LECTURES ON GEODESICS IN THE SPACE OF KAEHLER METRICS, LECTURE 3: HRMA, HCMA AND TORIC KAEHLER MANIFOLDS

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It is very hard to find computable examples of geodesics in the space of Kaehler metrics. It seems that they come in two types: (i) geodesics of toric Kaehler metrics, the topic of Lecture 2; (iii) geodesics coming from special test configurations and Hele-Shaw flows. Besides Donaldson's construction (reviewed in §1) there are not many examples of geodesic rays. Almost the only other explicit examples are toric cases or 'test configurations'. This lecture is devoted to the HRMA and to toric Kahler manifolds where the toric geodesic equation reduces to the HRMA.

The only (embedded) toric Kaehler manifolds of complex dimension one are the Riemannian metrics on $S^2 = \mathbb{CP}^1$ which are invariant under rotations around the third axis. Thus we are interested in " one parameter families of surfaces of revolution which are geodesics in the Mabuchi-Semmes-Donaldson metric.' It is not obvious, but surfaces of revolution (i.e. toric Kaehler metrics on S^2) form a totally geodesic submanifold $\mathcal{H}_{\mathbf{T}^m}$ of $\mathcal{H}_\omega$.

The reason that one can explicitly solve the MSD geodesic equation in the toric case is that the HCMA may be linearized by the Legendre transform. Roughly speaking, we transfer the problem from Kaehler potentials to *symplectic potentials*, where the equation for geodesics becomes linear and solvable. All of the difficulty lies in Legendre transforming back and dealing with the singularities that arise from this transform.

The torus invarince in complex dimension one is S^1 invariance and it reduces the HCMA to the HRMA. A toy model for the HRMA is the equation $\det \mathrm{Hess} f = 0$ for a function $f(x,t)$ on the upper half plane $t > 0$. The initial value problem is briefly reviewed in §2.

1. Donaldson Example

This example is from [Dkahler], Let $\dot\varphi_0 = h$. Let $U_r = h^{-1}(r,1]$. Identify $\mathbb{C}$ with $T_q\mathbb{CP}^1$ (south pole). Let $\alpha_r : \mathbf{D} \to U_r$ be the unique Riemann map with $\alpha_r(0)$ equal to the south pole. Rotation of the disc defines a circle action on ∂U_r. Key idea: *Define ω_0 so that the Hamiltonian flow of h on each level coincides with the S^1 action.* Here, the coordinates $(t, e^{i\theta})$ on the cylinder $[0,\infty] \times S^1$ correspond to $e^{-t}e^{i\theta}$.

For each $v \in S^2$ there exists a unique $e^{is(v)} \in S^1$ so that $\alpha_{h(v)}(0, e^{is(v)}) = v$. Put

$$f_t(v) = \alpha_{h(v)}(t, e^{is(v)})$$

in polar coordinates. Thus,

$$\Gamma_v(t,s) = \alpha_{h(v)}(e^{-t}e^{is(v)}e^{is}).$$

The leaf through $v \times \{0\}$ is the image of this map. For fixed t one rotates the circle of radius e^{-t}. For fixed s one has the image of a radial line. Hence $f_t(v)$ takes v up the radial line through v. Then define $\omega_t = f_t^{-1*}\omega$.

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That ω_t is a geodesic ray (for $t > 0$) follows from the fact that $f_t^* \omega_t = \omega$ and that f_t is obtained by analytic continuation of the Hamilton flow of h with respect to ω. It follows that $\omega_t = \omega + dd^c \varphi_t$ and that $\partial\bar\partial\Phi$ (the spacetime potential) solves the HCMA.

How 'rare' is this construction? There are two types of rarities. First, it is rare that if one has a perfect Morse function h and a symplectic form ω on S^2, then the orbits of X_h^ω all have the same period. Such an h is known as an action variable. Secondly, it is extremely rare that the the Riemann mapping function for the superlevel sets U_r of h induce the same parameterization of $S^1 \to \partial U_r$ as given by the Hamilton orbit parametrization of this curve. This is a much more stringent condition than the orbits having the fixed period 2π.

However for every perfect Morse function h, this construction finds an ω for which (ω, h) is the initial data of a geodesic ray ω_t. The argument does not obviously run in the reverse direction, i.e. given ω it is not clear that there exists such an h.

2. Cauchy problem for the HRMA in the upper half plane

This section is from [Fo1, Fo2]. The HRMA on the upper half plane is,

$$\begin{cases} \det D^2\Phi(x,t) = 0, \quad (x,t)\mathbb{R} \times \mathbb{R}_+ \\ \\ \Phi|_{t=0} = u \qquad\qquad \partial_t \Phi = v \ \ (t=0). \end{cases}$$

Proposition **2.1.** *If $\Phi \in C^3$, and $\partial_x^2 \Phi(x,t) > 0$ for all (x,t) then the null foliation of $D^2\Phi$ consists of straight lines along which Φ is affine linear.*

Proof. The main statement is that the leaves are totally geodesic. Let T be a smooth vector field tangent to the null foliation. The null foliation has dimension one. Then $\nabla_T d\Phi = 0$

Let Z be any vector field. Then

$$\begin{aligned} D^2\Phi(Z, \nabla_T T) = (\nabla_Z d\Phi)(\nabla_T T) \quad &= \quad = T(\nabla_Z d\Phi)(T)) - (\nabla_T \nabla_Z d\Phi)(T) \\ \\ &= \quad 0 - (\nabla_Z \nabla_T du)(T)) - (\nabla_{[T,Z]} d\Phi)(T) = 0. \end{aligned}$$

Here we use that $\nabla_Z d\Phi(T) = D^2\Phi(T, Z) = 0$ since $T \in \ker D^2\Phi$ and that $R(Z, T) = 0$ since the metric is Euclidean. Further, $\nabla_T d\Phi(T) = 0$.

Then if $\gamma(t)$ is an integral curve (necessarily a line) we have

$$\frac{d^2}{dt^2} \Phi(\gamma(t)) = D^2\Phi(\dot\gamma, \dot\gamma) = 0.$$

□

For the initial value problem, $D^2\Phi(x, 0)$ is completely determined by the initial data and the equation $\det D^2\Phi(x, 0) = 0$ (which determines $\partial_t^2 \Phi(x, 0)$. The straight lines (characteristics) are generated by $\ker D^2\Phi(x, 0)$ along the $t = 0$ axis. There lifespan of the solution is the minimal time T before straight lines in $\ker D^2\Phi(x, 0)$ starting at $t = 0$ intersect. At such a time, it is impossible to have the solution affine linear along two intersecting characteristics. Each transports the initial data to the crossing point and the values in general are different.

If the initial null directions point away from each other, and no characteristic lines cross, then there is a global solution.

3. Toric Kaehler metrics

This section is from [RZAIM]. We now generalize the HRMA discussion to any toric Kaehler manifold. All of the important objects in the theory of geodesics are explicitly computable. In particular, we have several (equivalent) formulae for the Moser maps f_s. The orbits $f_s(z)$ are the characteristics and the lifespan of a solution again depends on whether the characteristics cross.

As will be explained below, the HCMA on a toric Kaehler manifold reduces to the HRMA on the real points x of the open orbit, which is equivalent to $\mathbb{R}^n$. As explained below, $u = \mathcal{L}\varphi_0$ is the symplectic potential, (the Legendre transform of φ_0 and $\mu_\tau = \nabla\varphi_\tau$ is the moment map corresponding to the symplectic form ω_τ of the geodesic

$$\omega_\tau = f_\tau^{-1*}\omega_0.$$

Remark: Possibly confusing notational issue: We usually reserve the notation φ for a relative Kaehler potential, as in $\omega_\varphi = \omega_0 + i\partial\bar\partial\varphi$. But on the open orbit there exist a Kaehler potential for ω_0 as well, and the combined potential is denoted ψ as in $\omega = i\partial\bar\partial\psi$. In this introduction we write φ but later we write ψ.

Before getting started, we list the key results and formulae. Undefined notation will be defined in later sections.

3.1. **Key formulae.** The Hamiltonian flow is given by

$$\begin{cases} \mu_0 \circ exptX^{\omega_{\varphi_0}}_{\dot\varphi_0} \circ \mu_0^{-1}(y,\theta) &= \nabla\psi_0 \circ \exp tX^{\omega_{\varphi_0}}_{\dot\varphi_0} \circ (\nabla\psi_0)^{-1}.(y,\theta) \\ &= (y, \theta - t\nabla_y\dot\varphi_0 \circ (\nabla\psi_0)^{-1}),\ (y,\theta)\ (P\setminus\partial P)\times(S^1)^n, \\ \exp tX^{\omega_{\varphi_0}}_{\dot\varphi_0}.(x,\theta) = &= (x, \theta - t(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0),\ (x,\theta)\in\ M_{\mathbb{R}}\times(S^1)^n. \end{cases}$$

The analytic continuation in time is thus the Moser flow,

$$f_\tau(z) = \exp -\sqrt{-1}\tau X^{\omega_{\varphi_0}}_{\dot\varphi_0}(z) = z - \tau(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0, \quad \tau\in S_\infty. \tag{1}$$

Other formulae are given in Lemma 6.4:

$$\begin{cases} (i) & f_\tau(e^\rho) = e^\rho - \tau(D^2\varphi_0)^{-1}\nabla_\rho\dot\varphi_0(e^\rho),\ \text{Lemma (6.2)} \\ (ii) & f_\tau(e^\rho) = e^\rho + \tau\nabla\dot u_0(\nabla_\rho\varphi_0(e^\rho)) = \nabla u_0\circ\nabla\varphi_0(e^\rho) + \tau\nabla\dot u_0(\nabla_\rho\varphi_0(e^\rho)),\quad (47), \\ (iii) & f_\tau = \mu_0^{-1}\circ\mu_\tau(e^\rho) = \nabla u_s\circ(\nabla u_0)^{-1},\quad \text{all}\ \ s\geq 0,\ \ (38) \\ (iv) & f_\tau(\nabla u_0(x)) = \nabla_x u_0(x) + \tau\nabla_x\dot u_0(x) = \nabla u_\tau\circ(\nabla u_0)^{-1},\quad \text{all}\ \ \tau\geq 0\quad (40). \end{cases}$$

Note that

$$y = \nabla\varphi_0(x),\ \ x = \nabla u_0(y)\implies \nabla_y = \nabla^2_y u_0.\nabla_x. \tag{2}$$

Also,

$$(\nabla^2\varphi_0)^{-1}\nabla_x\dot\varphi_0 = \nabla_y\dot u_0. \tag{3}$$

3.2. **Main results.** The first result (Theorem 6.8) is that

$$e^\rho \to \nabla u_0 \circ \nabla \varphi_0(e^\rho) + \tau \nabla \dot{u}(\nabla_\rho \varphi_0(e^\rho))$$

is invertible if and only if $\nabla u_0 + \tau \nabla \dot{u}_\tau$ is invertible if and only if $u_0 + \tau \dot{u}_0$ is convex. If there exists T so that $u_0 + \tau \dot{u}_0$ fails to be convex for $\tau \geq T$ then the life span of the solution of the geodesic equation is T.

The proof is based on the identity,

Lemma **1.**

$$\nabla^2 u_0(y).\nabla_x f_s(\nabla u_0(y)) = \nabla_y^2(u_0 + s\dot{u}_0).$$

To prove the identity, we start with (1),

$$f_s(z) = z - s(\nabla^2 \psi_0)^{-1} \nabla_x \dot{\varphi}_0, \quad \tau \in S_\infty, z \in M_{\rm o}, \tag{4}$$

or in terms of the moment coordinates ((iv))

$$f_s(\nabla u_0(y)) = \nabla_y u_0(y) + s\nabla_y \dot{u}_0, \quad s \in \mathbb{R}_+, y \in P \setminus \partial P. \tag{5}$$

Applying the gradient with respect to y to this equation we obtain

$$\nabla^2 u_0(y).\nabla_x f_s(\nabla u_0(y)) = \nabla_y^2(u_0 + s\dot{u}_0).$$

Since $\nabla^2 u_0$ is invertible for $y \in P \setminus \partial P$, it follows that the gradient of f_s is invertible at $z \in M_{\rm o}$ if and only if $u_0 + s\dot{u}_0$ is strictly convex on $P \setminus \partial P$. More details are given in §6.3.

The second result is Theorem 7.2: the HMRA reduces to a Hamilton-Jacobi equation. Let

$$F(\sigma, \xi) = \sigma - \dot{\varphi}_0((\nabla \varphi_0)^{-1}(\xi)),$$

where $\sigma \in \mathbb{R}, \xi \in \mathbb{R}^n$. Then if $\sigma = \dot{\varphi}_\tau$ and $\xi = \nabla \varphi_\tau$, one has

$$F(\dot{\varphi}_\tau, \nabla \varphi_\tau) = 0 \iff \dot{\varphi}_\tau \circ f_\tau = \dot{\varphi}_0.$$

This follows from the explicit formulae, modulo mainly regularity considerations and existence of an inverse to f_τ. Namely, it satisfies the conservation law, $\dot{\varphi}_\tau - \dot{\varphi}_0 \circ f_\tau^{-1} = 0$. But this means that the 'vertical projection' of the Lagrangian submanifold $\{(s, \dot{\varphi}_s, z, d\varphi_s)\}$, namely has a projection to $(s, \dot{\varphi}_s, z, d\varphi_s) \to (\dot{\varphi}_s, d\varphi_s)$ takes its image in the set $\{F = 0\}$, and the Lebesgue measure of the image is therefore zero. Details are in §6.3.

The origin of these formulae is that the Hamiltonian flow G^t of $\dot{\varphi}_0$ is linear in "action-angle variables" $G^t(I, \theta) = (I, \theta + t\nabla_I H(I))$ when we express $\dot{\varphi}_0 = H(I)$. In fact, $I = \mu(e^\rho)$ and $H = -\dot{u}_0$. Analytic continuation to imaginary time switches the component in which the flow is linear so that $G^{is}(I, \theta) = (I - sJ\nabla_I H(I), \theta)$.

Regarding (i), $(D^2\varphi_0)^{-1}\nabla_\rho \dot{\varphi}_0(e^\rho)$ is the ω_0-metric gradient of $\dot{\varphi}_0$ so

$$(D^2\varphi_0)^{-1}\nabla_\rho \dot{\varphi}_0(e^\rho) = JX^{\omega_0}_{\dot{\varphi}_0},$$

and (i) says that the imaginary time continuation of the Hamiltonian flow of $X^{\omega_0}_{\dot{\varphi}_0}$ is linear motion in the direction, $JX^{\omega_0}_{\dot{\varphi}_0}$.

3.3. **Background.** In this section we review toric Kaehler manifolds. The main ideas are from [A, G].

A toric Kähler manifold is a Kähler manifold (M, J, ω) on which the complex torus $(\mathbb{C}^*)^m$ acts holomorphically with an open orbit M^o. Choosing a basepoint m_0 on the open orbit identifies $M^o \equiv (\mathbb{C}^*)^m$ and give the point $z = e^{\rho/2+i\varphi}m_0$ the holomorphic coordinates

$$z = e^{\rho/2+i\varphi} \in (\mathbb{C}^*)^m, \quad \rho, \varphi \in \mathbb{R}^m. \tag{6}$$

The real torus $\mathbf{T}^m \subset (\mathbb{C}^*)^m$ acts in a Hamiltonian fashion with respect to ω. Its moment map $\mu = \mu_\omega : M \to P \subset \mathbf{t}^* \simeq \mathbb{R}^m$ (where $\mathbf{t}$ is the Lie algebra of $\mathbf{T}^m$) with respect to ω defines a singular torus fibration over a convex lattice polytope P; as in the introduction, P is understood to be the closed polytope. We recall that the moment map of a Hamiltonian torus action with respect to a symplectic form ω is the map $\mu_\omega : M \to \mathbf{t}^*$ defined by $d\langle \mu_\omega(z), \xi\rangle = \iota_{\xi^\#}\omega$ where $\xi^\#$ is the vector field on M induced by the vector $\xi \in \mathbf{t}$. Over the open orbit one thus has a symplectic identification

$$\mu : M^o \simeq P^o \times \mathbf{T}^m.$$

We let x denote the Euclidean coordinates on P. The components $(I_1, \dots, I_m)$ of the moment map are called action variables for the torus action. The symplectically dual variables on $\mathbf{T}^m$ are called the angle variables. Given a basis of $\mathbf{t}$ or equivalently of the action variables, we denote by $\{\frac{\partial}{\partial\theta_j}\}$ the corresponding generators (Hamiltonian vector fields) of the $\mathbf{T}^m$ action. Under the complex structure J, we also obtain generators $\frac{\partial}{\partial\rho_j}$ of the $\mathbb{R}^m_+$ action.

The action variables are globally defined smooth functions but fail to be coordinates at points where the generators of the $\mathbf{T}^m$ action vanish. We denote the set of such points by $\mathcal{D}$ and refer to it as the divisor at infinity. If $p \in \mathcal{D}$ and $\mathbf{T}^m_p$ denotes the isotropy group of p, then the generating vector fields of $\mathbf{T}^m_p$ become linearly dependent at P.

We assume M is smooth and that P is a Delzant polytope. It is defined by a set of linear inequalities

$$\ell_r(x) := \langle x, v_r\rangle - \lambda_r \geq 0, \ r = 1, ..., d,$$

where v_r is a primitive element of the lattice and inward-pointing normal to the r-th $(m-1)$-dimensional facet $F_r = \{\ell_r = 0\}$ of P. We recall that a facet is a highest dimensional face of a polytope. The inverse image $\mu^{-1}(\partial P)$ of the boundary of P is the divisor at infinity $\mathcal{D} \subset M$. For $x \in \partial P$ we denote by

$$\mathcal{F}(x) = \{r : \ell_r(x) = 0\}$$

the set of facets containing x. To measure when $x \in P$ is near the boundary we further define

$$\mathcal{F}_\varepsilon(x) = \{r : |\ell_r(x)| < \varepsilon\}. \tag{7}$$

4. Kähler potential in the open orbit

On a simply connected open set, a Kähler metric may be locally expressed as $\omega = 2i\partial\bar\partial\varphi$ where φ is a locally defined function which is unique up to the addition $\varphi \to \varphi + f(z) + \overline{f(z)}$ of the real part of a holomorphic or antiholomorphic function f. Here, $a \in \mathbb{R}$ is a real constant which depends on the choice of coordinates. Thus, a Kähler metric $\omega \in \mathcal{H}$ has a

Kähler potential φ over the open orbit $M^o \subset M$. Invariance under the real torus action implies that φ only depends on the ρ-variables, so that we may write it in the form

$$\varphi(z) = \varphi(\rho) = F(e^\rho). \tag{8}$$

The notation $\varphi(z) = \varphi(\rho)$ is an abuse of notation, but is standard. For instance, the Fubini-Study Kähler potential is $\varphi(z) = \log(1 + |z|^2) = \log(1 + e^\rho) = F(e^\rho)$. Note that the Kähler potential $\log(1 + |z|^2)$ extends to $\mathbb{C}^m$ from the open orbit $(\mathbb{C}^*)^m$, although the coordinates (ρ, θ) are only valid on the open orbit. This is a typical situation.

On the open orbit, we then have

$$\omega_\varphi = \frac{i}{2} \sum_{j,k} \frac{\partial^2 \varphi(\rho)}{\partial \rho_k \partial \rho_j} \frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k} = \sum_{j,k} \frac{\partial^2 \varphi(\rho)}{\partial \rho_k \partial \rho_j} d\rho_j \wedge d\theta_k \tag{9}$$

Positivity of ω_φ implies that $\varphi(\rho) = F(e^\rho)$ is a strictly convex function of $\rho \in \mathbb{R}^n$. The moment map with respect to ω_φ is given on the open orbit by

$$\mu_{\omega_\varphi}(z_1, \dots, z_m) = \nabla_\rho \varphi(\rho) = \nabla_\rho F(e^{\rho_1}, \dots, e^{\rho_m}), \quad (z = e^{\rho/2 + i\theta}). \tag{10}$$

The formula (10) follows from the fact that the generators $\frac{\partial}{\partial \theta_j}$ of the $\mathbf{T}^m$ actions are Hamiltonian vector fields with respect to ω_φ with Hamiltonians $\frac{\partial \varphi(\rho)}{\partial \rho_j}$, since

$$\iota_{\frac{\partial}{\partial \theta_j}} \omega_\varphi = d \frac{\partial \varphi}{\partial \rho_j}. \tag{11}$$

The moment map is a homeomorphism from $\rho \in \mathbb{R}^m$ to the interior P^o of P and extends as a smooth map from $M \to \bar{P}$ with critical points on the divisor at infinity $\mathcal{D}$. Hence, the Hamiltonians (11) extend to $\mathcal{D}$.

Note that the local Kähler potential on the open orbit is not the same as the global smooth *relative Kähler potential* with respect to a background Kähler metric ω_0. That is, given a reference metric ω_0 with Kähler potential φ_0, it follows by the $\partial\bar{\partial}$ lemma that $\omega = \omega_0 + dd^c\varphi$ with $\varphi \in C^\infty(M)$. The Kähler potential φ on the open orbit defines a singular potential on M which satisfies $dd^c\varphi = \omega + H$ where H is a fixed current supported on $\mathcal{D}$. We generally denote Kähler potentials by φ and in each context explain which type we mean.

5. Legendre transform and symplectic potential

The Legendre transform $\mathcal{L}\varphi$ of the open-orbit Kähler potential φ, a convex function on $\mathbb{R}^m$ in logarithmic coordinates, is the so-called dual symplectic potential

$$u_\varphi(x) = \mathcal{L}\varphi(x), \tag{12}$$

a convex function on the convex polytope P. Under this Legendre transform, the complex Monge-Ampère equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $\ddot{u} = 0$ and is thus solved by

$$u_t = u_{\varphi_0} + t(u_{\varphi_1} - u_{\varphi_0}). \tag{13}$$

Hence the solution φ_t of the geodesic equation on $\mathcal{H}$ is solved in the toric setting by

$$\varphi_t = \mathcal{L}^{-1} u_t. \tag{14}$$

Our goal is to show that $\varphi_k(t; z) \to \mathcal{L}^{-1} u_t$ as in (13) in a strong sense.

The Legendre conjugate of a continuous function $f = f(x)$ on $\mathbb{R}^n$ is defined by

$$\mathcal{L}f(y) = f^\star(y) := \sup_{x\in\mathbb{R}^n} \big(\langle x, y\rangle - f(x)\big).$$

For simplicity, we will refer to $f^\star$ sometimes as the Legendre dual, or just dual, of f. Usually, f is assumed to be convex but it need not be– and in our applications, it often is not.

5.1. **Generalities on the Legendre transform.** Consider the Legendre transform f^* of a convex functions f on $\mathbb{R}$. Then $f^*(\xi) = x_\xi\xi - f(x_\xi)$ where $f'(x_\xi) = \xi$. One thinks of $f^*(\xi)$ as a function of 'slopes' ξ and x_ξ is the unique point where the graph of f has slope ξ. The tangent line at this point is $y = \xi(x-x_\xi)+f(x_\xi)$ and its y-intersect is $-\xi x_\xi+f(x_\xi) = -f^*(\xi)$. Here $x_\xi = (f')^{-1}(\xi)$. I.e. $f'(x_\xi) = \xi$. Write $x_\xi = g(\xi)$. Then $f'(g(\xi)) = \xi$ so $g'(\xi) = \frac{1}{f''(g(\xi))}$. Note also that $\frac{d}{d\xi}f^*(\xi) = g(\xi) + (\xi - f'(g(\xi)))g'(\xi) = g(\xi)$ and

$$\frac{d^2}{d\xi^2}f^*(\xi) = \frac{dg}{d\xi} = \frac{1}{f''(g(\xi)}.$$

Hence f^* is convex and f, f^* have innverse derivatives and Hessians.

One often writes $x\xi = f(x)+f^*(\xi)$ but it is understood that there is only one independent variable and either $x = x_\xi = g(\xi)$ or the inverse.

5.2. **Symplectic potential.** By (9), a $\mathbf{T}^m$-invariant Kähler potential defines a real convex function on $\rho \in \mathbb{R}^m$. Its Legendre dual is the *symplectic potential* u_φ: for $x \in P$ there is a unique ρ such that $\mu_\varphi(e^{\rho/2}) = \nabla_\rho\varphi = x$. Then the Legendre transform is defined to be the convex function

$$u_\varphi(x) = \langle x, \rho_x\rangle - \varphi(\rho_x), \quad e^{\rho_x/2} = \mu_\varphi^{-1}(x) \iff \rho_x = 2\log\mu_\varphi^{-1}(x) \tag{15}$$

on P. The gradient $\nabla_x u_\varphi$ is an inverse to μ_{ω_φ} on $M_\mathbb{R}$ on the open orbit, or equivalently on P, in the sense that $\nabla u_\varphi(\mu_{\omega_\varphi}(z)) = z$ as long as $\mu_{\omega_\varphi}(z) \notin \partial P$.

The symplectic potential has canonical logarithmic singularities on ∂P. There is a one-to-one correspondence between $\mathbf{T}^m_\mathbb{R}$-invariant Kähler potentials ψ on M_P and symplectic potentials u in the class S of continuous convex functions on $\bar{P}$ such that $u - u_0$ is smooth on $\bar{P}$ where

$$u_0(x) = \sum_k \ell_k(x)\log\ell_k(x). \tag{16}$$

Thus, $u_\varphi(x) = u_0(x) + f_\varphi(x)$ where $f_\varphi \in C^\infty(\bar{P})$. We note that u_0 and u_φ are convex, that $u_0 = 0$ on ∂P and hence $u_\varphi = f_\varphi$ on ∂P. By convexity, $\max_P u_0 = 0$.

We denote by $G_\varphi = \nabla^2_x u_\varphi$ the Hessian of the symplectic potential. It has simple poles on ∂P. It follows that $\nabla^2_\rho\varphi$ has a kernel along $\mathcal{D}$. The kernel of $G_\varphi^{-1}(x)$ on $T_x\partial P$ is the linear span of the normals μ_r for $r \in \mathcal{F}(x)$. We also denote by $H_\varphi(\rho) = \nabla^2_\rho\varphi(e^\rho)$ the Hessian of the Kähler potential on the open orbit in ρ coordinates. By Legendre duality,

$$H_\varphi(\rho) = G_\varphi^{-1}(x), \quad \mu(e^\rho) = x. \tag{17}$$

This relation may be extended to $\mathcal{D} \to \partial P$. The kernel of the left side is the Lie algebra of the isotropy group G_p of any point $p \in \mu^{-1}(x)$.

5.3. **Repetition.** Since it is $\mathbf{T}$-invariant, the Kähler potential may be identified with a smooth strictly convex function on $\mathbb{R}^n$ in logarithmic coordinates. Therefore its gradient $\nabla\psi$ is one-to-one onto the convex polytope $P = \overline{\mathrm{Im}\,\nabla\psi}$ and one has the following explicit expression for its Legendre dual ([Ro])

$$u(y) = \psi^\star(y) = \langle y, (\nabla\psi)^{-1}(y)\rangle - \psi\circ(\nabla\psi)^{-1}(y), \tag{18}$$

which is a smooth strictly convex function on P, satisfying

$$\nabla u(y) = (\nabla\psi)^{-1}(y), \tag{19}$$

and

$$\left(\nabla^2 u(y)\right)^{-1} = \nabla^2\psi((\nabla\psi)^{-1}(y)). \tag{20}$$

Following Guillemin, the function u is called the symplectic potential of the metric $\sqrt{-1}\partial\bar\partial\psi$. The space of all symplectic potentials is denoted by $\mathcal{LH}(\mathbf{T})$. Put

$$u_G := \sum_{k=1}^d l_k \log l_k, \tag{21}$$

where l_k are the defining linear functionals of the polytope P (we refer to [**?**, §4.2] for notation). A result of Guillemin states that for any symplectic potential u the difference $u - u_G$ is a smooth function on P (that is, up to the boundary). In other words,

$$\mathcal{LH}(\mathbf{T}) = \{u \in C^\infty(P\setminus\partial P)\,:\, u = u_G + F, \quad \text{with } F\in C^\infty(P)\}. \tag{22}$$

5.4. **Simplest example.** Let ω_{FS} denote the Fubini-Study form of constant Ricci curvature 1 on the Riemann sphere, given locally by

$$\omega_{\text{FS}} = \frac{\sqrt{-1}}{\pi}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}.$$

The associated open-orbit Kähler potential can be taken as

$$\psi_0(x) = \log(1+|z|^2) - \frac{1}{2}\mathrm{Re}\, z = \log(1+e^{2x}) - x. \tag{23}$$

The corresponding moment polytope is $[-1,1]$, and the symplectic potential dual to ψ_0 can be computed via the moment map $y(x) = \psi_0'(x) \in [-1,1]$,

$$u_0(y) = (1+y)\log(1+y) + (1-y)\log(1-y), \quad y\in[-1,1]. \tag{24}$$

Let $\dot\varphi_0 \in C^\infty(S^2)$ be given and set

$$\dot u_0 = -\dot\varphi_0((\psi_0')^{-1}(\,\cdot\,)) = -\dot\psi_0((\psi_0')^{-1}(\,\cdot\,)) \in C^\infty([-1,1]).$$

6. Geodesics in the space of toric Kaehler metrics: Linearization of the Monge-Ampère equation

On a a toric manifold manifold (M^n,ω), torus-invariant Cauchy data $(\omega_{\varphi_0},\dot\varphi_0)$ on the open-orbit, $M_o \equiv (\mathbb{C}^\star)^n$ has the form, $\omega_{\varphi_0} = \sqrt{-1}\partial\bar\partial\psi_0$, where ψ_0 is the open-orbit potential.

The HCMA in this setting reduces to a HRMA:

$$\begin{cases} \mathrm{MA}\,\psi = \det \nabla^2 \psi = 0, & \text{on } [0,T]\times\mathbb{R}^n, \\ \psi(0,\,\cdot\,) = \psi_0(\,\cdot\,), & \text{on } \mathbb{R}^n, \\ \dfrac{\partial \psi}{\partial s}(0,\,\cdot\,) = \dot{\varphi}_0(\,\cdot\,), & \text{on } \mathbb{R}^n, \end{cases} \tag{25}$$

for a convex function ψ on $[0,T]\times\mathbb{R}^n$ (properly defined for Lipschitz functions)

Recall that on a symplectic toric manifold the Legendre transform

$$f \mapsto \mathcal{L}f = f^\star$$

is a bijection between the set of $\mathbf{T}$-invariant Kähler potentials on the open orbit $M_0 \simeq (\mathbb{C}^n)^\star$ of the (complex) torus action

$$\mathcal{H}(\mathbf{T}) := \{\psi \in C^\infty(\mathbb{R}^n) \,:\, \sqrt{-1}\partial\bar{\partial}\psi = \omega_\varphi|_{M_0}$$

$$\text{with } \varphi \in \mathcal{H}_\omega \text{and} \overline{\mathrm{Im}\,\nabla\psi} = \mathrm{P}\},$$

and the set of 'symplectic potentials' on the moment polytope $P \subset \mathbb{R}^n$

$$\mathcal{L}\mathcal{H}_{\mathbf{T}^m} := \{u \in C^\infty(P \setminus \partial P) \cap C^0(P) \,:\, u = \psi^\star\},$$

with $\psi \in \mathcal{H}_{\mathbf{T}^m}$.

The Legendre transform linearizes the Monge-Ampère geodesic equation.

PROPOSITION **6.1.** *Let M_P^c be a toric variety. Then under the Legendre transform $\varphi \to u_\varphi$, the complex Monge-Ampére equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $u'' = 0$. Hence the Legendre transform of a geodesic φ_t has the form $u_t = u_0 + t(u_1 - u_0)$.*

Proof. It suffices to show that the energy functional

$$E = \int_0^1 \int_M \dot{\varphi}_t^2 d\mu_{\varphi_t} dt \tag{26}$$

is Euclidean on paths of symplectic potentials. For each t let us pushforward the integral $\int_M \dot{\varphi}_t^2 d\mu_\varphi$ under the moment map μ_{φ_t}. The integrand is by assumption invariant under the real torus action, so the pushforward is a diffeomorphism on the real points. The volume measure $d\mu_{\varphi_t}$ pushes forward to dx. The function $\partial_t\varphi_t(\rho)$ pushes forward to the function $\psi_t(x) = \dot{\varphi}_t(\rho_{x,t})$ where $\mu_{\varphi_t}(\rho_{x,t}) = x$. By (15), the symplectic potential at time t is

$$u_t(x) = \langle x, \rho_{x,t}\rangle - \varphi_t(\rho_{x,t}).$$

We note that

$$\dot{u}_t = \langle x, \partial_t\rho_{x,t}\rangle - \dot{\varphi}_t(\rho_{x,t}) - \langle \nabla_\rho\varphi_t(\rho_{x,t}), \partial_t\rho_{x,t}\rangle. \tag{27}$$

The outer terms cancel, and thus, our integral is just

$$\int_0^1 \int_P |\dot{u}_t|^2 dx dt.$$

Clearly the Euler-Lagrange equations are linear.

□

6.1. **The Cauchy problem for the symplectic potential.** It is well-known that the Legendre transform linearizes the HRMA. This fact also has a geometric interpretation that we now briefly review.

Let (M, ω) be a toric Kähler manifold of complex dimension n and let $\mathbf{T} = (S^1)^n$ denote the real torus of dimension n which acts on (M, ω) in a Hamiltonian fashion. We denote by $\mathcal{H}(\mathbf{T})$ the class of $\mathbf{T}$-invariant Kähler metrics in the cohomology class of ω. On the open-orbit of $\mathbf{T}^{\mathbb{C}} = (\mathbb{C}^\star)^n$, a $\mathbf{T}$-invariant Kähler metric has a Kähler potential ψ and we also write $\psi \in \mathcal{H}(\mathbf{T})$. The Legendre transform is an isometry between $(\mathcal{H}(\mathbf{T}), g_{L^2})$ and $(\mathcal{L}\mathcal{H}(\mathbf{T}), L^2(P))$. It transforms the Christoffel symbols of $(\mathcal{H}(\mathbf{T}), g_{L^2})$ to zero and thus linearizes the Monge-Ampère equation to the equation $\ddot{u} = 0$. differential of the Legendre transform acts as minus the identity, that is if η_s is a curve in $\mathcal{H}(\mathbf{T})$ and if $u_s := \eta_s^\star$ are the corresponding symplectic potentials then

$$\dot{\eta}_s = -\dot{u}_s \circ \nabla \eta_s. \tag{28}$$

Therefore the IVP on $(\mathcal{H}(\mathbf{T}), g_{L^2})$ is transformed to the following initial value problem for geodesics in the space of symplectic potentials:

$$\ddot{u} = 0, \quad , u_0 = \psi_0^\star, \quad \dot{u}_0 = -\dot{\psi}_0 \circ (\nabla \psi_0)^{-1}. \tag{29}$$

It may now seem that solving the geodesic equation with toric initial data is trivial: use that $\mathcal{L}$ is an isometry from $\mathcal{H}$ to $L^2(P)$ and thus transforms the IVP geodesic equation to the linear equation

$$\ddot{u} = 0, \quad u_0 = \psi_0^\star, \quad \dot{u}_0 = -\dot{\psi}_0 \circ (\nabla \psi_0)^{-1}, \tag{30}$$

whose solution is given by

$$u_s := u_0 + s\dot{u}_0.$$

The problem is transforming back. It gives a simple illustration of many of the problems involved in solving the geodesic equation.

Define the convex lifespan by

$$T^{\rm cvx}_{span}(\psi_0, \dot{\psi}_0) := \sup \{ \, s \, : \, \psi_0^\star - s\dot{\psi}_0 \circ (\nabla \psi_0)^{-1} \text{is convex on P}$$

If $s < T^{cvx}_{span}$, i.e., u_s is strictly convex and hence belongs to $\mathcal{L}\mathcal{H}_{\mathbf{T}^m}$, it is well-known that the IVP for geodesics has an explicit solution,

$$\psi(s, x) = \psi_s(x) := (u_0 + s\dot{u}_0)^\star(x),$$

$$s \in [0, T_{span}), \; x \in \mathbb{R}^n.$$

But the Legendre transform potential φ ceases to solve the HCMA for any $T > T^{cvx}_{span}$. Let $\Delta(\psi) := \{ \, (s, x) \, : \, \psi \in \mathcal{H}_{\mathbf{T}^m}$ is finite, differentiable at $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^n \}$ denote the regular locus of ψ, and let

$$\Sigma_{sing} := \mathbb{R}_+ \times \mathbb{R}^n \; - \Delta(\psi),$$

denote its singular locus. Since ψ is everywhere finite, the former is dense while the latter has Lebesgue measure zero in $\mathbb{R}_+ \times \mathbb{R}^n$. Set,

$$\Sigma_{sing}(T) := [0, T] \times \mathbb{R}^n \; - \Delta(\psi).$$

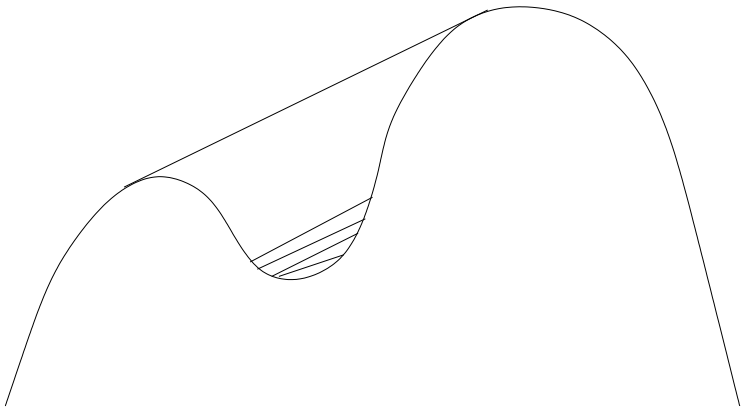

FIGURE 2. The graphs of $-\dot{u}_0$ and $-(\dot{u}_0^{**})$ over P.

6.2. **Complexifying Hamiltonian flows on toric manifolds.**

LEMMA **6.2.** *Let (M, J, ω) be a toric Kähler manifold. Given a toric Kähler potential φ_0 let ψ_0 be a smooth strictly convex function on $\mathbb{R}^n$ such that over the open orbit $\omega_{\varphi_0} = \sqrt{-1}\partial\bar{\partial}\psi_0$, and let $\dot{\varphi}_0$ be a smooth torus-invariant function on M. For every $z \in M_{\rm o}$, the orbit of the Hamiltonian vector field $X_{\dot{\varphi}_0}^{\omega_{\varphi_0}}$ admits an analytic continuation to the strip S_∞. Moreover, it is given explicitly by*

$$f_\tau(z) = \exp -\sqrt{-1}\tau X_{\dot{\varphi}_0}^{\omega_{\varphi_0}} : z \mapsto z - \tau(\nabla^2\psi_0)^{-1}\nabla_x\dot{\varphi}_0, \quad \tau \in S_\infty. \tag{31}$$

This expression remains valid on the divisor at infinity if we restrict to the orbit coordinates $\tilde{x}$ on a slice containing z.

Here (and in similar expressions below) by $(\nabla^2\psi_0)^{-1}\nabla_x\dot{\varphi}_0$ we mean the usual matrix multiplication of the matrix $(\nabla^2\psi_0)^{-1}(x)$ and the vector $\nabla_x\dot{\varphi}_0(x)$.

Proof. The moment coordinates y on the polytope P and the angular coordinates on the regular orbits are action-angle coordinates for the $(S^1)^n$ Hamiltonian action on (M, ω_{φ_0}), in other words

$$(\nabla\psi_0)_\star\omega_{\varphi_0} = \sum_{j=1}^n dy_j \wedge d\theta_j, \quad \text{over} \quad (P \setminus \partial P) \times (S^1)^n, \tag{32}$$

where $(\nabla\psi_0)_\star$ denotes the push-forward under the diffeomorphism $\nabla\psi_0$. The Hamiltonian vector field of $\dot{\varphi}_0$ is given in these coordinates by

$$(\nabla\psi_0)_\star X_{\dot{\varphi}_0}^{\omega_{\varphi_0}} = -\sum_{j=1}^n \frac{\partial\dot{\varphi}_0}{\partial y_j}\big((\nabla\psi_0)^{-1}(y)\big)\frac{\partial}{\partial\theta_j}, \quad y \in P \setminus \partial P. \tag{33}$$

Therefore the Hamiltonian flow of $X_{\dot{\varphi}_0}^{\omega_{\varphi_0}}$ is given, in terms of the moment coordinates, by

$$\nabla\psi_0 \circ \exp tX_{\dot{\varphi}_0}^{\omega_{\varphi_0}} \circ (\nabla\psi_0)^{-1}.(y,\theta) = (y, \theta - t\nabla_y\dot{\varphi}_0 \circ (\nabla\psi_0)^{-1}), \quad \text{over} \quad (P \setminus \partial P) \times (S^1)^n, \tag{34}$$

and in terms of the coordinates on $M_{\rm o}$ by

$$\exp tX_{\dot{\varphi}_0}^{\omega_{\varphi_0}}.(x,\theta) = (x, \theta - t(\nabla^2\psi_0)^{-1}\nabla_x\dot{\varphi}_0).$$

It therefore admits a holomorphic extension to a map $\exp \sqrt{-1}\tau X^{\omega_{\varphi_0}}_{\dot\varphi_0}, \tau = s + \sqrt{-1}t$, given in these coordinates by

$$\exp -\sqrt{-1}\tau X^{\omega_{\varphi_0}}_{\dot\varphi_0}.(x,\theta) = (x - s(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0\,,\ \theta - t(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0), \quad s \in \mathbb{R}_+, t \in \mathbb{R}. \tag{35}$$

For each $z \in M_{\rm o}$, this is a holomorphic map of S_∞ into $M_{\rm o} \subset M$ since in terms of the complex coordinates $z_j := x_j + \sqrt{-1}\theta_j$ it is given by an affine map

$$\tau \mapsto z - \tau(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0, \quad \tau \in S_\infty.$$

□

6.3. **Moser flows on toric manifolds.** Having derived an explicit expression for the analytic continuations of the Hamiltonian orbits for all imaginary time, we now turn to investigate the invertibility of the resulting Moser maps.

LEMMA **6.3.** *Let (M, J, ω) be a toric Kähler manifold. Given a toric Kähler potential φ_0 let ψ_0 be a smooth strictly convex function on $\mathbb{R}^n$ such that over the open orbit $\omega_{\varphi_0} = \sqrt{-1}\partial\bar\partial\psi_0$, and let $\dot\varphi_0$ be a smooth torus-invariant function on M. The Moser maps $f_s(z) = \exp -\sqrt{-1}sX^{\omega_{\varphi_0}}_{\dot\varphi_0}.z$ defined by Lemma 6.2 are smoothly invertible if and only if*

$$s < T^{\rm cvx}_{\rm span} := \sup\{\, a > 0 : \psi_0^\star - a\dot\varphi_0 \circ (\nabla\psi_0)^{-1} \text{ is convex}\}. \tag{36}$$

Note that the formula for $T^{\rm cvx}_{\rm span}$ is well-defined independently of the choice of the open-orbit Kähler potential ψ_0 for ω_{φ_0}.

LEMMA **6.4.** *Let ψ_s be a smooth solution of the HRMA (25), and let f_s denote the associated Moser diffeomorphisms given by Lemma 6.2. Then on the open-orbit,*

$$f_s^{-1} = (\nabla\psi_0)^{-1} \circ \nabla\psi_s = \nabla u_0 \circ (\nabla u_s)^{-1}, \quad s \in [0, T^{\rm cvx}_{\rm span}), \tag{37}$$

and if we let $u_s(y) = u_0(y) + s\dot u_0(y)$, then

$$f_s = \nabla u_s \circ (\nabla u_0)^{-1}, \quad \textit{all}\ \ s \geq 0. \tag{38}$$

These expressions remain valid globally on M if we use the Euclidean gradient in the orbit coordinates $\tilde x$ along each slice.

Proof. From the proof of Lemma 6.2 (cf.(35)) we have the following formula for the Moser maps, restricted to the open orbit,

$$f_s(z) = z - s(\nabla^2\psi_0)^{-1}\nabla_x\dot\varphi_0, \quad \tau \in S_\infty, z \in M_{\rm o}, \tag{39}$$

or in terms of the moment coordinates

$$f_s(\nabla u_0(y)) = \nabla_y u_0(y) + s\nabla_y \dot u_0, \quad s \in \mathbb{R}_+, y \in P \setminus \partial P. \tag{40}$$

Since $\nabla_y = \nabla^2_y u_0.\nabla_x$, applying the gradient with respect to y to this equation we obtain

$$\nabla^2 u_0(y).\nabla_x f_s(\nabla u_0(y)) = \nabla^2_y(u_0 + s\dot u_0).$$

Since $\nabla^2 u_0$ is invertible for $y \in P \setminus \partial P$, it follows that the gradient of f_s is invertible at $z \in M_{\rm o}$ if and only if $u_0 + s\dot u_0$ is strictly convex on $P \setminus \partial P$. The analysis for $z \in M \setminus M_{\rm o}$ is similar, following the technicalities outlined in the proof of Lemma 6.2. Since by definition $u_0 = \psi_0^\star$, we obtain (36). □

6.4. **Review of convex analysis.** A geometric, definition of Monge-Ampere mass is due to Alexandrov, and uses the notion of a subdifferential of a convex function.

PROPOSITION **6.5.** (See [RT], Section 2) *For any convex function f, the measure $\mathcal{MA}\, f$, defined by*

$$(\mathcal{MA}\, f)(E) := \text{Lebesgue measure of } \partial f(E),$$

is a Borel measure.

The following result of Rauch-Taylor links these two definitions and will be crucial below.

THEOREM **6.6.** (See [RT], Proposition 3.4) *For every convex function f on $\mathbb{R}^{n+1}$ one has the equality of Borel measures $\mathrm{MA} f = \mathcal{MA}\, f$. In particular, the real Monge-Ampère measure is zero if and only if the image of the subdifferential map has Lebesgue measure zero in $\mathbb{R}^{n+1}$.*

Recall the following duality between differentiability and strict convexity.

LEMMA **6.7.** (See [Ro], Theorem 26.3.) *A closed proper convex function is essentially strictly convex if and only if its Legendre dual is essentially smooth.*

THEOREM **6.8.** *Suppose the initial Kaehler potential $\psi_0 \in \mathcal{H}(T)$ and velocity $\dot{\varphi}_0 \in T_{\psi_0}\mathcal{H}(T)$ are torus-invariant.Then:*
(i) The Cauchy problem for HCMA has a unique smooth solution for any $T \leq T_{\mathrm{span}}$ given by

$$\varphi_t = \psi_t - \psi_0 \in \mathcal{H}_\omega, \quad t \in [0, T], \tag{41}$$

where

$$\psi_t = \mathcal{L}(\mathcal{L}\psi_0 - t\dot{\varphi}_0 \circ (\nabla\psi_0)^{-1}) \in \mathcal{H}(T), \tag{42}$$

and

$$T_{\mathrm{span}} := \sup\{t > 0 : \mathcal{L}\psi_0 - t\dot{\varphi}_0 \circ (\nabla\psi_0)^{-1} \text{ is convex}\}.$$

Moreover, there exists no admissible solution to the Cauchy problems for any $T > T_{\mathrm{span}}$.

PROPOSITION **6.9.** *Let (M, ω) be a Kähler toric manifold and let $\dot{\varphi}_0$ be a torus-invariant function.*

- *Each Hamilton orbit $\Gamma_z(it) = \exp tX^\omega_{\dot{\varphi}_0}(I, \theta)z$ admits a global holomorphic extension in time t to a half-plane;*
- *One can always solve HCMA leafwise along any complexified orbit;*
- *However, the map $f_s(z) = \Gamma_z(s)$ is a diffeomorphism if and only if $u_0 + \tau\dot{u}$ is convex.*

Thus, only the third problem with Donaldson's formal solution arises in the toric case, but it is enough to destroy the IVP for $(\omega_{\varphi_0}, \dot{\varphi}_0)$ unless $-\dot{u}_0(x) = \dot{\varphi}_0(\nabla\varphi_0^{-1}(x))$ is convex.

To see that orbits analytically continue, we write them in action-angle variables. We denote by $\beta(e^{i\theta}) \cdot z$ the action of $e^{i\theta} \in \mathbf{T}^m$ on M Let $\mu_0 = (I_1, \ldots, I_m)$ be the moment map with respect to ω of the $\mathbf{T}^m$ action, i.e. $(I_1, \ldots, I_m)(\rho) = \mu_0(e^{\rho/2})$. Then the orbits of the $\mathbf{T}^m$ action are given in these action-angle coordinates by

$$\beta(e^{i\theta}) \cdot (I, \theta) = (I, \theta + \theta_0).$$

The real torus $\mathbf{T}^m$ acts holomorphically on M and that its complexification $(\mathbb{C}^*)^m$ acts holomorphically.

The Hamiltonian flow of $\dot{\varphi}_0$ w.r.t. ω_{φ_0} can be explicitly solved in "action-angle" coordinates (I, θ). Action coordinates= moment map, angle coordinates come from real torus action.

PROPOSITION **6.10.** *Each orbit admits a analytic continuation in t. Df fails to be invertible once $u + \tau\dot{u}$ fails to be convex.*

Proof. The orbit is

$$\exp tX^{\omega_{\varphi_0}}_{\dot\varphi_0}(z_0) = \rho_0/2 + i\theta_0 + it\nabla_I\dot\varphi_0(z_0).$$

It is linear in t, hence admits an AC in t. So "good" hypothesis is satsified.

THEOREM **1.** *(i) The Legendre potential ψ solves the HRMA on the dense regular locus,*

$$\mathrm{MA}\psi = 0 \quad \text{on} \quad \Delta(\psi) \subset \mathbb{R}_+ \times \mathbb{R}^{\mathrm{n}}.$$

In addition, $[0, T^{cvx}_{span}] \times \mathbb{R}^n \subset \Delta(\psi)$.
(ii) Whenever $T > T^{cvx}_{span}$, ψ fails to solve the HRMA. In particular, the Monge-Ampère measure of ψ charges the set $\Sigma_{sing}(T)$ with positive mass,

$$\int_{[0,T]\times\mathbb{R}^n} \mathrm{MA}\psi = \int_{\Sigma_{sing}(T)} \mathrm{MA}\psi > 0.$$

Equivalently, $\varphi = \varphi_\infty$ ceases to solve the HCMA when $T > T^{cvx}_{span}$. However, it does solve the HCMA on a dense set in $S_T \times M$.

6.5. **Ideas of proofs.**

PROPOSITION **6.11.** *Let (M, ω) be a Kähler toric manifold and let $\dot\varphi_0$ be a torus-invariant function.*

- *Each Hamilton orbit $\Gamma_z(it) = \exp tX^\omega_{\dot\varphi_0}(I, \theta)z$ admits a global holomorphic extension in time t to a half-plane;*
- *One can always solve HCMA leafwise along any complexified orbit;*
- *However, the map $f_s(z) = \Gamma_z(s)$ is a diffeomorphism if and only if $u_0 + \tau\dot{u}$ is convex.*

Thus, only the third problem with Donaldson's formal solution arises in the toric case, but it is enough to destroy the IVP for $(\omega_{\varphi_0}, \dot\varphi_0)$ unless $-\dot{u}_0(x) = \dot\varphi_0(\nabla\varphi_0^{-1}(x))$ is convex.

The real torus $\mathbf{T}^m$ acts holomorphically on M and that its complexification $(\mathbb{C}^*)^m$ acts holomorphically.

To see that orbits analytically continue, we write them in action-angle variables. We denote by $\beta(e^{i\theta}) \cdot z$ the action of $e^{i\theta} \in \mathbf{T}^m$ on M Let $\mu_0 = (I_1, \dots, I_m)$ be the moment map with respect to ω of the $\mathbf{T}^m$ action, i.e. $(I_1, \dots, I_m)(\rho) = \mu_0(e^{\rho/2})$. Then the orbits of the $\mathbf{T}^m$ action are given in these action-angle coordinates by

$$\beta(e^{i\theta}) \cdot (I, \theta) = (I, \theta + \theta_0).$$

6.6. **Loss of convexity = intersection of characteristics.** We compute the derivative Df_τ in action-angle variables to obtain,

$$Df_\tau = \begin{pmatrix} I & 0 \\ \\ 0 & \frac{\partial\rho}{\partial I} - \tau\frac{\partial^2\hat{H}}{\partial I_j \partial I_k} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \\ 0 & \nabla^2_I(u_0 + \tau\dot{u}) \end{pmatrix}.$$

Here, we observe that $\hat{H} = -\dot{u}$ and that $\partial\rho/\partial I = \nabla^2 u_0(I)$. Indeed, $\rho(I)$ is the inverse of the moment map $\mu(\rho)$ so $\partial\rho/\partial I$ is the inverse of $D\mu$. But the inverse of μ is $\nabla_x u_0$ and the inverse of $D\mu = \nabla^2\rho$ is $\nabla^2_x u_0$.

It follows that

$$\det Df_\tau(I,\theta) = 0 \iff \det(\nabla_I^2(u_0 + \tau\dot{u})) = 0.$$

Clearly, f_τ is a diffeomorphism wherever $u_0 + \tau\dot{u}$ is convex. On the other hand, it fails to be a diffeomorphism at points (θ, I) where $\det(\nabla_I^2(u_0 + \tau\dot{u})) = 0$.

Since $\omega_t = f_{-t}^*\omega_0$ we see that ω_t and φ_t have singularities wherever f_t fails to be invertible. □

7. HRMA and the Hamilton-Jacobi equation

Next, we show that in the case of the HRMA the leafwise obstruction vanishes and characterizes the Legendre transform subsolution among all subsolutions of the Cauchy problem.

Proposition **7.1.** *(i) The Legendre transform potential, given by*

$$\psi_L(s,x) := (\psi_0^\star - s\dot{\varphi}_0 \circ (\nabla\psi_0)^{-1})^\star(z), \quad x \in \mathbb{R}^n,\ s \in \mathbb{R}_+, \tag{43}$$

is the unique admissible leafwise subsolution to the HRMA (25) for all $T > 0$.
(ii) The corresponding unique admissible leafwise subsolution to the HCMA is given by

$$\varphi_L(s + \sqrt{-1}t, e^{x+\sqrt{-1}\theta}) := \psi_L(s,x) - \psi_0(x). \tag{44}$$

The proof uses the following characterization of the HRMA in terms of a Hamilton–Jacobi equation:

Theorem **7.2.** (HRMA and Hamilton–Jacobi) *$\eta \in C^1([0,T] \times \mathbb{R}^n)$ is an admissible weak solution of the HRMA (25) if and only if it is a classical solution of the Hamilton–Jacobi equation*

$$F(\nabla\eta) = 0, \qquad \eta(0,\cdot) = \psi_0, \tag{45}$$

where $F(\sigma,\xi) = \sigma - \dot{\psi}_0 \circ (\nabla\psi_0)^{-1}(\xi)$, where $\sigma \in \mathbb{R}, \xi \in \mathbb{R}^n$.

Recall that the initial Neumann data $\dot{\psi}_0$ of the HRMA (25) is a bounded function on $\mathbb{R}^n$ obtained by restricting the global Neumann data $\dot{\varphi}_0$ on the toric manifold to the open-orbit.

Proof of Theorem 7.2. Given the Cauchy data $(\psi_0, \dot{\psi}_0)$ of (25), we set

$$\dot{u}_0 := -\dot{\psi}_0 \circ (\nabla\psi_0)^{-1}.$$

Lemma **7.3.** *Let η be a C^1 admissible solution for the HRMA (25). Define the set-valued map,*

$$G : s \in \mathbb{R}_+ \mapsto \text{Im } \nabla\eta(\{s\} \times \mathbb{R}^n) \subset \mathbb{R}^{n+1}.$$

Then $G(s) = G(0) = \{(-\dot{u}_0(y), y) : y \in P \setminus \partial P\}$, for each $s \in [0,T)$.

The proof is in [RZAIM].

Thus, by Lemma 7.3 and the differentiability assumption, for each $(s,x) \in [0,T] \times \mathbb{R}^n$ there exists a unique $y \in P \setminus \partial P$ such that

$$\Big(\frac{\partial\eta}{\partial s}(s,x), \nabla_x\eta(s,x)\Big) = (-\dot{u}_0(y), y),$$

or, in other words,

$$\frac{\partial\eta}{\partial s}(s,x) = -\dot{u}_0 \circ \nabla_x\eta(s,x), \tag{46}$$

which concludes the proof of one direction of Theorem 7.2.

For the converse, suppose that $\eta \in C^1([0,T]\times\mathbb{R}^n)$ is a solution of the Hamilton–Jacobi equation. Then $\operatorname{Im}\nabla\eta \subset G(0)$, and since $G(0)$ has zero Lebesgue measure in $\mathbb{R}^{n+1}$, η is a weak solution of the HRMA. □

PROPOSITION **7.4.** *Let (M,J,ω_{φ_0}) be a toric Kähler manifold, and let $\dot\varphi_0 \in C^\infty(M)$ be torus-invariant. Assume that the corresponding Cauchy problem for the HCMA is T-good. Then any C^1 $\pi_2^\star\omega$-psh solution of the HCMA up to time T is the unique T-leafwise subsolution.*

Proof of Proposition 7.4. Let ψ_L denote the leafwise subsolution of the HRMA (25) given by Proposition 7.1 and let η be a C^1 admissible solution of (25). Both ψ_L and η are convex functions on $[0,T]\times\mathbb{R}^n$. By Theorem 7.2 both ψ_L and η are solutions of the Hamilton–Jacobi equation (45). The method of characteristics implies that C^1 solutions of (45) are unique as long as the characteristics of the equation do not intersect each other. The equation for the projected characteristic curves $\mathbf{x}(s)$ is

$$\mathbf{x}(s) = \big(s, x_0 + s\nabla\dot u_0(\nabla\psi_0(x_0))\big).$$

Thus, the projected characteristic do not intersect as long as the map $(s,x) \mapsto (s, x + s\nabla\dot u_0(\nabla\psi_0(x)))$ is invertible, or equivalently as long as

$$x \mapsto \nabla u_0 \circ \nabla\psi_0(x) + s\nabla\dot u_0 \circ \nabla\psi_0(x) \tag{47}$$

is invertible on $\mathbb{R}^n$; this is precisely as long as $\nabla u_0 + s\nabla\dot u_0$ is invertible on $P\setminus\partial P$, or as long as $u_0 + s\dot u_0$ is strictly convex, i.e., precisely for $s < T^{\rm cvx}_{\rm span}$.

□

8. Appendix on Moser maps and analytic continuation of orbits

Finally, we relate the orbits of the Moser map, the Hamiltonian orbits, and the characteristics in $\mathbb{R}^{n+1}$ of the HRMA. The following generalizes to weak C^1 solutions of the HRMA the well-known 'conservation law' of smooth solutions of the HCMA.

DEFINITION **8.1.** *Assume f_s is invertible as a C^1 map. Define $\dot\varphi_s$, ω_s resp. X_s by*

$$\dot\varphi_s := \dot\varphi_0\circ f_s^{-1}, \quad \omega_s = (f_s^{-1})^\star\omega_0, \quad \frac{d}{dt} f_{s+\sqrt{-1}t} =: X_s\circ f_{s+\sqrt{-1}t}, \quad \varphi(s+\sqrt{-1}t) := \varphi_0 + \int_0^s \dot\varphi_\sigma d\sigma.$$

Since $\Gamma_z : S_T \to M$ is a holomorphic map we have

$$\frac{d}{dt}\Gamma_z(s+\sqrt{-1}t) = -J\frac{d}{ds}\Gamma_z(s+\sqrt{-1}t) \implies \frac{d}{ds} f_{s+\sqrt{-1}t}(z) = -JX_s\circ f_{s+\sqrt{-1}t}. \tag{48}$$

PROPOSITION **8.2.** *Let η be a C^1 weak solution of the HRMA (25), and let $\varphi = \eta - \psi_0$, considered as a function M. Also, let f_s be the Moser maps $f_s(z)$ Then*

$$\dot\varphi_s \circ f_s = \dot\varphi_0.$$

Further, the f_s-orbits $(s, f_s(x))$ are the leaves of the real Monge–Ampère foliation, namely the projected characteristics of the Hamilton–Jacobi equation (45).

Proof. By combining (46) and and Proposition 7.4, one sees that this equation is equivalent to the Hamilton–Jacobi equation in Theorem 7.2.

To prove the last statement we note that the leaves of the Monge–Ampère foliation are orbits of the complexified Hamiltonian action $\exp tX_{\dot\varphi_0}^{\omega_{\varphi_0}}$. The real orbits lie on the orbits of

the Hamiltonian $(S^1)^n$-action and the real slice of this torus orbit is a point. Hence the real slice is the imaginary time orbit, i.e., the orbit of f_s. □

LEMMA **8.3.** *Suppose that f_τ is defined by analytic continuation of the Hamilton flow of $(\dot\varphi_0, \omega_{\varphi_0})$. Then,*

$$f_{s+\sqrt{-1}t} = h_{s+\sqrt{-1}t} \circ f_s. \tag{49}$$

Consequently, $X_s = -JX^{\omega_s}_{\dot\varphi_s}$, so that ; $\frac{d}{ds} f_{s+\sqrt{-1}t}(z) = -JX^{\omega_s}_{\dot\varphi_s} \circ f_{s+\sqrt{-1}t}$ (see (48) and Definition 8.1).

9. Appendix: Lagrangian submanifolds and Legendre transformation

Let $\Lambda \subset T^*\mathbb{R}^d$ be a Lagrangian submanifold, so that

$$\Lambda = \{(x, \xi = \nabla S(x)) : x \in U\}.$$

The transformation from a generating function $S(x)$ to a generating function $S*(\xi)$ is a Legendre transform. If Λ can be represented as is projectible to the ξ- variables we define $F(\xi) = S(x)\langle \xi, x\rangle$ and look for its critical points $x_\xi \in U$. These points solve $d_x S(x_\xi) = \xi$, so by our assumption on W there exists a unique solution $x_\xi \in U$, corresponding to the unique point $(x_\xi, \xi) \in \Lambda$ with momentum coordinate ξ. If we now take $S^*(\xi) = S(x_\xi) - \langle x_\xi, \xi\rangle$, we find that it indeed generates Λ:

$$\nabla_\xi S^*(\xi) = \frac{\partial x_\xi}{\partial \xi}\,(\nabla_x S(x_\xi - \xi) - x_\xi = -x_\xi.$$

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