

Quiz 7 Rubric

Problem

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable with f' bounded on (a, b) . Prove that there exists $c \neq 0$ such that the function

$$g(x) := x + cf(x)$$

is injective.

Rubric

- I chose this problem expecting usage of MVT, however the professor has informed me that it was proven that $f'(x) > 0$ implies f strictly increasing. So do not check for this.
 - Consequently I would be more lenient to solutions that *do* use the MVT. (There will be some though they should be correct...)
- Taking some c with $|c| < \frac{1}{M}$, where $|f'| \leq M$, shows some good intuition and should receive at least **5 points**.
- If the solution is mostly correct, except for choosing $c = \pm \frac{1}{M}$ and having an inequality erroneously strict somewhere, I would still award at least **8 points**. (This approach would be fine if they took M to be a *strict* upper bound on $|f'|$, i.e. $|f'| < M$.)

Solution 1

Since f' bounded there exists $M > 0$ such that $|f'| \leq M$. Now take $c = \frac{1}{10M}$. We claim this works. If g is not injective, then there exist x, y , $x \neq y$ such that $g(x) = g(y)$. So

$$x + cf(x) = y + cf(y)$$

and

$$x - y = -c(f(x) - f(y)).$$

Thus

$$\frac{-1}{c} = \frac{f(x) - f(y)}{x - y}.$$

By the MVT, there exists z between x and y such that

$$\frac{-1}{c} = f'(z).$$

However,

$$M \geq |f'(z)| = \frac{1}{c} = 10M,$$

which is absurd.

Solution 2

(The below lemma can be used without proof.)

Lemma 7.1

If $g : (a, b) \rightarrow \mathbb{R}$ is differentiable and $g'(x) > 0$ for all $x \in (a, b)$, then g is injective.

Proof. If not, then $g(x) = g(y)$ for some x and y distinct. By Rolle's Theorem we find that there is some z between x and y such that $g'(z) = 0$. This contradicts the assumptions.

Alternatively the student may claim that g is (strictly) increasing. To prove this, they must suppose otherwise so that $g(x) \geq g(y)$ for some $x < y$. Then $g(y) - g(x) \leq 0$ hence $\frac{g(y)-g(x)}{y-x} \leq 0$. By MVT we then get z such that $g'(z) \leq 0$, which is absurd. $\square$

Now we solve the problem. Letting M be an upper bound on $|f'|$, we take $c = \frac{1}{2M}$. Notice that $g(x) = x + cf(x)$ is differentiable, and its derivative is $g'(x) = 1 + cf'(x)$. Now

$$1 + cf'(x) = 1 + \frac{f'(x)}{2M} \geq 1 - \frac{M}{2M} = \frac{1}{2} > 0.$$

So $g' > 0$. By the Lemma we conclude that g is injective.