Quiz 2 Rubric

Problem

Let $\{a_n\}_n$ be a sequence of real numbers.

- (1) Suppose $a_n \to L$. Show that $\frac{1}{n}\sum_{k=1}^{n} a_k \to L$.
- (2) Does the converse hold? If yes, give the proof. If no, give a counterexample.

Rubric

If a submission does not remotely resemble any of the presented solutions, it is likely a 0.

- (2 pt) Fix $\varepsilon > 0$ and find N_ε witnessing the convergence $a_n \to L$. If the student does not explicitly write "fix $\varepsilon > 0$", comment but do not deduct.
- (2 pts) Split the sum from 1 to N_ε and from N_ε to n. Comment but do not deduct for minor off-by-1 indexing errors.
- (2 pts) Use the bound $|a_n - L| < \varepsilon \; \forall n \geq N_\varepsilon$ to obtain a bound on $\sum_{k=N_\varepsilon}^{n} |a_k - L|$ (as in Solution 1) or a bound on $\sum_{k=N_\varepsilon}^{n} a_k$ (as in Solution 2 and Solution 3).
- (2 pts) Finish the proof by either appealing to the limsup (as in Solution 1), both liminf and limsup (as in Solution 2), or finding an N so large that $\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| < \varepsilon$ for all $n \geq N$.
- (2 pts) Give a valid counterexample which shows that the converse is false. Give 0.5 pts of partial credit for the right answer but no given counterexample.

The solutions that follow are just for the first part. An easy counterexample for the second part is given by $a_n = (-1)^n$. The limit of averages is 0 but the limit of a_n does not exist.

Solution 1

Fix $\varepsilon > 0$. Find N_ε so large that $|a_n - L| < \varepsilon$ for all $n > N_\varepsilon$.

For all $n > N_\varepsilon$ we may write

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k.$$

Subtracting L gives

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| &= \left|\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k\right) - L\right| \\
&= \left|\frac{1}{n}\sum_{k=1}^{N_\varepsilon}(a_k - L) + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n}(a_k - L)\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{N_\varepsilon}|a_k - L| + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n}|a_k - L|. \qquad \text{(Triangle ineq.)}
\end{aligned}$$

But since $|a_k - L| < \varepsilon$ for all $k > N_\varepsilon$, we have that $\sum_{k=N_\varepsilon+1}^{n}|a_k - L| \leq \sum_{k=N_\varepsilon+1}^{n}\varepsilon = \varepsilon(n - N_\varepsilon)$. Hence

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| \leq \frac{1}{n}\sum_{k=1}^{N_\varepsilon}|a_k - L| + \frac{\varepsilon(n - N_\varepsilon)}{n}$$

for all $n > N_\varepsilon$. Sending $n \to \infty$,

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| \leq \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon}|a_k - L| + \frac{\varepsilon(n - N_\varepsilon)}{n}\right) = 0 + \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so

$$0 \leq \liminf_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| \leq \limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| = 0,$$

which entails that the above limit exists and is equal to 0.

Solution 2

Fix $\varepsilon > 0$. Find N_ε so large that $|a_n - L| < \varepsilon$ for all $n > N_\varepsilon$.

For all $n > N_\varepsilon$ we may write

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k.$$

Sending $n \to \infty$ gives us that

$$\begin{aligned}\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k\right) &\leq \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \leq \limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k\right).\end{aligned}$$

We bound each side in turn. We have that

$$\begin{aligned}\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k\right) &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \limsup_{n\to\infty}\frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k \\ &= 0 + \limsup_{n\to\infty}\frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} (L+\varepsilon) \\ &= \limsup_{n\to\infty}\frac{1}{n}(n-N_\varepsilon)(L+\varepsilon) \\ &= L+\varepsilon,\end{aligned}$$

and similarly

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k\right) \geq L-\varepsilon.$$

Thus

$$L-\varepsilon \leq \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \leq L+\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we in fact have that

$$L \leq \liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \leq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k \leq L,$$

so the limit exists and is equal to L.

Solution 3

Fix $\varepsilon > 0$. Find N_ε so large that $|a_n - L| < \frac{\varepsilon}{100}$ for all $n > N_\varepsilon$.

For all $n > N_\varepsilon$ we may write

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{N_\varepsilon} a_k + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} a_k.$$

Subtracting L (as in Solution 1) gives

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| \leq \frac{1}{n}\sum_{k=1}^{N_\varepsilon} |a_k - L| + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} |a_k - L|.$$

Since $|a_k - L| < \frac{\varepsilon}{100}$ for all $k < N_\varepsilon$, we have that

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| &\leq \frac{1}{n}\sum_{k=1}^{N_\varepsilon} |a_k - L| + \frac{1}{n}\sum_{k=N_\varepsilon+1}^{n} \frac{\varepsilon}{100} \\ &= \frac{1}{n}\sum_{k=1}^{N_\varepsilon} |a_k - L| + \frac{n - N_\varepsilon}{n}\cdot\frac{\varepsilon}{100} \\ &\leq \frac{1}{n}\sum_{k=1}^{N_\varepsilon} |a_k - L| + \frac{\varepsilon}{100}..\end{aligned}$$

Now:

- Let $N_1 = \dfrac{100}{\varepsilon\sum_{k=1}^{N_\varepsilon}|a_k - L|}$, so that $\frac{1}{n}\sum_{k=1}^{N_\varepsilon}|a_k - L| < \frac{\varepsilon}{100}$ for all $n > N_1$.
- Let $N = \max(N_1, N_\varepsilon)$.

Then for all $n > N$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k - L\right| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{100} < \varepsilon.$$

We are done by definition of a limit.

Rubric Notes: Obtaining a final bound of 2ε is acceptable. N_1 need not be chosen explicitly but its existence should be stated. Its existence can be implicitly stated by writing "for all n large enough".