

Problem 1

Part 1

For $x \neq 0$ the denominator has a faster rate of growth, so $f_n(x) \rightarrow 0$. When $x = 0$, $f_n(0) = 0 \rightarrow 0$. So $f(x) \equiv 0$.

Part 2

It is not uniform. Since $f_n(x) = f_1(nx)$, and $x \mapsto nx$ is a bijection on the domain $[0, \infty)$, it follows that $\sup_{[0, \infty)} |f_n - 0| = \sup_{[0, \infty)} |f_1| > 0$ for all n , so in particular $\lim_{n \rightarrow \infty} \sup_{[0, \infty)} |f_n - 0| \neq 0$.

Part 3

For all $1 \leq x$,

$$\begin{aligned} |f_n(x)| &= \left| \frac{nx}{1+n^2x^2} \right| \leq \frac{n|x|}{1+n^2|x|^2} \leq \frac{n|x|^2}{1+n^2|x|^2} \\ &= \frac{1}{n} \cdot \frac{n^2|x|^2}{1+n^2|x|^2} \leq \frac{1}{n}, \end{aligned}$$

so $\sup_{x \geq 1} |f_n(x)| \rightarrow 0$.

Problem 2

Part 1

$$R = \limsup_{n \rightarrow \infty} \frac{1}{\left| \frac{1}{n2^n} \right|^{1/n}} = \limsup_{n \rightarrow \infty} n^{1/n} \cdot 2 = 2$$

Part 2

The endpoints are $x = 1$ and $x = 5$. At $x = 5$ we have the harmonic series which diverges. At $x = 1$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by alternating series test. So the answer is $[1, 5)$.

Problem 3

Place a corner of this box at a point (x, y, z) on the part of the ellipsoid lying within the $(+, +, +)$ octant. Then the volume of the box is $(2x)(2y)(2z) = 8xyz$. The compactness of the ellipsoid and continuity of $(x, y, z) \mapsto xyz$ implies the existence of a maximum. Moreover the gradient

$$\nabla \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = (2x/a^2, 2y/b^2, 2z/c^2)$$

never vanishes on the ellipsoid. So at the maximizer, which we shall call (x, y, z) , the equations from Lagrange multipliers hold:

$$\begin{cases} 8yz = \frac{2x}{a^2} \lambda \\ 8xz = \frac{2y}{b^2} \lambda \\ 8xy = \frac{2z}{c^2} \lambda \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases}$$

Multiplying the first, second, and third equations by x , y , and z respectively, we find that

$$\frac{x^2}{a^2} \lambda = \frac{y^2}{b^2} \lambda = \frac{z^2}{c^2} \lambda.$$

Now take cases on λ . If $\lambda = 0$, then going back to the original equations we find that at least two of $\{x, y, z\}$ must be zero, in which case the volume is $8xyz = 0$ which is one possible extremum (clearly not the maximum). If instead $\lambda \neq 0$, then

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

By the fourth equation, we in fact get that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}.$$

So $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, and $z = c/\sqrt{3}$. This gives a maximum volume of $8abc/\sqrt{27}$.

Problem 4

There are a few fun ways to do this using complex numbers, but let's not. Integrating by parts,

$$\begin{aligned} I &:= \int_0^\infty e^{-x} \cos x \, dx = -e^{-x} \cos x \Big|_{x=0}^\infty - \int_0^\infty -e^{-x} (-\sin x) \, dx \\ &= 1 - \int_0^\infty e^{-x} \sin x \, dx = 1 - \left(-e^{-x} \sin x \Big|_{x=0}^\infty - \int_0^\infty -e^{-x} \cos x \, dx \right) \\ &= 1 - 0 - \int_0^\infty e^{-x} \cos x \, dx = 1 - I, \end{aligned}$$

so $2I = 1$ i.e. $I = 0.5$.

Problem 5

Let's apply Green's theorem. Since

$$\text{curl}(x^2y - y^3, x^3 + xy^2) = (3x^2 + y^2) - (x^2 - 3y^2) = 2x^2 + 4y^2,$$

we have that

$$\begin{aligned} \int_C (x^2y - y^3, x^3 + xy^2) \cdot ds &= \int_{x=0}^1 \int_{y=x^2}^x 2x^2 + 4y^2 \, dy \, dx \\ &= \int_{x=0}^1 2x^2(x - x^2) + \int_{y=x^2}^x 4y^2 \, dy \, dx = \int_{x=0}^1 2x^2(x - x^2) + \frac{4}{3}(x^3 - x^6) \, dx \\ &= \int_{x=0}^1 \frac{10}{3}x^3 - 2x^4 - \frac{4}{3}x^6 \, dx = \frac{5}{6} - \frac{2}{5} - \frac{4}{21} \\ &= \frac{13}{30} - \frac{4}{21} = \frac{91 - 40}{210} = \frac{51}{210} = \frac{17}{70} \end{aligned}$$

Though that wasn't much easier than just doing it directly. Split up C into C_1 , the $y = x^2$ part, and C_2 , the $y = x$ part. Then, from the parametrization $t \mapsto (t, t^2)$,

$$\int_{C_1} (x^2y - y^3, x^3 + xy^2) \cdot ds = \int_0^1 (t^4 - t^6, t^3 + t^5) \cdot (1, 2t) dt = \int_0^1 t^4 - t^6 + 2t^4 + 2t^6 \, dt = \frac{3}{5} + \frac{1}{7},$$

and from the parametrization $t \mapsto (t, t)$,

$$\int_{-C_2} (x^2y - y^3, x^3 + xy^2) \cdot ds = \int_0^1 (t^3 - t^3, t^3 + t^3) \cdot (1, 1) \, dt = \int_0^1 2t^3 \, dt = \frac{1}{2}.$$

So the full integral is

$$\frac{3}{5} + \frac{1}{7} - \frac{1}{2} = \frac{42 + 10 - 35}{70} = \frac{17}{70}.$$

Bleh.