinary time derivatives. There is a finite threshold

$$\Lambda_{\text{res}} = \Lambda_{\text{res}}(b, \beta) \geq \max\{\Lambda_{\text{cas}}(b, \beta), \Lambda_{\text{pkt}}(b, \beta)\}$$

obtained by taking the maximum over this finite set of output pairs, such that, whenever $\Lambda \geq \Lambda_{\text{res}}$,

$$(147) \quad \|\partial_t^m E_{q+1}^{\text{pert}}\|_{C^k} \leq \frac{1}{2} \lambda^{-1/4} \quad (m+k \leq m_{q+1}).$$

Proof. Use the named decomposition (139). Before powers of L and finite constants are absorbed, its three parts satisfy, for $1 \leq j \leq M$,

$$\begin{aligned}
\|\partial_t^m \mathcal{A}_0\|_{C^k} &\leq C_\beta L^C \lambda^{k+m\gamma-1+\beta-2/\beta}, \\
\|\partial_t^m \mathcal{E}_j\|_{C^k} &\leq C_{\beta,j} L^C \left(\lambda^{-1+(2k+1-j)\beta} \right. \\
&\quad \left. + \lambda^{m+2(k+1)(1-\beta)-M\beta} + \lambda^{m+2(k+1)(1-\beta)+1-M\beta} \right), \\
(148) \quad \|\partial_t^m \mathcal{Q}_M\|_{C^k} &\leq C_\beta L^C \lambda^{k-1+(1-M)\beta+m\gamma}.
\end{aligned}$$

On $\text{supp } w'$, one has $0 \leq t - s \leq L^{-8}$. Equation (101) and (100) give

$$\left| \frac{\dot{a}}{a} \right| \leq C\Lambda^b, \quad \frac{a(t)}{a(s)} \leq \exp(C\Lambda^b L^{-8}) \leq C_\beta.$$

Thus $a(t) \leq C_\beta \lambda^{-2/\beta}$ on $\text{supp } w'$. The first estimate in (148) follows from this bound and (125); derivatives of w cost only powers of L . For $\mathcal{E}_j$, the first displayed summand is the zero-harmonic coefficient $\text{Ev}_{\lambda,X}(\Pi_0 \mathbf{N}_j)$, estimated by (38). In the order of (135), the second displayed summand bounds $(u_{\mathbb{T}} - u^{[M]})[\zeta_j] \cdot \nabla P$, while the third bounds the analogous terms for the packets $\zeta_j, \Theta_j, \Theta_{j-1}$ multiplied by $\nabla \Theta_j$ or $\nabla \zeta_j$. Leibniz' rule and (34) give, respectively,

$$CL^C \lambda^{m+2(k+1)(1-\beta)-M\beta}, \quad CL^C \lambda^{m+2(k+1)(1-\beta)-1+2\beta-M\beta},$$

the latter is bounded by the displayed exponent with $+1$. These estimates use (34), with output orders $T = m$, $K = k + 1$, and the coefficient bounds from Lemma 5.7; the normalization by $a(t)$ in its proof matches the constant-size packet hypotheses. The grouped remainder is global and is differentiated only by ordinary time derivatives. Finally, the estimate for $\mathcal{Q}_M$ is (145) at $j = M$, followed by (125) to include derivatives falling on the oscillatory factor. Its coefficient demand is $p' + k' \leq m + k + M + 3$, which is exactly the terminal range supplied by the additional orders in Lemma 5.9.

For $m, k \in \mathbb{N}_0$ with $m + k \leq m_{q+1}$, the exponents in the three named estimates are bounded above as follows:

$$\begin{aligned}
(149) \quad \mathcal{A}_0 : & \quad -\frac{7}{4\beta} - 11 + \beta, \\
\mathcal{E}_j : & \quad -\frac{1}{2} - 20\beta, \quad -\frac{1}{2\beta} - 18 - 2\beta, \quad -\frac{1}{2\beta} - 17 - 2\beta, \\
\mathcal{Q}_M : & \quad -\frac{3}{4\beta} - 11 + \beta.
\end{aligned}$$

Indeed, $M\beta \geq \beta^{-1}$, $m + 2k \leq 2m_{q+1}$, and $j \geq 1$ in the $\mathcal{E}_j$ zero-harmonic term. The number of corrections and harmonics depends only on β . A finite threshold therefore absorbs all logarithmic factors and proves (147). $\square$

5.4. The cutoff commutator and exact stage recursion. Recall the cutoff commutator $\mathcal{C}_\mu$ from (14). It is odd and has zero mean when h is odd. Since $u_{\mathbb{T}}$ preserves Fourier support,

$$(150) \quad \text{supp } \widehat{\mathcal{C}_\mu(h)} \subset B_{4\mu}.$$

If $\text{supp } \widehat{h} \subset B_{2\nu}$ and $\mu \geq 4\nu$, then

$$S_\mu h = h, \quad S_\mu(u_{\mathbb{T}}(h) \cdot \nabla h) = u_{\mathbb{T}}(h) \cdot \nabla h,$$

and hence

$$(151) \quad \mathcal{C}_\mu(h) = 0.$$

Lemma 5.10 (Mixed cutoff commutator). *Let $\mu \geq 1$ and $h \in C^\infty([0, 1] \times \mathbb{T}^2)$. For integers $m, k \geq 0$ and $s_0 \geq 1$,*

$$(152) \quad \|\partial_t^m \mathcal{C}_\mu(h)\|_{C^k} \leq C_{m,k,s_0} \mu^{-s_0} \left(\max_{0 \leq p \leq m} \|\partial_t^p h\|_{C^{k+s_0+4}} \right)^2.$$

Proof. Put $h_{>} = (I - S_\mu)h$. Exact expansion gives

$$\mathcal{C}_\mu(h) = (I - S_\mu)(u_{\mathbb{T}}(h) \cdot \nabla h) - u_{\mathbb{T}}(h_{>}) \cdot \nabla h - u_{\mathbb{T}}(h) \cdot \nabla h_{>} + u_{\mathbb{T}}(h_{>}) \cdot \nabla h_{>}.$$

Set

$$\mathfrak{H} := \max_{0 \leq p \leq m} \|\partial_t^p h\|_{C^{k+s_0+4}}.$$

After applying ∂_t^m and Leibniz' rule, the periodic Schauder and tail estimates give, for every resulting summand,

$$\|(I - S_\mu)(u_{\mathbb{T}}(f) \cdot \nabla g)\|_{C^k} + \|u_{\mathbb{T}}((I - S_\mu)f) \cdot \nabla g\|_{C^k} \leq C \mu^{-s_0} \mathfrak{H}^2,$$

where f and g are time derivatives of h . The same bound holds when the tail falls on the differentiated factor; a term with two tails is smaller because $\mu \geq 1$. The explicit gradient uses one derivative, and (197) uses three more beyond the target order. This proves (152). $\square$

Corollary 5.11 (Stage commutator estimate). *Assume the adjacent-layer setup in Definition 4.3, and take $h = \rho^{(q)}$. Assume $\beta \leq 1/40$, and set*

$$m_{q+1} := \left\lfloor \frac{1}{4\beta} \right\rfloor - 10.$$

There is a finite threshold

$$\Lambda_{\text{com}} = \Lambda_{\text{com}}(b, \beta)$$

such that, whenever $\Lambda \geq \Lambda_{\text{com}}$ and $m, k \in \mathbb{N}_0$ satisfy $m + k \leq m_{q+1}$,

$$(153) \quad \|\partial_t^m \mathcal{C}_\mu(\rho^{(q)})\|_{C^k} \leq \frac{1}{2} \lambda^{-1/4}.$$

Proof. Choose

$$s_0 = k + 2m + 4 + \left\lceil \frac{2}{\beta} \right\rceil.$$

The largest mixed parent order required is

$$k + s_0 + 4 + 2m = 2k + 4m + 8 + \left\lceil \frac{2}{\beta} \right\rceil \leq \frac{3}{\beta} - 31 < \frac{4}{\beta^2},$$

where $m + k \leq m_{q+1}$ and $\beta \leq 1/40$ were used. The stage bound (54), together with $K_q \geq d(\beta)$, therefore applies. Its crude form, together with $\lambda_q \leq \lambda^{\beta/8}$, bounds the square in (152) by

$$4\lambda^{\frac{\beta}{4}(k+s_0+4+2m)}.$$

Since $\mu^{-s_0} = \lambda^{-\beta s_0/2}$, the resulting exponent is

$$\frac{\beta}{4}(k + 2m + 4 - s_0) = -\frac{\beta}{4} \left\lceil \frac{2}{\beta} \right\rceil \leq -\frac{1}{2}.$$

Thus increasing the threshold absorbs only $4C_{m,k,s_0}$ and proves (153). The comparison $3/\beta - 31 < 4/\beta^2$ holds for $\beta \leq 1/8$; the stronger assumption $\beta \leq 1/40$ is used here only to ensure $m_{q+1} \geq 0$. $\square$

Set

$$(154) \quad G_{q+1} := \mathcal{C}_{\mu_{q+1}}(\rho^{(q)}) + E_{q+1}^{\text{pert}}.$$

The new stage is

$$(155) \quad \boxed{\rho^{(q+1)} := P_{q+1} + \theta_{q+1} = S_{\mu_{q+1}} \rho^{(q)} + \theta_{q+1}.}$$

Its force is defined by

$$(156) \quad \boxed{F^{(q+1)} := S_{\mu_{q+1}} F^{(q)} + G_{q+1}.}$$

Indeed, the stage- q equation and (14) imply

$$\partial_t P_{q+1} + u_{\mathbb{T}}(P_{q+1}) \cdot \nabla P_{q+1} = S_{\mu_{q+1}} F^{(q)} + \mathcal{C}_{\mu_{q+1}}(\rho^{(q)}).$$

Adding (140) proves

$$(157) \quad \partial_t \rho^{(q+1)} + u_{\mathbb{T}}(\rho^{(q+1)}) \cdot \nabla \rho^{(q+1)} = F^{(q+1)},$$

with every cross term accounted for.

5.5. The one-step result.

Lemma 5.12 (Activation before the terminal interval and terminal normalization). *Assume the adjacent-layer setup in Definition 4.3. Let s_{q+1} and a_{q+1} be supplied by Lemmas 4.8 and 4.9. Set $p_{\text{term}} = e(\pi/2 + \lambda^{-\beta/8})$, and let X_{q+1} be defined by Lemma 4.6. There is a finite threshold*

$$\Lambda_{\text{activation}} = \Lambda_{\text{activation}}(b, \beta)$$

such that, whenever $\Lambda \geq \Lambda_{\text{activation}}$,

$$(158) \quad s_{q+1} + L^{-8} < t_{q+1} = 1 - \tilde{C} \lambda^{-3\beta/4}.$$

Moreover, uniformly for $t \in I_{q+1}$,

$$(159) \quad |a_{q+1}(t) - 1| + |\nabla X_{q+1}(0, t) - p_{\text{term}}| \leq CL^C \lambda^{-3\beta/4} = o(1).$$

Proof. Write $s := s_{q+1}$, $a := a_{q+1}$, and $X := X_{q+1}$. The lower bound in (97) gives

$$1 - s \geq \frac{1}{6C_0} \Lambda^{-3b/4}.$$

Equation (66) gives

$$L = k_- \beta \Lambda^{b/8}, \quad L^{-8} = (k_- \beta)^{-8} \Lambda^{-b}.$$

Consequently $L^{-8} = o(\Lambda^{-3b/4})$, while

$$\lambda^{-3\beta/4} = \exp\left(-\frac{3\beta L}{4}\right) = o(\Lambda^{-3b/4}).$$

This proves (158) above a finite threshold.

On I_{q+1} , Lemma 5.1 gives

$$|c_P(0, t)| + \|D_x v(t)\|_{C^0} \leq CL^C.$$

Since $a(1) = 1$, integration of $\dot{a}/a = -c_P(0, t)$ yields

$$|\log a(t)| \leq CL^C(1 - t) \leq CL^C \lambda^{-3\beta/4}.$$

The phase jet solves

$$\partial_t \nabla X(0, t) = -D_x v(0, t)^\top \nabla X(0, t), \quad \nabla X(0, 1) = p_{\text{term}}.$$

A short-time Grönwall estimate proves (159). $\square$

Proposition 5.13 (One-step extension). *Let an admissible family be given through level q . Choose*

$$\beta_{q+1} \in \left[\frac{\beta_q}{2}, \beta_q\right]$$

and an integer $k_{q+1} \geq d(\beta_{q+1})$. Assume $K_q \geq d(\beta_{q+1})$, and define λ_{q+1} by (63). For the fields defined by (155), let $\mathcal{B}_{q+1}(J)$ denote the property

$$\sup_{t \in [0, 1]} \|\partial_t^m \rho^{(q+1)}(t)\|_{C^j} \leq 2\lambda_{q+1}^{j+2m} \quad \text{for all } j, m \in \mathbb{N}_0 \text{ with } 1 \leq j + 2m \leq J,$$

and define

$$(160) \quad K_{q+1} := \max(\{0\} \cup \{J \in \mathbb{N}_0 : J \leq k_{q+1}, \quad \mathcal{B}_{q+1}(J)\}).$$

There is a finite threshold

$$\lambda_{\text{th}, \mathbb{T}} = \lambda_{\text{th}, \mathbb{T}}(\beta_q, \beta_{q+1}, k_{q+1}; f, \chi, W).$$

Choose it to dominate $\Lambda_{\text{sep}}, \Lambda_{\text{cas}}, \Lambda_{\text{activation}}$, and, when $\beta_{q+1} \leq 1/40$, $\Lambda_{\text{res}}, \Lambda_{\text{com}}$, together with the finitely many absorptions for $j + 2m \leq k_{q+1}$. The first group is independent of derivative order; only the last finite group depends on k_{q+1} . Whenever $\lambda_q \geq \lambda_{\text{th}, \mathbb{T}}$, the fields in (155) and (156) extend the admissible family through level $q + 1$. The activation interval ends before I_{q+1} by (158), and the new perturbation and its exact residual are smooth and vanish to infinite order in time at s_{q+1} . The following additional estimates hold uniformly for $t \in [0, 1]$.

- (i) For every fixed $j, m \geq 0$, the a priori constants $C_{j,m}(\beta_{q+1})$ and $N_{j,m}(\beta_{q+1})$ fixed in Definition 4.1 and (56) give exactly

$$(161) \quad \|\partial_t^m \theta_{q+1}(t)\|_{C^j} \leq C_{j,m}(\beta_{q+1}) L_{q+1}^{N_{j,m}(\beta_{q+1})} \lambda_{q+1}^{j+m(1-2\beta_{q+1})-1+\beta_{q+1}}.$$

These fixed-order estimates require no enlargement of $\lambda_{\text{th}, \mathbb{T}}$. The prescribed family already takes the maximum over the possible parent exponents $\beta_q \in [\beta_{q+1}, 2\beta_{q+1}]$, so its constants depend only on β_{q+1} . For every pair with $j + 2m \leq k_{q+1}$,

$$(162) \quad \|\partial_t^m \theta_{q+1}(t)\|_{C^j} \leq \lambda_{q+1}^{j+2m} \quad (j + 2m \leq k_{q+1}).$$

In particular, for $0 \leq j \leq k_{q+1}$,

$$(163) \quad \|\theta_{q+1}(t)\|_{C^j} \leq \lambda_{q+1}^j, \quad \|\theta_{q+1}(t)\|_{C^0} \leq \lambda_{q+1}^{-1+2\beta_{q+1}}, \quad \|\theta_{q+1}(t)\|_{C^1} \leq \lambda_{q+1}^{2\beta_{q+1}}.$$

- (ii) If $\beta_{q+1} \leq 1/40$, define $m_{q+1} := \lfloor (4\beta_{q+1})^{-1} \rfloor - 10$. The force increment satisfies

$$(164) \quad \|\partial_t^m G_{q+1}(t)\|_{C^k} \leq \lambda_{q+1}^{-1/4} \quad (m + k \leq m_{q+1}).$$

- (iii) One has

$$(165) \quad K_{q+1} \geq \min\{K_q, k_{q+1}\} \geq d(\beta_{q+1}).$$

Proof. We verify the conclusions in the order in which their ingredients enter the construction. The exact cascade identity and the cutoff-commutator algebra give (157). The seed support, the flow inclusion (116), the infinite-order vanishing of the activation cutoff at s_{q+1} , and the parity statements in Lemma 5.7 show that the new fields are smooth, odd, and mean zero, and give the support and infinite-order-vanishing assertions required by Definition 4.1.

The identity $S_\mu P_q = P_q$ leaves the level- q reference-angle law (60) exact. The newest-perturbation cutoff-tail estimate (109) perturbs by only $O(\lambda_q^{-4})$ the parent gradient and velocity-Jacobian geometry entering the child equation. Lemma 4.8 gives s_{q+1} and transversality; Lemma 4.9 and Corollary 4.10 give the no-collapse and seed bounds, including (115). The perturbation-flow bounds come from Lemmas 4.5 and 4.6.

Proposition 3.4 and Lemma 5.7, together with Corollary 5.3, organized by the within-child induction in Lemma 5.8, give (161) with exactly the a priori family. Indeed, at time order zero the cutoff-parent input comes from the constant-free $j \leq 2, m = 0$ stage bound and Bernstein's inequality. At time order $m \geq 1$, the coefficient pass uses parent and defect time orders at most $m - 1$. Higher cutoff-parent spatial orders again follow from Bernstein's inequality, and the flow and phase thresholds are independent of their derivative order. Thus the raw constants $\tilde{C}_{j,m}(b, \beta), \tilde{N}_{j,m}(b, \beta)$ generated at (j, m) use only the lower-time constants allowed in (56) and are bounded by the prescribed $C_{j,m}(\beta), N_{j,m}(\beta)$. No order-dependent constant is absorbed into a frequency gain or

enlarges the all-order threshold. At time order zero this proves (163) after increasing the finite threshold. More generally, if $j + 2m \leq k_{q+1}$, then

$$C_{j,m}(\beta) L^{N_{j,m}(\beta)} \lambda^{j+m(1-2\beta)-1+\beta} \leq \frac{1}{2} \lambda^{j+2m}$$

above one finite threshold, simultaneously for all such pairs. This proves (162) and part (i). Equations (147), proved in Lemma 5.9, and (153) give (164). For these applications, the enlarged cascade orders are supplied by Lemma 5.1 and Lemmas 4.5–4.6; the final grouped-residual orders are supplied by Corollary 4.7, while the commutator's uncut derivatives come directly from (54), as checked in the final subsection of Appendix A. This proves part (ii).

Lemma 5.12 shows that the activation cutoff equals one throughout I_{q+1} . Oddness gives $X_{q+1}(0, t) = 0$, and (120) gives

$$(166) \quad \nabla \zeta_1(0, t) = a_{q+1}(t) \lambda_{q+1}^{\beta_{q+1}} \nabla X_{q+1}(0, t).$$

Set

$$(167) \quad A_{q+1}(t) = B_{q+1}(t) := a_{q+1}(t) \lambda_{q+1}^{\beta_{q+1}} |\nabla X_{q+1}(0, t)|.$$

Writing $\nabla X_{q+1}(0, t) / |\nabla X_{q+1}(0, t)| = e(\alpha_{q+1}(t))$, the leading origin-jet identity gives

$$(168) \quad D_x u^{[0]}[\zeta_1](0, t) = C_0 B_{q+1}(t) \cos \alpha_{q+1}(t) \mathcal{M}(\alpha_{q+1}(t)).$$

The terminal normalization places both principal amplitudes in $[\frac{3}{4} \lambda_{q+1}^{\beta_{q+1}}, \frac{5}{4} \lambda_{q+1}^{\beta_{q+1}}]$ above a finite threshold.

For $j \geq 2$, the correction estimate and one oscillatory-factor derivative give

$$|\nabla \zeta_j(0, t)| + \|D_x u^{[M]}[\zeta_j](0, t)\| \leq C_\beta L^C \lambda^{(2-j)\beta} \leq C_\beta L^C.$$

For the principal correction, every nonleading expansion term gains $\lambda^{-\beta}$:

$$\|D_x(u^{[M]} - u^{[0]})(\zeta_1)(0, t)\| \leq C_\beta L^C \lambda^{-\beta}.$$

For every $1 \leq j \leq M$, the grouped exact velocity remainder satisfies

$$\|D_x(u_{\mathbb{T}} - u^{[M]})(\zeta_j)\|_{C^0} \leq C_\beta L^C \lambda^{3-(j+2)\beta-M\beta} \leq C_\beta L^C \lambda^{-5}.$$

Thus the sum of the higher-correction density jets and all nonleading velocity jets is bounded by $C_\beta L^C$. A final threshold makes this at most the errors required in (58)–(59), proving the complete origin geometry.

Finally, the $j = 1, 2$, $m = 0$ cases of (54) at the preceding stage give

$$\|P_{q+1}\|_{C^1} + \|u_{\mathbb{T}}(P_{q+1})\|_{C^1} \leq C \lambda_q^2.$$

Equation (63) makes this fixed power at most $\lambda_{q+1}^{\beta_{q+1}/8}$ above a finite threshold. This is the cutoff-parent bound in (57).

Parent admissibility and the terminal-time separation give

$$\tilde{C} \lambda_{q+1}^{-3\beta_{q+1}/4} \leq \frac{1}{2} \tilde{C} \lambda_q^{-3\beta_q/4} \leq \frac{1}{2}.$$

Thus (45) propagates to the child, so its terminal window also lies in $[0, 1]$.

Put $J_* = \min\{K_q, k_{q+1}\}$. If $1 \leq j + 2m \leq J_*$, cutoff boundedness and (54) at level q give

$$\|S_\mu \partial_t^m \rho^{(q)}\|_{C^j} \leq 2C_j \Lambda^{j+2m}.$$

There are finitely many such pairs, so frequency separation makes this at most $\frac{1}{2} \lambda^{j+2m}$ simultaneously for all of them. The new-perturbation bound above supplies the other half. Hence (165) follows from (155), proving part (iii). The scale selection, angle law, activation-interval separation, derivative bounds, and origin geometry verified above establish every remaining clause of the admissible-family definition. $\square$

6. GLOBAL ITERATION AND BLOW-UP

Retain the fixed cutoff χ , spatial profile f , and activation profile W from Sections 3 and 5. The initial base oscillation parameter will be chosen so that every perturbation support is contained in the coordinate chart of Proposition 3.4. The all-order mixed estimates used in the iteration are proved in Appendix A.

6.1. Initialization and the recursive exponent choice. Set

$$(169) \quad \beta_0 := \frac{1}{8}, \quad k_0 = K_0 := 257 = d(\beta_0), \quad \alpha_0 := \frac{\pi}{2} + \lambda_0^{-1/64}.$$

For sufficiently large λ_0 , define in the central chart

$$(170) \quad \theta_0(x) := \lambda_0^{-7/8} f(\lambda_0^{1/4} x) \sin(\lambda_0 e(\alpha_0) \cdot x), \quad P_0 := 0, \quad \rho^{(0)} := \theta_0,$$

and extend θ_0 periodically by zero. Put

$$(171) \quad F^{(0)} := u_{\mathbb{T}}(\theta_0) \cdot \nabla \theta_0.$$

Lemma 6.1 (Periodic base layer). *There is a finite threshold $\lambda_*(f, \chi)$ such that, whenever $\lambda_0 \geq \lambda_*(f, \chi)$, the fields in (170)–(171) form an admissible level-zero stage.*

Proof. Increase $\lambda_*(f, \chi)$, if necessary, so that $\lambda_*(f, \chi) \geq e$ and, whenever $\lambda_0 \geq \lambda_*(f, \chi)$,

$$\tilde{C} \lambda_0^{-3/32} \leq 1.$$

Since $\beta_0 = 1/8$, this is (45) and gives $I_0 \subset [0, 1]$. The envelope in (170) is supported strictly inside the coordinate chart. Hence its extension by zero is smooth even though the local phase covector need not belong to the integer lattice. The density is odd and therefore has zero mean. Its support lies in $B_{\lambda_0^{-1/4}}$, and

$$\|\theta_0\|_{C^j} \leq C_{f,j} \lambda_0^{j-7/8} \leq \lambda_0^j \quad (0 \leq j \leq 257),$$

above a finite threshold. The pair is stationary and satisfies the level-zero equation by (171). Moreover, $F^{(0)}$ is odd because $u_{\mathbb{T}}(\theta_0)$ is odd and $\nabla \theta_0$ is even.

Since $f = 1$ near the origin,

$$\nabla \theta_0(0) = \lambda_0^{1/8} e(\alpha_0).$$

On I_0 , define the witnesses in the admissible origin geometry by

$$\begin{aligned} A_0(t) &= B_0(t) = \lambda_0^{1/8}, \\ \alpha_0(t) &\equiv \frac{\pi}{2} + \lambda_0^{-1/64}, \\ E_0^\rho(t) &= 0, \\ E_0^u(t) &= D_x u_{\mathbb{T}}(\theta_0)(0) - C_0 \lambda_0^{1/8} \cos \alpha_0 \mathcal{M}(\alpha_0). \end{aligned}$$

For this application, set $P = 0$, $v := u_{\mathbb{T}}(P)$, and $\mathcal{G} := \nabla P$, so $v = \mathcal{G} = 0$. Apply Proposition 3.4 with this P , the linear phase $e(\alpha_0) \cdot x$, and $\beta_0 = 1/8$; its hypotheses hold with $v = \mathcal{G} = 0$, exactly linear phase on the amplitude support, $\text{supp } f(\lambda_0^{1/4} \cdot) \Subset B_{r_0}$, and $\nu_1 = \lambda_0 \geq e$, by Lemma 3.2 and Corollary 3.3. Its leading Euclidean term gives the shear matrix in (59). The smooth periodic-kernel contribution is absorbed into the grouped remainder after exactly M_0 integrations by parts. The nonleading expansion terms, grouped remainder, and smooth-kernel term contribute entrywise to the Jacobian error at most

$$C_f \lambda_0^{-1/8}, \quad C_f \lambda_0^{-43/8}, \quad C_f \lambda_0^{-391/8},$$

respectively. Thus, above a finite threshold, each of the four components of E_0^u has absolute value at most $\lambda_0^{1/64}$. Profile scaling gives the support condition and the sharp perturbation estimates, while stationarity gives every positive-time mixed bound. The preceding bounds verify the remaining

level-zero clauses of Definition 4.1; in particular, $P_0 = 0$ gives the required previous-stage C^1 estimates and the constant reference-angle law. $\square$

The exponent β_q controls the size of the q -th perturbation and the number of derivative estimates required at that scale. The integer k_q is the largest prescribed weighted mixed-derivative order for that perturbation; K_q is the largest weighted mixed-derivative order verified for the complete stage. At a fixed exponent we raise k_q as the base oscillation parameters grow. Once K_q is large enough to authorize the smaller exponent and the corresponding finite threshold has been reached, we replace β_q by $\beta_q/2$.

Fix a finite threshold function

$$\lambda_{\text{th},\mathbb{T}}(\beta, \beta', k; f, \chi, W), \quad 0 < \frac{\beta}{2} \leq \beta' \leq \beta \leq \frac{1}{8}, \quad k \geq d(\beta'),$$

which is nondecreasing in k and dominates every finite requirement in Proposition 5.13. With parent exponent β and child exponent β' , it dominates

$$\begin{aligned} &\Lambda_{\text{sep}}(\beta), \quad \Lambda_{\text{geom}}(\beta, \beta'), \quad \Lambda_{\text{flow}}(\beta, \beta'), \quad \Lambda_{\perp}(\beta, \beta'), \\ &\Lambda_{\text{gain}}(\beta, \beta'), \quad \Lambda_{\text{seed}}(\beta, \beta'), \quad \Lambda_{\text{pkt}}(\beta, \beta'), \quad \Lambda_{\text{cas}}(\beta, \beta'), \\ &\Lambda_{\text{res}}(\beta, \beta'), \quad \Lambda_{\text{com}}(\beta, \beta'), \quad \Lambda_{\text{activation}}(\beta, \beta'), \end{aligned}$$

whenever the named threshold is used, together with the finite scale, support, profile, and mixed-order absorptions for $j + 2m \leq k$. Each depends only on $(\beta, \beta', k; f, \chi, W)$. The arbitrary fixed-order sharp estimates have order-dependent constants but impose no additional threshold. Increase λ_0 , if necessary, so that

$$\lambda_0 \geq \max \{ \lambda_*(f, \chi), \lambda_{\text{th},\mathbb{T}}(\beta_0, \beta_0, d(\beta_0); f, \chi, W) \}.$$

Suppose that level q has been constructed and write $\beta = \beta_q$. If

$$(172) \quad K_q \geq d(\beta/2) \quad \text{and} \quad \lambda_q \geq \max \{ \lambda_{\text{th},\mathbb{T}}(\beta, \beta/2, d(\beta/2); f, \chi, W), \lambda_{\text{th},\mathbb{T}}(\beta/2, \beta/2, d(\beta/2); f, \chi, W) \},$$

choose

$$(173) \quad \beta_{q+1} := \frac{\beta_q}{2}, \quad k_{q+1} := d(\beta_{q+1}).$$

Otherwise choose

$$(174) \quad \beta_{q+1} := \beta_q, \quad k_{q+1} := \max \{ k \in \mathbb{N} : d(\beta_q) \leq k \leq q + d(\beta_q), \lambda_q \geq \lambda_{\text{th},\mathbb{T}}(\beta_q, \beta_q, k; f, \chi, W) \}.$$

The admissible set in (174) is nonempty. At $q = 0$, the candidate $d(\beta_0)$ is authorized by the base threshold. If level q follows a halving, the second threshold in (172) authorizes $d(\beta_q)$. If it follows a retention step, the previously chosen k_q remains admissible: it lies below the enlarged cap, while

$$\lambda_q \geq \lambda_{q-1} \geq \lambda_{\text{th},\mathbb{T}}(\beta_q, \beta_q, k_q; f, \chi, W).$$

Once β_{q+1} has been fixed, define λ_{q+1} by (63), apply Proposition 5.13, and define K_{q+1} by (160).

Lemma 6.2 (Continuation). *Let the parameter and mixed-derivative-order sequences be generated from (169) by (172)–(174), (63), and (160). The recursion continues for every q . On every infinite consecutive set of indices on which β_q is constant, one has $k_q \rightarrow \infty$.*

Proof. At a halving step, condition (172) authorizes both that step and the first retained step at the new exponent. The nonemptiness verified immediately after (174) handles every retention step. Thus the recursion cannot stop.

Fix $\bar{\beta}$ and a derivative order $k \geq d(\bar{\beta})$. The number $\lambda_{\text{th},\mathbb{T}}(\bar{\beta}, \bar{\beta}, k; f, \chi, W)$ is finite. By (48), $\lambda_{q+1} \geq 4\lambda_q$, so $\lambda_q \rightarrow \infty$; the cap $q + d(\bar{\beta})$ also tends to infinity. Consequently $k_q \geq k$ at all sufficiently late indices of a constant- $\bar{\beta}$ block. $\square$

6.2. Iterated cutoff formulas and parameter growth. The density formula (61) was derived directly from the finite-stage invariant. The same cutoff induction gives, for every $Q \geq 1$,

$$(175) \quad F^{(Q)} = S_{\mu_1} F^{(0)} + \sum_{i=1}^{Q-1} S_{\mu_{i+1}} G_i + G_Q.$$

Indeed, $\mu_j \geq 2\mu_i$ whenever $j > i$, so

$$S_{\mu_j} S_{\mu_i} = S_{\mu_i}.$$

Thus every perturbation and every force increment is truncated exactly once and is then fixed. In particular,

$$\text{supp } \widehat{S_{\mu_{i+1}} \theta_i} \subset B_{2\mu_{i+1}}, \quad \text{supp } \widehat{S_{\mu_{i+1}} G_i} \subset B_{2\mu_{i+1}}.$$

Moreover, (150) and $\mu_{i+1} \geq 4\mu_i$ imply

$$S_{\mu_{i+1}} \mathcal{C}_{\mu_i}(\rho^{(i-1)}) = \mathcal{C}_{\mu_i}(\rho^{(i-1)}).$$

Lemma 6.3 (Decay of the scale exponent). *For the parameter and mixed-derivative-order sequences generated by (172)–(174), (63), and (160), one has*

$$(176) \quad \beta_q \downarrow 0, \quad x_q := \beta_q \log \lambda_q \rightarrow \infty.$$

After increasing the base threshold,

$$(177) \quad x_{q+1} \geq 64x_q, \quad L_q^4 \leq \lambda_q^{\beta_q}, \quad r_{q+1} \leq \lambda_q^{-8}.$$

Consequently $r_q \rightarrow 0$, $\underline{t}_q \uparrow 1$, and the intervals $[\underline{t}_q, \underline{t}_{q+1}]$ cover a terminal neighborhood of one.

Proof. If $\beta_q \not\rightarrow 0$, then, because each step either retains β_q or replaces it by $\beta_q/2$, only finitely many halvings occur. Hence $\beta_q = \bar{\beta} > 0$ for every sufficiently large q . Since $\beta_{q+1} = \beta_q$ at every sufficiently late stage, the halving criterion fails there. Its frequency condition already holds, so its first condition fails and $K_q < d(\bar{\beta}/2)$, hence

$$K_q \leq D_* := \left\lceil \frac{16}{\bar{\beta}^2} \right\rceil,$$

at all sufficiently late stages. Lemma 6.2 gives $k_q > D_* + 1$. Put $n = K_q + 1$. By maximality in (160), there are integers $j, m \geq 0$ with $j + 2m = n$, and a time τ_q , such that

$$(178) \quad \|\partial_t^m \rho^{(q)}(\tau_q)\|_{C^j} > 2\lambda_q^n.$$

Choose q_0 so that $k_i > D_* + 1$ for every $i \geq q_0$. Let C_{head} bound the fixed contribution of indices $i < q_0$, uniformly over the finitely many possible values of n . The cutoff operators commute with ∂_t and have a uniform C^j -operator norm for each $j \leq D_* + 1$. Hence (61) and the perturbation mixed estimates (53) give

$$\|\partial_t^m \rho^{(q)}(\tau_q)\|_{C^j} \leq C_{\text{head}} + C_{D_*} \sum_{i=q_0}^{q-1} \lambda_i^n + \lambda_q^n.$$

On a constant- $\bar{\beta}$ block, the deterministic frequency law implies $\lambda_q/\lambda_{q-1} \rightarrow \infty$. Since n ranges over the finite set $\{1, \dots, D_* + 1\}$,

$$C_{\text{head}} + C_{D_*} \sum_{i=q_0}^{q-1} \lambda_i^n = o(\lambda_q^n),$$

uniformly in the possible n . This contradicts (178). Therefore $\beta_q \downarrow 0$.

The selected frequency separation gives

$$\lambda_{q+1}^{\beta_{q+1}/8} \geq 4\lambda_q.$$

Taking logarithms and using $\beta_q \leq 1/8$ yields

$$x_{q+1} \geq 8 \log \lambda_q = \frac{8}{\beta_q} x_q \geq 64 x_q.$$

Equation (63) also gives

$$L_{q+1} = \frac{\beta_{q+1}}{192\pi} \exp(x_q/8), \quad \log L_{q+1} \leq \frac{x_q}{8}.$$

After arranging the base case, $4 \log L_{q+1} \leq x_{q+1}$, which proves $L_{q+1}^4 \leq \lambda_{q+1}^{\beta_{q+1}}$. Therefore

$$r_{q+1} = \lambda_{q+1}^{-2\beta_{q+1}} L_{q+1}^4 \leq \lambda_{q+1}^{-\beta_{q+1}} \leq \lambda_q^{-8}.$$

The remaining conclusions follow from (40) and the neighboring time-scale separation. $\square$

6.3. The limiting density and force. Define

$$(179) \quad \rho := S_{\mu_1} \theta_0 + \sum_{i=1}^{\infty} S_{\mu_{i+1}} \theta_i, \quad F := S_{\mu_1} F^{(0)} + \sum_{i=1}^{\infty} S_{\mu_{i+1}} G_i.$$

Proposition 6.4 (Global limit). *For the admissible stages produced by the recursive exponent choice, let ρ, F be defined by (179). Then $F \in C^\infty([0, 1] \times \mathbb{T}^2)$. For every $T < 1$, the sequences $(\rho^{(q)})_q$ and $(F^{(q)})_q$ are eventually constant in the stage index on $[0, T]$, the limiting density is smooth there, and*

$$(180) \quad \partial_t \rho + u_{\mathbb{T}}(\rho) \cdot \nabla \rho = F \quad \text{on } [0, T] \times \mathbb{T}^2.$$

The initial datum is

$$(181) \quad \rho_{\text{in}} = \rho(0) = S_{\mu_1} \theta_0.$$

The function ρ_{in} is smooth, odd, and has zero spatial mean. For every $t \in [0, 1]$, the function $F(t, \cdot)$ is odd and has zero spatial mean. There is one function

$$\rho_* \in \bigcap_{0 \leq \eta < 1} C^\eta(\mathbb{T}^2)$$

such that, for every $0 \leq \eta < 1$,

$$(182) \quad \rho(t) \longrightarrow \rho_* \quad \text{in } C^\eta(\mathbb{T}^2) \quad \text{as } t \uparrow 1.$$

Proof. For $\eta = 0$, cutoff boundedness follows from (196); for $0 < \eta < 1$, it follows from (198). Interpolation between the two sharp estimates in (163) gives

$$\|S_{\mu_{i+1}} \theta_i\|_{C^\eta} \leq C_\eta \lambda_i^{-1+2\beta_i+\eta}.$$

By (176), the exponent on the right is at most $-(1-\eta)/2$ for every sufficiently large i . The first series in (179) therefore converges uniformly in C^η , including at $t = 1$, because $\lambda_{i+1} \geq 4\lambda_i$ by (48). Set $\rho_* := \rho(1)$. Uniform smallness of the series tails and continuity of every finite head prove (182); this definition of ρ_* is independent of η .

Fix $m, k \geq 0$. Since $\beta_i \rightarrow 0$, choose $i_0 = i_0(m, k)$ so that for every $i \geq i_0$,

$$\beta_i \leq \frac{1}{40}, \quad m + k \leq m_i := \left\lfloor \frac{1}{4\beta_i} \right\rfloor - 10.$$

By admissibility and (156), the base term and the finitely many summands with $i < i_0$, including every term formed while $\beta_i > 1/40$, are smooth and form a finite head of (179). On the remaining tail, Proposition 5.13 and (164) give

$$\|\partial_t^m G_i\|_{C^k} \leq \lambda_i^{-1/4},$$

throughout this tail. The second series in (179) therefore converges in $C_t^m C_x^k$: the cutoffs are uniformly bounded on every fixed C^k , commute with time derivatives, and $\lambda_{i+1} \geq 4\lambda_i$ makes $\sum_i \lambda_i^{-1/4}$ finite. Since m, k are arbitrary,

$$F \in C^\infty([0, 1] \times \mathbb{T}^2).$$

Fix $T < 1$. Choose q sufficiently large that every perturbation with index at least $q - 1$ vanishes on $[0, T]$. Then $\rho^{(q-1)}$ is already Fourier truncated to $B_{2\mu_{q-1}}$, and (201) puts its nonlinearity in $B_{4\mu_{q-1}}$. Consequently,

$$S_{\mu_q} \rho^{(q-1)} = \rho^{(q-1)}, \quad \mathcal{C}_{\mu_q}(\rho^{(q-1)}) = 0 \quad \text{on } [0, T].$$

The perturbation residual E_q^{pert} also vanishes there.

The newest untruncated term of $F^{(q-1)}$ is then a cutoff commutator at scale μ_{q-1} , and (150) places it in $B_{4\mu_{q-1}}$. Every older force piece already has a lower Fourier cutoff. Since $\mu_q \geq 4\mu_{q-1}$,

$$S_{\mu_q} F^{(q-1)} = F^{(q-1)} \quad \text{on } [0, T].$$

Equations (155) and (156) now show that

$$\rho^{(q)} = \rho^{(q-1)}, \quad F^{(q)} = F^{(q-1)} \quad \text{on } [0, T].$$

These identities propagate by induction. Suppose $n \geq q$ and $\rho^{(n)} = \rho^{(q-1)}$, $F^{(n)} = F^{(q-1)}$ on $[0, T]$. The common density and force have Fourier supports in $B_{2\mu_{q-1}}$ and $B_{4\mu_{q-1}}$, respectively. Since $\mu_{n+1} \geq \mu_q \geq 4\mu_{q-1}$, the next cutoff fixes both, $\mathcal{C}_{\mu_{n+1}}(\rho^{(n)}) = 0$, and $\theta_{n+1} = E_{n+1}^{\text{pert}} = 0$ on $[0, T]$. The two stage recursions therefore give the same identities at level $n + 1$. Thus both sequences are eventually constant in the stage index on $[0, T]$. The corresponding partial sums in (179) are $\rho^{(Q)} - \theta_Q$ and $F^{(Q)} - G_Q$. For sufficiently large Q , both θ_Q and G_Q vanish on $[0, T]$, so the two series agree there with the eventual stages. The finite-stage equation then gives (180) without passage through a nonlinear limit.

Every perturbation after the base layer vanishes at $t = 0$, which proves (181). Each θ_i , $F^{(0)}$, and G_i is odd, and the even cutoffs preserve oddness. Uniform convergence therefore preserves oddness in both limiting series; every odd periodic scalar has zero spatial mean. The first cutoff also makes ρ_{in} smooth.

At time zero, $E_1^{\text{pert}} = 0$, and

$$S_{\mu_1} F^{(0)} + \mathcal{C}_{\mu_1}(\theta_0) = u_{\mathbb{T}}(S_{\mu_1} \theta_0) \cdot \nabla S_{\mu_1} \theta_0.$$

The commutator is supported in $B_{4\mu_1}$, so $S_{\mu_2} \mathcal{C}_{\mu_1}(\theta_0) = \mathcal{C}_{\mu_1}(\theta_0)$. For $i \geq 2$, the density $\rho^{(i-1)}(0)$ is Fourier truncated to $B_{2\mu_{i-1}}$, while $\mu_i \geq 4\mu_{i-1}$. Thus the Fourier truncated commutator identity (151) gives $\mathcal{C}_{\mu_i}(\rho^{(i-1)})(0) = 0$; moreover, $E_i^{\text{pert}}(0) = 0$. Hence $G_i(0) = 0$ for every $i \geq 2$. Every perturbation with positive index and every force increment G_i with $i \geq 2$ vanishes on a neighborhood of $t = 0$, so $\partial_t \rho(0) = 0$ and

$$F(0) = u_{\mathbb{T}}(\rho_{\text{in}}) \cdot \nabla \rho_{\text{in}}.$$

Thus the limiting equation also holds at the initial time. $\square$

6.4. Density-gradient lower bound on terminal intervals. For

$$t \in [t_q, t_{q+1}],$$

every perturbation above $q + 1$ vanishes. The iterated cutoff formula therefore gives

$$(183) \quad \rho(t) = P_q(t) + S_{\mu_{q+1}} \theta_q(t) + S_{\mu_{q+2}} \theta_{q+1}(t).$$

Set

$$(184) \quad y_q(t) := 2r_{q+1} e(\alpha_q(t)).$$

Since $\text{supp } \theta_{q+1}(t) \subset B_{r_{q+1}}$ and $r_{q+1} \leq \lambda_q^{-8}$, this point lies outside the support of θ_{q+1} , but remains close enough to the origin to retain the level- q geometry. For the density estimate only the cutoff tail of θ_{q+1} remains; for the velocity estimate, $u_{\mathbb{T}}(\theta_{q+1})$ is nonlocal and must also be bounded. All constants in the cutoff-tail estimates below are absolute after the fixed cutoff is chosen; in particular, they are independent of q , β_q , and β_{q+1} .

Proposition 6.5 (Density-gradient lower bound). *For the limiting density ρ produced by the recursive exponent choice, there is $q_* \in \mathbb{N}$ such that, for every $q \geq q_*$ and every $t \in [t_q, t_{q+1}]$,*

$$(185) \quad \|\nabla \rho(t)\|_{L^\infty} \geq \frac{1}{8} \lambda_q^{\beta_q}.$$

Consequently,

$$(186) \quad \lim_{t \uparrow 1} \|\nabla \rho(t)\|_{L^\infty} = \infty.$$

Proof. For large q , the point $y_q(t)$ lies in the fixed coordinate chart, and the child θ_{q+1} vanishes in a neighborhood of it.

Apply (197) with $j = 1$, $s = 3$, the perturbation C^7 estimate (53), and $\mu_{q+2} \geq \lambda_{q+1}^4$. This gives

$$(187) \quad \|\nabla(S_{\mu_{q+2}} - I)\theta_{q+1}\|_{L^\infty} \leq C\lambda_{q+1}^{-5}.$$

The cutoff-tail estimate (72) gives

$$\|\nabla(S_{\mu_{q+1}} - I)\theta_q\|_{L^\infty} \leq C\lambda_q^{-4}.$$

By the mean-value theorem, (184), the C^2 case of (163), and (177),

$$(188) \quad |\partial_{e(\alpha_q(t))}\theta_q(y_q(t), t) - \partial_{e(\alpha_q(t))}\theta_q(0, t)| \leq 2r_{q+1}\|\theta_q(t)\|_{C^2} \leq 2\lambda_q^{-6}.$$

The cutoff-parent bound (57) gives

$$\|P_q(t)\|_{C^1} \leq \lambda_q^{\beta_q/8}.$$

Combining (183), the three preceding error estimates, and (58), we obtain

$$|\partial_{e(\alpha_q(t))}\rho(y_q(t), t)| \geq \frac{1}{2}\lambda_q^{\beta_q} - C\lambda_q^{\beta_q/8} - C\lambda_q^{-4} - C\lambda_q^{-6} - C\lambda_{q+1}^{-5}.$$

This is at least $\lambda_q^{\beta_q}/8$ for all sufficiently large q , uniformly throughout the interval. Lemma 6.3 proves (186). $\square$

6.5. Velocity-gradient lower bound on terminal intervals.

Proposition 6.6 (Velocity-gradient lower bound). *For the limiting density ρ produced by the recursive exponent choice, there are $c > 0$ and $q_* \in \mathbb{N}$ such that, for every $q \geq q_*$ and every $t \in [t_q, t_{q+1}]$,*

$$(189) \quad \|D_x u_{\mathbb{T}}(\rho(t))\|_{L^\infty} \geq c\lambda_q^{7\beta_q/8}.$$

Consequently,

$$(190) \quad \lim_{t \uparrow 1} \|D_x u_{\mathbb{T}}(\rho(t))\|_{L^\infty} = \infty.$$

Proof. Since

$$\mathcal{M}(\alpha) = -e^\perp(\alpha)e(\alpha)^\top,$$

its entry in the first row and second column is $\sin^2 \alpha$. The angle bounds in Lemma 4.4, the bound $B_q(t) \geq \frac{1}{2}\lambda_q^{\beta_q}$ from Definition 4.1, and (59) give

$$C_0 B_q |\cos \alpha_q| \sin^2 \alpha_q \geq \frac{C_0}{4} (1 - o(1)) \lambda_q^{7\beta_q/8},$$

uniformly in the terminal interval. Retaining fixed slack, we obtain

$$(191) \quad |\partial_2 u_{\mathbb{T},1}(\theta_q)(0,t)| \geq \frac{C_0}{8\pi} \lambda_q^{7\beta_q/8} - \lambda_q^{\beta_q/8}.$$

The mean-value theorem, (195) applied to each $\partial_a \theta_q$, the C^3 case of (163), and (177) give

$$|D_x u_{\mathbb{T}}(\theta_q)(y_q(t), t) - D_x u_{\mathbb{T}}(\theta_q)(0, t)| \leq C r_{q+1} \lambda_q^3 \leq C \lambda_q^{-5}.$$

The cutoff tail (72) contributes at most $C \lambda_q^{-4}$, while (57) bounds the cutoff-parent contribution by $\lambda_q^{\beta_q/8}$.

Since $|y_q(t)| = 2r_{q+1}$ and $\text{supp } \theta_{q+1}(t) \subset B_{r_{q+1}}$,

$$\text{dist}_{\mathbb{T}^2}(y_q(t), \text{supp } \theta_{q+1}(t)) \geq r_{q+1}$$

for all sufficiently large q . The differentiated periodic kernel is the differentiated Euclidean kernel, bounded by $C|h|^{-3}$, plus a smooth function. Integration over the child support therefore gives

$$|D_x u_{\mathbb{T}}(\theta_{q+1})(y_q(t), t)| \leq C \frac{\|\theta_{q+1}(t)\|_{C^0}}{r_{q+1}} + C \|\theta_{q+1}(t)\|_{L^1}.$$

The perturbation-size estimate (163) and the definition of r_{q+1} now give

$$\begin{aligned} |D_x u_{\mathbb{T}}(\theta_{q+1})(y_q(t), t)| &\leq C L_{q+1}^{-4} \lambda_{q+1}^{-1+4\beta_{q+1}} + C \|\theta_{q+1}(t)\|_{L^1} \\ &\leq 1, \end{aligned}$$

for all sufficiently large q , because $\beta_{q+1} \leq 1/8$. Applying (197) with $j = 2$, $s = 3$, and the perturbation C^8 estimate (53), followed by the periodic Schauder estimate, gives

$$(192) \quad \|D_x u_{\mathbb{T}}((S_{\mu_{q+2}} - I)\theta_{q+1})\|_{L^\infty} \leq C \lambda_{q+1}^{-4}.$$

Combining (183) with (191) and the preceding error estimates yields

$$\|D_x u_{\mathbb{T}}(\rho(t))\|_{L^\infty} \geq c \lambda_q^{7\beta_q/8},$$

for every sufficiently large q , uniformly throughout the interval. Since the intervals cover a terminal neighborhood of one and $\lambda_q^{7\beta_q/8} \rightarrow \infty$, this proves (190). $\square$

Proof of Theorem 2.1. Lemma 6.1, Proposition 5.13, and the recursive exponent choice construct admissible stages for every q . Proposition 6.4 gives an odd smooth datum, an odd force in $C^\infty([0, 1] \times \mathbb{T}^2)$, a classical solution on $[0, 1)$, and convergence to an endpoint in every C^η , $0 \leq \eta < 1$. Propositions 6.5 and 6.6 give the two gradient divergences asserted in the theorem. $\square$

7. ACKNOWLEDGMENTS

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APPENDIX A. MIXED ESTIMATES AND SMOOTHNESS OF THE FORCE

At the step from level q to level $q + 1$, use the one-step aliases

$$\begin{aligned} \beta &= \beta_{q+1}, \quad \lambda = \lambda_{q+1}, \quad L = L_{q+1}, \quad M = M_{q+1} = \lceil \beta^{-2} \rceil, \quad \mu = \mu_{q+1}, \quad r = r_{q+1}, \\ P &= P_{q+1}, \quad v = v_q = u_{\mathbb{T}}(P), \quad \Phi = \Phi_q, \quad X = X_{q+1}, \quad \alpha = \alpha_{q+1}, \quad a = a_{q+1}, \quad s = s_{q+1}, \\ \gamma &= 1 - 2\beta, \quad D = \partial_t + v \cdot \nabla. \end{aligned}$$

The estimates below distinguish derivatives of perturbation coefficients from derivatives of the oscillatory factor. A coefficient derivative costs only powers of L and $\lambda^{2\beta}$ per spatial derivative, whereas an ordinary time derivative of the oscillatory factor costs λ^γ .

We first derive all-order bounds for the cutoff parent and its normalized coefficient propagator. We then verify the endpoint and derivative-range details for the two-pass mixed-time induction stated in Lemma 5.8. The final subsection verifies the two finite uncut-parent ranges authorized by $K_q \geq d(\beta)$: the final grouped velocity remainder and the cutoff commutator.

A.1. All-orders parent estimates.

Proof of Lemma 5.1. At time order zero, the constant-free stage bound gives

$$\|\rho^{(q)}\|_{C^2} \leq 2\lambda_q^2.$$

The transfer identity turns λ_q^2 into a power of the child logarithm L , and the periodic Schauder estimate gives the corresponding velocity bound. If $q = 0$, all positive time derivatives vanish.

Now assume $q \geq 1$ and fix $1 \leq p \leq T$. Differentiating (61) and using commutation with the cutoffs gives

$$\partial_t^p \rho^{(q)} = S_{\mu_1} \partial_t^p \theta_0 + \sum_{i=1}^{q-1} S_{\mu_{i+1}} \partial_t^p \theta_i + \partial_t^p \theta_q.$$

For every $0 \leq i \leq q$, (55) gives

$$\|\partial_t^p \theta_i\|_{C^2} \leq C_{2,p}(\beta_i) L_i^{N_{2,p}(\beta_i)} \lambda_i^{1+\beta_i+p(1-2\beta_i)}.$$

Monotonicity gives $\beta_i \in [\beta_q, 1/8]$. The a priori envelope (56) therefore bounds all constants and logarithmic exponents in terms of β_q and p . Moreover,

$$1 + \beta_i + p(1 - 2\beta_i) \leq \frac{9}{8} + p.$$

Thus a single exponent may be used before summation. For fixed $A > 0$ and $N \geq 0$, lacunarity gives

$$\sum_{i=0}^q L_i^N \lambda_i^A \leq C_{A,N} L_q^N \lambda_q^A.$$

Apply this with $A = 9/8 + p$. The transfer identity (65) converts the resulting parent-frequency power into a power of the child logarithm L , proving the density part of (110). The periodic Schauder estimate gives the velocity part.

Since $P = S_\mu \rho^{(q)}$ and S_μ commutes with ordinary time derivatives, the Fourier-cutoff Bernstein inequality and the preceding C^2 bound give

$$\|\partial_t^p P\|_{C^{j+1}} + \|\partial_t^p v\|_{C^{j+1}} \leq C_{p,j} L^{C_{p,j}} \mu^j.$$

This proves (111)–(112). Oddness gives $v(0, t) = 0$, so the mean-value theorem proves (113).

Finally, induct on $p + j$, using, for $\nu \in \{1, 2\}$, $[D, \partial_{x_\nu}] = -(\partial_{x_\nu} v) \cdot \nabla$. All factors are evaluated on B_r . The ordinary cutoff bounds control spatial derivatives, while the transport factor satisfies $\|v\|_{C^0(B_r)} \leq CL^C r$; hence its extra derivative costs at most $r\mu \leq 1$, up to a power of L . This proves (114). Each positive material order uses ordinary parent time derivatives no higher than that order.

In addition, $D^p v$ is odd, so it vanishes at the origin; its local C^0 bound follows from its C^1 bound and the mean-value theorem. $\square$

The preceding lemma supplies arbitrary spatial orders from the Fourier cutoff and the a priori low-order parent bound. The inequality $K_q \geq d(\beta)$ is used only for the finite uncut-parent derivative ranges in the residual and cutoff-commutator estimates.

A.2. The weighted coefficient propagator.

Proof of Lemma 5.2. Put $\tilde{c}_P(x, t) = c_P(x, t) - c_P(0, t)$. Since P is odd, $\tilde{c}_P$ is even and has zero constant and linear jets at the origin. Lemma 5.1 and Taylor's theorem give, for $y \in B_{\rho_{\text{ch}}}$,

$$|\partial_t^m \tilde{c}_P(y, t)| \leq C_m L^{C_m} \mu^2 |y|^2,$$

whereas, globally,

$$\|\partial_t^m \tilde{c}_P\|_{C^k(\mathbb{T}^2)} \leq C_{m,k} L^{C_{m,k}} \mu^k \quad (k \geq 1).$$

Here $c_P = m_0(t) \cdot \nabla P$. Differentiating the angle equation (90) and using Lemma 5.1 controls every time derivative of $m_0(t)$ by a power of L ; at positive order m , only parent time orders through $m-1$ occur. This proves the two displayed bounds. The same estimates hold with material derivatives in place of ordinary time derivatives. Indeed, material differentiation preserves evenness, and the material bounds in Lemma 5.1 give the same Taylor estimate at the origin. Let $\tilde{\Phi}(\tau; t, x)$ denote the coordinate lift furnished by Lemma 4.5, invoked with $J \geq \max\{S+T+1, 3\}$, and use the same notation $\tilde{c}_P$ for its periodic coordinate lift. For $x \in B_r$ and τ between σ and t , (77) gives

$$|\tilde{\Phi}(\tau; t, x)| \leq CL^2 r.$$

The fixed threshold Λ_{flow} is chosen so that

$$\sup_{\substack{x \in B_r \\ \tau \text{ between } \sigma \text{ and } t}} |\tilde{c}_P(\tilde{\Phi}(\tau; t, x), \tau)| \leq CL^C \mu^2 r^2 \leq 1.$$

For each fixed $m \geq 1$, every required ordinary or material derivative satisfies

$$\sup_{\substack{x \in \Omega_t \\ \tau \text{ between } \sigma \text{ and } t}} |\mathcal{D}^m \tilde{c}_P(\tilde{\Phi}(\tau; t, x), \tau)| \leq C_m L^{C_m} \mu^2 r^2 \leq C_m L^{C_m}, \quad \mathcal{D} \in \{\partial_t, D\}.$$

Here $\mu^2 r^2 = \lambda^{-3\beta} L^8$; no positive-order smallness is used. Also $\mu^k \leq \lambda^{2k\beta}$. The characteristic formula is

$$\tilde{H}_{\sigma,t}[\varphi(\sigma)](x) = \varphi(\Phi(\sigma; t, x), \sigma) \exp\left(-\int_{\sigma}^t \tilde{c}_P(\Phi(\tau; t, x), \tau) d\tau\right).$$

The tube estimate gives the zeroth-order exponential bound. On $\Omega_{\tau} \Subset B_r$, the positive-order C^k -bounds above for $\tilde{c}_P$, together with (79), control every spatial derivative of the exponential. Since $\varphi(\sigma)$ is compactly supported in $\Omega_{\sigma} \Subset B_r$, its stated B_r -bounds control every nonzero term after composition. Finally, $\Phi(t; \sigma, \Omega_{\sigma}) = \Omega_t$ carries the support.

For a datum depending on the lower-endpoint variable σ , set $U_{\sigma,t} = \tilde{H}_{\sigma,t}[\varphi(\sigma)]$. Write

$$D_{\sigma} := \partial_{\sigma} + v(\cdot, \sigma) \cdot \nabla$$

for the material derivative acting on the lower-endpoint datum, and set $\mathcal{L}_{\sigma} := D_{\sigma} + \tilde{c}_P(\cdot, \sigma)$. The exact endpoint identities are

$$\partial_{\sigma} U_{\sigma,t} = \tilde{H}_{\sigma,t}[\mathcal{L}_{\sigma} \varphi(\sigma)], \quad D_t U_{\sigma,t} = -\tilde{c}_P(t) U_{\sigma,t}.$$

Iteration of the first identity gives

$$\partial_{\sigma}^n U_{\sigma,t} = \tilde{H}_{\sigma,t}[\mathcal{L}_{\sigma}^n \varphi(\sigma)].$$

Every datum factor in $\mathcal{L}_{\sigma}^n \varphi$ is $D_{\sigma}^m \varphi$ with $m \leq n$; every differentiated parent factor has material order at most $n-1$. Iteration of the second identity expresses $D_t^p \partial_{\sigma}^n U_{\sigma,t}$ as a finite sum of $\tilde{H}_{\sigma,t}[\mathcal{L}_{\sigma}^n \varphi]$

multiplied by products of $\tilde{c}_P, D\tilde{c}_P, \dots, D^{p-1}\tilde{c}_P$. Thus an output C^j norm uses datum pairs only with $m \leq n$ and spatial order at most j . The diagonal condition gives $m + j \leq n + j \leq p + n + j \leq S + T$. Every parent factor has ordinary time order at most $p + n - 1$, and all needed spatial orders are at most $p + n + j \leq S + T$. The preceding coefficient, flow, and product bounds therefore prove (123). The case $p = n = 0$ uses only the characteristic formula and order-zero parent bounds. $\square$

Proof of Corollary 5.3. To pass from material to ordinary derivatives, use $\partial_t = D - v \cdot \nabla$. On B_r , a transport factor acting on a coefficient costs

$$r\lambda^{2\beta} = L^4.$$

Commutators with spatial derivatives are controlled by (111), proving (124). If the derivative falls on the oscillatory factor, its cost is

$$r\lambda = L^4\lambda^{1-2\beta} = L^4\lambda^\gamma.$$

The phase and parameter bounds in Lemma 3.2 prove (125). $\square$

Proof of Lemma 5.4. The identity

$$H_{\sigma,t}[a(\sigma)\varphi(\sigma)] = a(t)\tilde{H}_{\sigma,t}[\varphi(\sigma)]$$

shows that $g = aJ$, where

$$J(t) = \int_s^t \tilde{H}_{\sigma,t}[\varphi(\sigma)] d\sigma.$$

Equivalently,

$$(D + \tilde{c}_P)J = \varphi, \quad J(s) = 0.$$

For each $0 \leq k \leq S + T$, apply the fixed-endpoint case of Lemma 5.2 with output parameters $(T, S) = (0, k)$. Its local datum dependence uses only the order-zero C^k bound for the Duhamel source φ . The resulting estimate is uniform in σ , and integration over $[s, t]$, whose length is at most one, gives

$$\|J\|_{C^k(B_r)} \leq C_k L^{C_k} \mathcal{A} \lambda^{2k\beta} \quad (0 \leq k \leq S + T).$$

For $m \geq 1$, applying D^{m-1} to the equation for J gives the exact recurrence

$$D^m J = D^{m-1} \varphi - \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (D^\ell \tilde{c}_P) D^{m-1-\ell} J.$$

Induction in m , using the Duhamel-source hypotheses and the parent bounds, yields

$$\|D^m J\|_{C^k(B_r)} \leq C_{m,k} L^{C_{m,k}} \mathcal{A} \lambda^{2k\beta} \quad (m \leq T, m + k \leq S + T).$$

Indeed, the Duhamel source term at this pair is $D^{m-1}\varphi$ in C^k , so its time-space index satisfies $(m-1) + k \leq S + T - 1$.

To pass to ordinary time derivatives, use $\partial_t = D - v \cdot \nabla$. An order- p ordinary derivative in C^j uses $D^m J$ through spatial order $j + p - m$, $0 \leq m \leq p$. The term $m = 0$ uses the undifferentiated Duhamel source φ through order $j + p$; for $m \geq 1$, the recurrence uses the Duhamel-source derivative pair $(m-1, j + p - m)$. Each extra spatial derivative is paired with a transport coefficient. Equations (111) and (113), together with

$$r\lambda^{2\beta} = L^4,$$

show that these conversions cost only powers of L . This proves the material and ordinary bounds for J . Finally, the amplitude equation gives $|\partial_t^m a| \leq C_m L^{C_m} a$; the product rule proves (127). Characteristic transport and (126) give

$$\text{supp } g(t) \subset \Phi(t; s, K_\varphi) \Subset \Omega_t,$$

which is the support assertion. $\square$

For requested output orders T, K , the cutoff-parent estimates supply every spatial derivative through

$$N_{\text{prof}} = K + M + T + 2, \quad N_{\text{phase}} = N_{\text{prof}} + 1.$$

The additional phase derivative is used when a differentiated phase term is formed; it is supplied by the cutoff before Proposition 3.4 is invoked.

A.3. Technical proof of the density-first mixed induction.

Proof of Lemma 5.8. The normalized seed in (146) has fixed pulled-back compact support, and Lemma 5.2 uses only its value at $\sigma = s$. Equations (38) and (39), followed by (136), give the order-zero alternating chain displayed after Lemma 5.8. We verify the Duhamel datum ranges and derivative endpoints needed at positive time order.

Fix the current step $q \rightarrow q + 1$, let $p \geq 1$, and suppose that its coefficient and structured-defect estimates are known at every time order below p . For an order- p output-time derivative with the propagator's lower endpoint fixed at s , the propagator uses datum time order zero and parent time orders at most $p - 1$, by Lemma 5.2. The flow, phase, and angle have the same lower-parent dependence by their evolution equations in Lemmas 4.5 and 4.6.

For $1 \leq j < M$, let h_j be the corresponding coefficient of $\mathcal{Q}_j$, and write the coefficient of the $(j + 1)$ -st correction as

$$g_{j+1}(t) = - \int_s^t H_{\sigma,t}[h_j(\sigma)] d\sigma, \quad \varphi_j = \frac{h_j}{a}.$$

Fix an output pair (p, k) with $k + p \leq \mathcal{K}_{j+1} + T$. The order-zero assertion $\mathbf{R}_{0,j}$ supplies φ_j through spatial order $k + p$. For $1 \leq m \leq p - 1$, the already proved assertion $\mathbf{R}_{m,j}$, together with the derivative bounds for a^{-1} , supplies

$$\|D^m \varphi_j\|_{C^{k'}} \leq C_{m,k',j} L^{C_{m,k',j}} \lambda^{-1+(1-j)\beta+2k'\beta} \quad (m + k' \leq k + p - 1).$$

These are exactly the Duhamel-source hypotheses of Lemma 5.4 with $(T, S) = (p, k)$. Thus $\mathbf{C}_{p,j+1}$ uses only $\mathbf{R}_{0,j}$ at spatial order $k + p$ and $\mathbf{R}_{m,j}$ at spatial order $k + p - 1 - m$, $1 \leq m \leq p - 1$. In the case $p = 1$, the positive-time Duhamel-source range is empty. Hence every order- p correction coefficient in the current child is obtained without an order- p structured defect.

In the defect pass, Lemma 5.1 supplies the same-order cutoff-parent inputs from the prescribed low-order sharp family and the Fourier cutoff. The finite expansion then estimates the common-phase and background interactions by (38)–(39), keeps the exact velocity remainder grouped as in (34), and applies Lemma 5.10. This proves every $\mathbf{R}_{p,j}$ and the order- p force-increment estimate for this step. No same-order defect is returned to the density pass, so induction in p closes.

The descending derivative orders are consistent because $\mathcal{K}_j = \mathcal{K}_{j+1} + M + T + 2$. For $k + p \leq \mathcal{K}_{j+1} + T$, the localized velocity expansion uses coefficient order

$$k + M + p + 2 \leq \mathcal{K}_j$$

and one additional phase derivative. Both are supplied by the parent and phase hypotheses. $\square$

Factor each correction coefficient as $g = a\tilde{g}$, as in the proof of Lemma 5.7. The normalized coefficient has the constant size required by (125); restoring the factor a gives (161) for every fixed $m, k \geq 0$. This is (55). In this estimate all order-dependent factors remain in $C_{k,m}(\beta)L^{N_{k,m}(\beta)}$; the a priori envelope (56) was chosen to dominate them, and none is absorbed into a frequency gain. The flow and phase thresholds are independent of the requested derivative order. Thus the sharp part of Proposition 5.13 requires no enlargement of its finite threshold for (k, m) . If $k + 2m \leq k_{q+1}$, then

$$(k + 2m) - (k + m(1 - 2\beta) - 1 + \beta) = m(1 + 2\beta) + 1 - \beta \geq \frac{7}{8}.$$

A finite threshold depending on k_{q+1} therefore absorbs the logarithmic factor and gives (162). The requested finite order k_{q+1} is arbitrary subject to $k_{q+1} \geq d(\beta)$: high spatial derivatives fall on the

cutoff parent and recursively generated perturbation coefficients, not directly on the uncut complete parent $\rho^{(q)}$.

A.4. The finite force-derivative range. Suppose

$$\beta \leq \frac{1}{40}, \quad m + k \leq m_{q+1} := \left\lfloor \frac{1}{4\beta} \right\rfloor - 10.$$

For the differentiated perturbation residual, apply Proposition 3.4 with $T = m$ and $K = k + 1$. The required phase order is

$$N_{\text{phase}} = M + m + k + 4.$$

Including the $2m + 2$ ordinary-to-material conversion margin in Corollary 4.7, the total unsmoothed order is

$$N_{\text{phase}} + 2m + 2 = M + k + 3m + 6.$$

Since $m + k \leq (4\beta)^{-1} - 10$,

$$M + k + 3m + 6 \leq \lceil \beta^{-2} \rceil + \frac{3}{4\beta} - 24 < \left\lceil \frac{4}{\beta^2} \right\rceil + 1 = d(\beta).$$

This count concerns only the grouped exact velocity remainder in (34). Thus $K_q \geq d(\beta)$ authorizes its finite uncut-parent range. The other residual terms use the high spatial orders of the cutoff parent from Lemma 5.1 and the phase estimates from Lemmas 4.5 and 4.6. Their constants may depend on the requested order, but their thresholds do not. The remaining uncut C^2 time derivatives in (110) come from the a priori sharp family.

The stage application in Corollary 5.11 requires mixed parent order at most

$$2k + 4m + 8 + \left\lceil \frac{2}{\beta} \right\rceil \leq \frac{3}{\beta} - 31 < \frac{4}{\beta^2}.$$

The required order is also strictly below $d(\beta)$. These derivatives are supplied by the constant-free stage bound (54) at level q , already available from admissibility before the child is constructed. Therefore (147) and (153) hold simultaneously and give

$$(193) \quad \|\partial_t^m G_{q+1}\|_{C^k} \leq \lambda_{q+1}^{-1/4} \quad (m + k \leq m_{q+1}).$$

The activation cutoff w vanishes to infinite order at s_{q+1} . Induction through the Duhamel cascade shows that θ_{q+1} and E_{q+1}^{pert} vanish to infinite order there. The term $\mathcal{C}_{\mu_{q+1}}(\rho^{(q)})$ contains no factor of the child activation cutoff w and is already smooth. Whenever $\theta_q = 0$,

$$\rho^{(q)} = P_q = S_{\mu_q} \rho^{(q-1)},$$

so $\text{supp } \widehat{\rho^{(q)}} \subset B_{2\mu_q}$. The separation (50) gives $\mu_{q+1} \geq 4\mu_q$, and (151) makes the commutator zero.

Equations (161), (162), and (193) are the mixed-order conclusions asserted in Proposition 5.13. Their constants and logarithmic exponents may depend on the fixed mixed order, but the all-order sharp estimate requires no further frequency threshold. The infinite-order vanishing of θ_{q+1} and E_{q+1}^{pert} at s_{q+1} follows from the activation profile and the Duhamel cascade, while the exact residual and cutoff-commutator identities give the classical stage equation.

APPENDIX B. PERIODIC MULTIPLIER, CUTOFF, AND KERNEL ESTIMATES

Lemma B.1 (Parity and periodic Schauder bounds). *Let $\rho \in C^\infty(\mathbb{T}^2)$. If $\rho(-x) = -\rho(x)$, then $u_{\mathbb{T}}(\rho)(-x) = -u_{\mathbb{T}}(\rho)(x)$. In particular,*

$$\partial^\gamma \rho(0) = \partial^\gamma u_{\mathbb{T}}(\rho)(0) = 0 \quad \text{when } |\gamma| \text{ is even.}$$

For every integer $j \geq 0$, every $0 < \eta < 1$, and every $f \in C^{j,\eta}(\mathbb{T}^2)$,

$$(194) \quad \|u_{\mathbb{T}}(f)\|_{C^{j,\eta}} \leq C_{j,\eta} \|f\|_{C^{j,\eta}}.$$

In particular,

$$(195) \quad \|D_x u_{\mathbb{T}}(f)\|_{L^\infty} \leq C\|f\|_{C^2}.$$

Proof. Let $\mathfrak{R}f(x) = f(-x)$. Evenness of this Fourier multiplier symbol gives $u_{\mathbb{T}}(\mathfrak{R}f) = \mathfrak{R}u_{\mathbb{T}}(f)$, which proves the parity assertion. On each periodic dyadic annulus, the rescaled Fourier multiplier symbol has a smooth compactly supported extension. Let K_n denote its periodized kernel. These kernels have uniformly bounded L^1 norm and integral zero, so

$$(K_n * f)(x) = \int_{\mathbb{T}^2} K_n(y)(f(x-y) - f(x)) dy.$$

Applying this first-difference identity and treating the finitely many low modes proves (194). Apply that estimate to each $\partial_a f$ with Hölder exponent $1/2$, and bound $\|\partial_a f\|_{C^{0,1/2}}$ by $C\|f\|_{C^2}$, to obtain (195). $\square$

Lemma B.2 (Cutoff tails and exact Fourier cutoffs). *Let $\mu \geq 1$ and $f, g \in C^\infty(\mathbb{T}^2)$. For integers $j, s \geq 0$,*

$$(196) \quad \|S_\mu f\|_{C^j} \leq C_j \|f\|_{C^j},$$

$$(197) \quad \|(I - S_\mu)f\|_{C^j} \leq C_{j,s} \mu^{-s} \|f\|_{C^{j+s+3}}.$$

For every $0 < \eta < 1$,

$$(198) \quad \|S_\mu f\|_{C^\eta} \leq C_\eta \|f\|_{C^\eta}.$$

The cutoff preserves parity and the spatial mean and commutes with spatial derivatives and $u_{\mathbb{T}}$. On smooth time-dependent families, it also commutes with ∂_t . Moreover,

$$(199) \quad \text{supp } \widehat{S_\mu f} \subset \{k : |k| \leq 2\mu\},$$

$$(200) \quad \nu \geq 2\mu \implies S_\nu S_\mu = S_\mu.$$

If $\text{supp } \widehat{f}, \text{supp } \widehat{g} \subset \{|k| \leq 2\mu\}$, then

$$(201) \quad \text{supp } u_{\mathbb{T}}(\widehat{f}) \cdot \nabla g \subset \{|k| \leq 4\mu\}.$$

Consequently, when $\nu \geq 4\mu$,

$$(202) \quad S_\nu(u_{\mathbb{T}}(f) \cdot \nabla g) = u_{\mathbb{T}}(f) \cdot \nabla g.$$

Proof. Poisson summation represents the kernel of S_μ as a periodization of a Schwartz function at scale μ^{-1} . Its $L^1(\mathbb{T}^2)$ norm is bounded uniformly in μ , which proves (196). The same kernel bound gives

$$|S_\mu f(x) - S_\mu f(y)| \leq C[f]_{C^\eta} |x - y|^\eta,$$

and hence (198). For the tail, Fourier inversion and three integrations beyond the requested derivative order give

$$\sum_{|k| \geq \mu} |k|^j |\widehat{f}(k)| \leq C_{j,s} \mu^{-s} \|f\|_{C^{j+s+3}},$$

and on the transition support $\mu \leq |k| \leq 2\mu$, we have the same estimate. This proves (197). Evenness gives parity preservation, $\chi(0) = 1$ gives mean preservation, multiplier form gives commutation with spatial derivatives, scalarity gives commutation with $u_{\mathbb{T}}$, and time independence gives commutation with ∂_t .

The support assertion follows directly from (13). On that support, $|k|/\nu \leq 1$ when $\nu \geq 2\mu$, proving exact nesting. The multiplier $u_{\mathbb{T}}$ does not enlarge Fourier support. Convolution of the two Fourier supports then gives (201), and the identity $\chi = 1$ on B_1 gives (202). $\square$

Lemma B.3 (Periodic kernel decomposition). *For $f \in C^\infty(\mathbb{T}^2)$ supported in U_0 , one has on U*

$$V_{\mathbb{T}}f = V_{\mathbb{R}}f + \mathcal{H}_0f,$$

where $V_{\mathbb{R}}$ is defined by (15), and

$$\mathcal{H}_0f(x) = \int_{U_0} H_0(x, y)f(y) dy \quad \text{with } H_0 \in C^\infty(U \times U_0).$$

There is also a kernel H_{far} , smooth on a neighborhood of $(\mathbb{T}^2 \setminus U) \times \overline{U_0}$, such that, for $x \notin U$,

$$V_{\mathbb{T}}f(x) = \int_{U_0} H_{\text{far}}(x, y)f(y) dy.$$

The same decomposition holds componentwise for $u_{\mathbb{T}}$ through (9).

Proof. Let $\mathcal{G}_{\mathbb{T}}$ be the mean-zero Green function of $-\Delta$ on $\mathbb{T}^2$. In a coordinate neighborhood of the diagonal,

$$\mathcal{G}_{\mathbb{T}}(x - y) = -\frac{1}{2\pi} \log |x - y| + h(x, y).$$

Indeed, subtracting the displayed logarithm cancels the delta mass in the Green equation; the difference solves a Poisson equation with smooth right-hand side, and interior elliptic regularity makes h smooth. Since $V_{\mathbb{T}} = 2\pi\partial_1(-\Delta)^{-1}$, its local convolution kernel with respect to ordinary Lebesgue measure is

$$-\frac{x_1 - y_1}{|x - y|^2} + \tilde{H}(x, y).$$

After the substitution $h = y - x$, the singular term is exactly (15). Insert a cutoff equal to one near the diagonal. The difference between the cutoff singular kernel and the full Euclidean kernel is smooth on $U \times U_0$. Since $\overline{U_0} \Subset U$, the periodic kernel is smooth on a neighborhood of $(\mathbb{T}^2 \setminus U) \times \overline{U_0}$, which gives H_{far} . Finally, since $\text{supp } f \Subset U_0$, integration by parts in y gives

$$(-\mathcal{H}_0(\partial_2 f), \mathcal{H}_0(\partial_1 f)) = \int_{U_0} (\partial_{y_2} H_0(x, y), -\partial_{y_1} H_0(x, y)) f(y) dy.$$

The same identity with H_{far} proves the componentwise assertion; all resulting kernels are smooth. $\square$

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ANTHROPIC, PBC, 500 HOWARD ST., SAN FRANCISCO, CA 94105, USA

Email address: `alpoge@fas.harvard.edu`

COURANT INSTITUTE SCHOOL OF MATHEMATICS, COMPUTING, AND DATA SCIENCE, NEW YORK UNIVERSITY, 251
MERCER STREET, NEW YORK, NY 10012, USA

Email address: `buckmaster@nyu.edu`

COURANT INSTITUTE SCHOOL OF MATHEMATICS, COMPUTING, AND DATA SCIENCE, NEW YORK UNIVERSITY, 251
MERCER STREET, NEW YORK, NY 10012, USA

Email address: `m.coiculescu@nyu.edu`