

Quiz 6 Rubric

Problem

Show that $f : [0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

Rubric

There are several ways to solve this problem. Choose the rubric/guideline that gives the most points. It is important to look for *understanding* and award partial credit for that.

For solutions following Solution 1:

- Give **4 points** for correctly applying the theorem which states that a continuous function on a compact set must be uniformly continuous.
- Give **4 points** for showing uniform continuity over $[N, \infty)$ for some N . (Give partial credit for an attempt at this.)
- Give **2 points** for correctly combining the results to deduce uniform continuity over $[0, \infty)$.

For solutions following Solution 2:

- Give **5 points** for correctly guessing that $\sqrt{x}$ is $\frac{1}{2}$ -Hölder continuous and noting that this is sufficient for uniform continuity. A proof of this sufficiency is not necessary.
- Give up to **5 points** based on the correctness of the proof of Hölder continuity.
- If the proof of Hölder continuity is mostly incorrect, consider giving 1-2 points of partial credit if they provide a proof that Hölder continuity implies uniform continuity.

General guidelines for the above and other solutions:

- If the student does not understand what uniform continuity is, they should typically receive a score in the interval **[0, 3]**, based on how much general understanding they demonstrate. Give exactly **4 points** if they prove continuity instead of uniform continuity.
- If the student correctly states uniform continuity (or clearly understands what it is) and makes an attempt to prove it, they should receive at least **5 points**.
 - An honest but otherwise hopeless approach should receive a score in **[5, 6]**.
 - A solution with at least one good idea should receive at least a **7**.
 - Solutions with a score of at least **8** should be at least kinda close to correct.

Solution 1

Fix $\varepsilon > 0$. Then since $[0, 200]$ is compact, $\sqrt{x}$ is uniformly continuous over $[0, 200]$. So there exists $\delta_1 > 0$ such that $|\sqrt{x} - \sqrt{y}| < \varepsilon$ for all $x, y \in [0, 200]$ with $|x - y| < \delta_1$.

Now consider $x, y \geq 100$. Then,

$$|\sqrt{x} - \sqrt{y}| \leq \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{\sqrt{100} + \sqrt{100}}|x - y| = \frac{1}{20}|x - y|,$$

so $\sqrt{x}$ is uniformly continuous over $[100, \infty)$.

To combine these results, we pick $\delta = \min(\delta_1, \varepsilon, 1)$. Then if $|x - y| < \delta$, we in particular have that $|x - y| < 1$, so either $x, y \in [0, 200]$ or $x, y \in [100, \infty)$. If the first case holds, then since $|x - y| < \delta_1$ we have $|\sqrt{x} - \sqrt{y}| < \varepsilon$ by definition of δ_1 . If the second case holds, then as we computed,

$$|\sqrt{x} - \sqrt{y}| \leq \frac{1}{20}|x - y| < \frac{\delta}{20} < \frac{\varepsilon}{20} < \varepsilon.$$

This completes the proof.

Note: It was important that we established uniform continuity for $[0, 200]$ and $[100, \infty)$, rather than for $[0, 100]$ and $[100, \infty)$. This is because otherwise something like $x = 100 - \delta/3$ and $y = 100 + \delta/3$ could happen, where the points are in different intervals and so neither of the uniform continuity results can apply.

Solution 2

We claim that $\sqrt{x}$ is $\frac{1}{2}$ -Hölder continuous. That is, there exists a constant $C > 0$ such that

$$|\sqrt{x} - \sqrt{y}| \leq C|x - y|^{1/2}$$

for all $x, y \geq 0$. If this holds, this implies that $\sqrt{x}$ is uniformly continuous.

We will, in fact, show the inequality for $C = 1$. Squaring both sides, it suffices to prove that

$$x + y - 2\sqrt{xy} \stackrel{?}{\leq} |x - y|.$$

Assume without loss of generality that $x \geq y$. Then we wish to prove that

$$x + y - 2\sqrt{xy} \stackrel{?}{\leq} x - y$$

or

$$2y \stackrel{?}{\leq} 2\sqrt{xy}$$

or

$$y \stackrel{?}{\leq} \sqrt{xy}$$

or

$$y^2 \stackrel{?}{\leq} xy$$

or $y \stackrel{?}{\leq} x$, which is true by assumption.

Note: One could have followed this solution by fixing $\varepsilon > 0$ and then choosing $\delta = \varepsilon^2$. The algebra in this solution implies that such a δ works.