

Quiz 8 Rubric

Problem

Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1, & \frac{1}{n+1} \leq x < \frac{1}{n} \text{ for } n \text{ odd,} \\ -1, & \frac{1}{n+1} \leq x < \frac{1}{n} \text{ for } n \text{ even,} \\ 0, & x = 0 \end{cases}.$$

Show that f is (Riemann) integrable over $[0, 1]$.

Rubric

- This was fairly easy. Solutions that are essentially correct should receive at least **8 points**.
- Deduct **1 point** if the student did not observe that f is bounded.
- Ideally the student should also point out that the domain $[0, 1]$ is bounded, but better to comment rather than deduct.
- To follow the sample solution, the student needs to observe that f is discontinuous exactly at $\{0\} \cup \{1/n : n = 2, 3, \dots\}$ and state that this has measure zero.
 - If the student misses the $x = 0$ discontinuity, comment but don't deduct.
 - The student can cite the fact that countable sets have measure zero. The student does not need to argue that the set is countable.
 - If the solution is incomplete and is following this exact outline, deduct **1 point** for each of the following items that are missing: (1) Stating the set of discontinuity. (2) Observing that it is countable. (3) Concluding it has measure zero.
 - There are, of course, other ways to argue that the set of discontinuities has measure zero.
- Some students may have tried showing it is integrable by showing that $U(f, \mathcal{P}) - L(f, \mathcal{P})$ can be made arbitrarily small. This can work but it needs to be done correctly.
- For solutions that are pretty incorrect, give up to **5 points** based on correct observations and understanding.

Solution 1

f is bounded on a bounded interval. Moreover f is discontinuous exactly when $x = 1/n$ (or also when $x = 0$), which is a countable set and thus has measure zero. So, by the theorem concerning Riemann integrability, we conclude that f is Riemann integrable.

Solution 2

Let $\mathcal{P}_n = \{[1/2, 1), [1/3, 1/2), \dots, [1/n, 1/(n-1)), [0, 1/n)\}$. Then

$$L(f, \mathcal{P}_n) = \frac{1}{n} \inf_{[0, 1/n)} f + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \inf_{[1/(k+1), 1/k)} f = \frac{-1}{n} + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) (-1)^{k+1}$$

and

$$U(f, \mathcal{P}_n) = \frac{1}{n} \sup_{[0, 1/n)} f + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \sup_{[1/(k+1), 1/k)} f = \frac{1}{n} + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) (-1)^{k+1}.$$

So

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) < \frac{2}{n}.$$

A standard argument then shows that

$$0 \leq \overline{\int_0^1} f - \underline{\int_0^1} f \leq \frac{2}{n}$$

for any n . Thus the lower and upper integrals are equal, hence f is Riemann integrable.