Quiz 5 Rubric

Problem

Show that every compact metric space is separable.

Rubric

I'd be surprised if there were any other way to solve this problem.

- (5 points) Using the compactness
  - Award **3 points** if the student tries to use compactness via its definition (every open cover has a finite subcover).
  - If this is done by placing a small ball centered at every point, instead award the full **5 points**.
  - Compactness can be sidestepped by using that the space is totally bounded. Award this full points.
- (5 points) Constructing the countable dense set
  - Award **3 points** for correctly claiming that a subset that the student constructs is countable and dense.
  - Award **2 points** for showing that the subset is indeed dense.
  - A rigorous proof that it is countable is not needed provided that it is indeed countable.

Solution

Let the compact metric space be (X, d). For each $n \in \mathbb{N}$, we have

$$\bigcup_{x \in X} B(x, 1/n) \supseteq X,$$

so $\{B(x, 1/n)\}_{x \in X}$ is an open cover of X. Since X is compact, it follows that there exists a finite subcover, $\{B(x_{n,k}, 1/n)\}_{k=1}^{m_n}$. Let

$$E := \{x_{n,k} : 1 \leq k \leq m_n,\ n \in \mathbb{N}\}.$$

It is not too hard to see that E is countable. We claim that E is dense (in X). To see this, take some $x \in X$ and $r > 0$. We will show that $B(x, r)$ intersects E.

Find $n \in \mathbb{N}$ such that $1/n < r$. Then

$$\bigcup_{k=1}^{m_n} B(x_{n,k}, 1/n) \supseteq X,$$

so since $x \in X$, we have $x \in B(x_{n,k}, 1/n)$ for some $1 \leq k \leq m_n$. But now $d(x, x_{n,k}) < 1/n < r$, so $x_{n,k} \in B(x, r)$, meaning that $B(x, r) \cap E \neq \emptyset$. This completes the proof.