Quiz 3 Rubric

Problem

For each $k, p \in \mathbb{N}$, define

$$s_{k,p} := \sum_{p<n\leq kp} \frac{1}{n}.$$

Show that for each $k \in \mathbb{N}$,

$$\lim_{p\to\infty} s_{k,p} = \log k.$$

Rubric

- (7 pts) Obtain a good bound on $\frac{1}{n}$ in terms of integration.
  - Give 4 pts partial credit for one bound but not the other.
  - Give 3 pts partial credit for recognizing that $s_{k,p} \approx \int_p^{kp} \frac{1}{x}\,dx$, but not making this more precise.
  - Give 5.5 pts partial credit if the correct bounds are obtained but no justification is given.
  - Give 5 pts partial credit if the correct bounds are obtained but reversed.
- (3 pts) Deduce that the limit is $\log k$.
  - The student need not say much to justify the deduction obtained from the squeeze rule. Only deduct points if they write something wrong.
  - The student may appeal to $\liminf$ and $\limsup$. This is perfectly acceptable.

If the student claims that $\lim_{p\to\infty} \sum_{p<n\leq kp} \frac{1}{n} - \int_p^{kp} \frac{1}{x}\,dx = 0$ without a correct justification, award a total of 6 pts.

There may be a solution based on defining the "error" E_k between the integral and the sum. I don't expect this to result in a novel solution. If correct, it should be gradable according to the rubric.

Solution

Since $\frac{1}{x}$ is decreasing for $x > 0$, we have the bounds

$$\int_{n}^{n+1} \frac{1}{x}\,dx \le \frac{1}{n} \le \int_{n-1}^{n} \frac{1}{x}\,dx$$

for every $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, summing from $n = p+1$ to $n = kp$ gives

$$\int_{p+1}^{kp+1} \frac{1}{x}\,dx \le s_{k,p} \le \int_{p}^{kp} \frac{1}{x}\,dx.$$

Evaluating the integrals,

$$\log\left(\frac{kp+1}{p+1}\right) \le s_{k,p} \le \log k.$$

By continuity of log, $\lim_{p\to\infty} \log\left(\frac{kp+1}{p+1}\right) = \log k$. So the desired claim is true by the squeeze rule.