Quiz 4 Rubric

Problem

Let $K, L \subseteq \mathbb{R}^d$ be non-empty compact sets. Show that the infimum

$$d(K, L) := \inf\{\|x - y\| : x \in K, y \in L\}$$

is obtained.

Rubric

Most methods will likely use compactness, continuity, and then extreme value theorem. Depending on the exact approach, these components will have varying challenge.

- For a correct method for which the continuity is the main difficulty and the compactness is trivial (e.g. Solution 1):
  - Award 6 points for identifying the methodology
  - Award 1-4 points based on the correctness of the proof of continuity
- For a correct method for which continuity is pretty easy but compactness is not so obvious (e.g. Solution 4):
  - Award 2 points for stating the continuous function. (If the continuous function is the norm function $x \mapsto \|x\|$, no proof is required.)
  - Award 1-5 points based on the correctness of the proof of compactness
  - Award 3 points for applying the extreme value theorem
- For a correct method for which continuity and compactness are both need some work (e.g. Solution 2):
  - 3 points for continuity
  - 4 points for compactness
  - 3 points for applying extreme value theorem
- For other solutions using extreme value theorem which do not resemble any of the provided solutions, weight the point values for each component of the solution according to their difficulty.
- If an outline of the proof is written and it should work in theory, award at least 3 points.
- For very different solutions (e.g. Solution 5) use your best judgment.

Solution 1

We begin by observing that

$$\inf_{x\in K, y\in L} \|x-y\| = \inf_{x\in K}\left(\inf_{y\in L}\|x-y\|\right).$$

So define a function $f : K \to \mathbb{R}$ via $f(x) = \inf_{y\in L}\|x-y\|$. (This is the "distance" between the point x and the set L.) We claim that f is continuous. Indeed, fix $x_0 \in K$. Fix $\varepsilon > 0$. Let $x \in K$ so that $\|x - x_0\| < \delta$, where δ will be chosen later. We will show that $|f(x) - f(x_0)| < \varepsilon$. Indeed, by definition of $f(x_0)$, we may find $y_0 \in L$ such that

$$f(x_0) \le \|x_0 - y_0\| < f(x_0) + \frac{\varepsilon}{100}.$$

It follows that

$$f(x) \le \|x-y_0\| \le \|x-x_0\| + \|x_0-y_0\| \le \|x-x_0\| + f(x_0) + \frac{\varepsilon}{100} \le f(x_0) + \delta + \frac{\varepsilon}{100}.$$

So $f(x) - f(x_0) \le \delta + \frac{\varepsilon}{100}$. By a symmetrical argument (instead find $y \in L$ so that $f(x) \le \|x-y\| < f(x) + \varepsilon/100$), we also have $f(x_0) - f(x) \le \delta + \frac{\varepsilon}{100}$. Hence

$$|f(x) - f(x_0)| \le \delta + \frac{\varepsilon}{100} < \varepsilon,$$

where we have now chosen $\delta = \varepsilon/7$.

Since f is continuous on a compact set, it has a minimum, which is what we needed.

Solution 1.5

We can prove f is continuous by instead showing that it is 1-Lipschitz. Pick $x_1, x_2 \in K$. We will show that $|f(x_1) - f(x_2)| \le \|x_1 - x_2\|$. For any $y_1, y_2 \in L$, we have

$$f(x_1) \le \|x_1 - y_1\| \le \|x_1 - x_2\| + \|x_2 - y_2\|,$$

so

$$f(x_1) - \|x_1 - x_2\| \le \|x_2 - y_2\|.$$

Taking the infimum over y_2 gives

$$f(x_1) - \|x_1 - x_2\| \le f(x_2)$$

so that $f(x_1) - f(x_2) \le \|x_1 - x_2\|$. An entirely symmetric argument (reversing the roles of x_1 and x_2) shows that $|f(x_1) - f(x_2)| \le \|x_1 - x_2\|$ as claimed.

Solution 2

This solution uses more than one type of norm so I will be specifying the norm used every time.

This solution also uses the extreme value theorem, but in this way:

- We show that the function $f : K \times L \to \mathbb{R}$ defined by $f(x, y) := \|x - y\|_{\mathbb{R}^d}$ is continuous (where the topology on $K \times L$ is inherited from $\mathbb{R}^{2d}$).

  (note that f has "type" $\mathbb{R}^{2d} \to \mathbb{R}$.)
- We show that $K \times L \subseteq \mathbb{R}^{2d}$ is compact (with respect to the topology of $\mathbb{R}^{2d}$).

To show that f is continuous, take a point $(x_0, y_0) \in K \times L$. Then for all $\varepsilon > 0$, we have, for all (x, y) with $\|(x, y) - (x_0, y_0)\|_{\mathbb{R}^{2d}} < \delta$ (with δ to be chosen later), that

$$\|(x, y) - (x_0, y_0)\|_{\mathbb{R}^{2d}} = \sqrt{\|x - x_0\|^2_{\mathbb{R}^d} + \|y - y_0\|^2_{\mathbb{R}^d}} < \delta,$$

so that

$$\sqrt{\|x - x_0\|^2_{\mathbb{R}^d} + 0} \leq \sqrt{\|x - x_0\|^2_{\mathbb{R}^d} + \|y - y_0\|^2_{\mathbb{R}^d}} < \delta$$

i.e. $\|x - x_0\|_{\mathbb{R}^d} < \delta$, and similarly $\|y - y_0\|_{\mathbb{R}^d} < \delta$. So

$$\|x - y\|_{\mathbb{R}^d} \leq \|x - x_0\|_{\mathbb{R}^d} + \|x_0 - y_0\|_{\mathbb{R}^d} + \|y_0 - y\|_{\mathbb{R}^d} \leq 2\delta + \|x_0 - y_0\|.$$

Similarly

$$\|x_0 - y_0\|_{\mathbb{R}^d} \leq \|x_0 - x\|_{\mathbb{R}^d} + \|x - y\|_{\mathbb{R}^d} + \|y - y_0\|_{\mathbb{R}^d} \leq 2\delta + \|x - y\|.$$

Combining these two inequalities allows us to conclude $|f(x) - f(y)| \leq 2\delta < \varepsilon$, where we have chosen $\delta = \varepsilon/20$.

To show that $K \times L$ is compact, we note that $K \times L \subseteq \mathbb{R}^{2d}$ and so, by Heine-Borel, it is sufficient to show that $K \times L$ is closed and bounded.

$K \times L$ is bounded because if $K \subseteq B(0, R_1)$ and $L \subseteq B(0, R_2)$, then $K \times L \subseteq B_{\mathbb{R}^{2d}}(0, \sqrt{R_1^2 + R_2^2})$.

To see that $K \times L$ is closed, we may show that the complement is open. Take $(x, y) \notin K \times L$. Then, this means that either $x \notin K$ or $y \notin L$. Without loss of generality, let us suppose $x \notin K$. Then since K^c is open, there exists $r > 0$ so that $B(x, r) \subseteq K^c$. We claim that $B((x, y), r) \subseteq (K \times L)^c$. This is because for any (x', y') which is distance $< r$ from (x, y), we have $\|x - x'\| < r$ (why?), so $x' \notin K$ and thus $(x', y') \notin K \times L$. This completes the proof.

Solution 2.5

We could also have shown that $K \times L$ is closed by showing that it contains its accumulation points. If (x, y) is an accumulation point of $K \times L$, then $B((x, y), 1/n) \setminus \{(x, y)\}$ intersects $K \times L$ for all n. Take a point (x_n, y_n) in the intersection. Evidently $(x_n, y_n) \to (x, y)$ so you can argue that $x_n \to x$ and $y_n \to y$. Since K and L are closed we must have $x \in K$ and $y \in L$, so $(x, y) \in K \times L$.

Solution 2.75

We could also have shown that $K \times L$ is closed by showing that it is sequentially closed — that is, if $(x_n, y_n) \in K \times L$ is Cauchy, and its limit is (x, y), then $(x, y) \in K \times L$. The proof is basically just the latter half of Solution 2.5.

Solution 3

Let us follow Solution 2, but prove compactness differently by showing that $K \times L$ is sequentially compact. Indeed, take $(x_n, y_n) \in K \times L$. Then we may extract a subsequence x_{n_k} which converges to $x_0 \in K$, and then extract a subsequence $y_{n_{k_j}}$ which converges to $y_0 \in L$. Then $(x_{n_{k_j}}, y_{n_{k_j}}) \to (x_0, y_0)$. (see also solution 5)

Solution 4

Define the difference set

$$K - L := \{x - y : x \in K, y \in L\}.$$

Then we wish to show that $\inf_{z \in K-L} \|z\|$ is obtained.

It is not hard to show that the norm function $f(z) := \|z\|$ is continuous (you can show that $|\|x\| - \|y\|| \leq \|x - y\|$ so that it's 1-Lipschitz, for example), so it suffices to show that $K - L$ is compact.

Since $K - L \subseteq \mathbb{R}^d$, we have by Heine-Borel that it is sufficient to show that $K - L$ is closed and bounded. To see that is bounded, find $K \subseteq B(0, R_1)$ and $L \subseteq B(0, R_2)$. Then for all $z \in K - L$, we may write z as $x - y$ with $x \in K$ and $y \in L$, and now

$$\|z\| \leq \|x\| + \|y\| \leq R_1 + R_2.$$

Thus $K - L \subseteq B(0, R_1 + R_2)$.

To see that $K - L$ is closed, we can show that it contains all its accumulation points. Suppose $z \in \text{acc}(K - L)$. Then for every $n \in \mathbb{N}$ we can find $x_n \in K$ and $y_n \in L$ such that $\|z - (x_n - y_n)\| < 1/n$. By compactness of K and L, we can extract subsequences: $x_{n_k} \to x_0 \in K$ and $y_{n_{k_j}} \to y_0 \in L$. This gives $x_{k_j} - y_{n_{k_j}} \to x_0 - y_0$ but also $\|z - (x_{k_j} - y_{k_j})\| \to 0$ so $x_{k_j} - y_{k_j} \to z$, so $z = x_0 - y_0 \in K - L$.

(well that's interesting, it seems like we had to use compactness to show that it's closed... could it be false if they are both not necessarily compact?)

Solution 5

The infimum exists and is finite, so there exists a sequence $\{(x_n, y_n)\}_n \in K \times L$ for which

$$\lim_{n\to\infty} \|x_n - y_n\| = \inf_{x \in K, y \in L} \|x - y\|.$$

Since K is compact, and $x_n \in K$, there exists a subsequence $\{x_{n_k}\}_k \in K$ such that $x_{n_k} \to x_0 \in K$.

Since L is compact, and $y_{n_k} \in L$, there exists a subsequence $\{y_{n_{k_j}}\}_j \in L$ such that $y_{n_{k_j}} \to y_0 \in L$. Subsequences preserve limits so $x_{n_{k_j}} \to x_0$.

Subsequences preserve limits, so

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{j\to\infty} \|x_{n_{k_j}} - y_{n_{k_j}}\|.$$

Since $x_{n_{k_j}} - y_{n_{k_j}} \to x_0 - y_0$, we may appeal to the continuity of the norm to conclude that the limit exists and is equal to $\|x_0 - y_0\|$. This obtains the infimum.