Problem 1

To make this easier to picture, let's change the curve to

$$y = \cosh x, \qquad -1 \le x \le 1$$

and then rotate this around the x-axis. It's the same thing.

Since I don't remember the formula they want us to use here, it will be easier to obtain the general surface area for an arbitrary $y = f(x)$ rotated about the x-axis. We should expect the surface area formula to simplify a bit due to the rotational symmetry, and working with a general f should make it easier to identify it.

To recap, let $f : [-1, 1] \to (0, \infty)$ be smooth and let Σ be the surface obtained by rotating the graph of f about the x-axis. To obtain the area of Σ, we start by obtaining a chart for Σ. A natural choice is $\phi : [-1, 1] \times (0, 2\pi] \to \Sigma$ where

$$\phi(t, \theta) := (t, f(t)\cos\theta, f(t)\sin\theta).$$

Now

$$\text{Area}(\Sigma) = \int_{[-1,1]\times(0,2\pi]} \sqrt{\det(D\phi(t,\theta)^T D\phi(t,\theta))}\, dt\, d\theta.$$

Computing the Jacobian matrix,

$$D\phi(t,\theta) = \begin{bmatrix} 1 & 0 \\ f'(t)\cos\theta & -f(t)\sin\theta \\ f'(t)\sin\theta & f(t)\cos\theta \end{bmatrix}.$$

Computing the Jacobian,

$$D\phi(t,\theta)^T D\phi(t,\theta) = \begin{bmatrix} 1 + f'(t)^2 & 0 \\ 0 & f(t)^2 \end{bmatrix},$$

$$\det D\phi(t,\theta)^T D\phi(t,\theta) = (1 + f'(t)^2) f(t)^2,$$

$$\sqrt{\det D\phi(t,\theta)^T D\phi(t,\theta)} = \sqrt{1 + f'(t)^2}\, f(t).$$

So

$$\text{Area}(\Sigma) = \int_{t=-1}^{1} \int_{\theta=0}^{2\pi} \sqrt{1 + f'(t)^2}\, f(t)\, d\theta\, dt = 2\pi \int_{t=-1}^{1} \sqrt{1 + f'(t)^2}\, f(t)\, dt.$$

Good. Now let's plug in $f(t) = \cosh t$. We have

$$\sqrt{1 + f'(t)^2}\, f(t) = \sqrt{1 + (\sinh t)^2} \cdot \cosh t = \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}} \cdot \frac{e^t + e^{-t}}{2}$$

$$= \frac{e^t + e^{-t}}{2} \cdot \frac{e^t + e^{-t}}{2} = \frac{e^{2t} + 2 + e^{-2t}}{4}.$$

Hence

$$\text{Area}(\Sigma) = 2\pi \int_{t=-1}^{1} \frac{e^{2t} + 2 + e^{-2t}}{4}\, dt = \frac{\pi}{2}\left(\frac{e^2 - e^{-2}}{2} + 4 + \frac{e^2 - e^{-2}}{2}\right) = \boxed{\pi(2 + \sinh 2)}.$$

Problem 2

Rename $(x_1, x_2, x_3) =: (x, y, z)$ because I'm a bad person. Then the goal is to minimize $f(x, y, z) := x^2 + y^2 + z^2$ given the constraint $g(x, y, z) = 0$ where $g(x, y, z) := xy + xz - yz/2 - 1$. (Of course, this is the *squared* distance, so we just need to remember to square root the answer we get.)

Before applying Lagrange multipliers we must do two things to justify its use.

First we must establish that a minimum exists. To quickly argue this, we observe that the set $S = \{g = 0\}$ is nonempty as $(1, 1, 0) \in S$. Next, since g is smooth, S must be a closed set, thus $S \cap \overline{B(0, 10)}$ is nonempty and compact. By continuity, f achieves a minimum value over $S \cap \overline{B(0, 10)}$. Whatever this minimum value is must be $\le$ any value for $f(x, y, z)$ for $(x, y, z) \in B(0, 10)^c$, so this is in fact a global minimum.

Second, we must establish that the gradient of the constraint has full rank at the minimizer. That is, if we let $(x, y, z) \in S$ be a point at which the constrained minimum of f is obtained, we must show that $\nabla g(x, y, z) \neq 0$. Computing,

$$\nabla g(x, y, z) = \begin{bmatrix} y + z \\ x - \frac{z}{2} \\ x - \frac{y}{2} \end{bmatrix}.$$

If this somehow were $= \vec{0}$, then it is not too difficult to show that $x = y = z = 0$. But $(0, 0, 0) \notin S$, which is a contradiction.

Thus it is safe to apply Lagrange multipliers *(for completeness one should also point out that f and g are of class C^1)*. This tells us that at the minimizer (x, y, z), there exists λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

. Note that $\nabla f(x, y, z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$. Including the constraint $g(x, y, z) = 0$, this gives a system of four equations:

$$\begin{cases} 2x = \lambda y + \lambda z \\ 2y = \lambda x - \frac{\lambda}{2} z \\ 2z = \lambda x - \frac{\lambda}{2} y \\ xy + xz - \frac{1}{2} yz = 1 \end{cases}$$

To solve this, view λ as a constant and examine the first three equations. This is a linear system of three variables $A\vec{x} = 0$, where

$$A = \begin{bmatrix} -2 & \lambda & \lambda \\ \lambda & -2 & -\frac{\lambda}{2} \\ \lambda & -\frac{\lambda}{2} & -2 \end{bmatrix}.$$

The key observation is that if A were invertible then $(x, y, z) = A^{-1}\vec{0} = \vec{0}$, but this cannot be the case! Thus $\det A = 0$. This lets us solve for λ. Computing,

$$\begin{aligned} 0 = \det A &= -2\left(4 - \frac{\lambda^2}{4}\right) - \lambda\left(-2\lambda + \frac{\lambda^2}{2}\right) + \lambda\left(-\frac{\lambda^2}{2} + 2\lambda\right) \\ &= -8 + \frac{\lambda^2}{2} + 2\lambda^2 - \frac{\lambda^3}{2} - \frac{\lambda^3}{2} + 2\lambda^2 \\ &= -\lambda^3 + \frac{9}{2}\lambda^2 - 8. \end{aligned}$$

We make the ansatz $\lambda = \mu/2$, then $\mu^3 - 9\mu^2 + 64 = 0$. By rational root theorem we now seek integer solutions for μ, particularly divisors of 64. Factor $64 = \mu^2(9 - \mu)$ to see that $\mu = 8$ as a solution, and $\mu^3 - 9\mu^2 + 64 = (\mu - 8)(\mu^2 - \mu - 8)$ which gives $\mu = \frac{1 \pm \sqrt{33}}{2}$ as additional solutions. Hence $\lambda = 4, \frac{1 \pm \sqrt{33}}{4}$.

This looks nasty, but there's one more trick to observe. Recall the three equations

$$\begin{cases} 2x = \lambda y + \lambda z \\ 2y = \lambda x - \frac{\lambda}{2} z \\ 2z = \lambda x - \frac{\lambda}{2} y \end{cases}.$$

Multiplying these by x, y, and z respectively, and then adding them up, gives

$$2x^2 + 2y^2 + 2z^2 = \lambda(2xy + 2xz - yz) = 2\lambda$$

because $xy + xz - \frac{1}{2}yz = 1$. So $x^2 + y^2 + z^2 = \lambda$, and so we actually just need to minimize the value of λ. Moreover, from this trick we see that it is necessary that $\lambda \geq 0$. So the only reasonable candidates are $\lambda = 2$ and $\lambda = \frac{1 + \sqrt{33}}{4}$. Since $\sqrt{33} < \sqrt{49}$, we find that $\frac{1 + \sqrt{33}}{4}$ is the smaller of the two. But before calling this in as the

(square of the) answer, we should establish that this is actually possible to obtain in that there are corresponding values of (x, y, z) satisfying the equations for this value of λ.

We can start with finding *some* solution to the first three equations. Since the roles of y and z are symmetric, we can take $y = z = 1$ and then this gives $x = \lambda$ and our previous work forces the second and third equations to be satisfied. Observing that $x, y, z > 0$ for this solution, we may then scale them down (since (ax, ay, az) is also a solution for any $a > 0$) to obtain a solution (x, y, z) which also satisfies $xy + yz - \frac{1}{2}yz = 1$. (Importantly, we had to check that $xy + yz - \frac{1}{2}yz > 0$ for the particular solution in order to ensure that such a scaling can work.)

So the value $\lambda = \frac{1+\sqrt{33}}{4}$ can indeed be obtained, giving a final answer of $\boxed{\frac{\sqrt{1+\sqrt{33}}}{2}}$.

Problem 3

Let's handle absolute convergence first. We claim this does not converge absolutely. Intuitively this is because $\cos(k^{1/3}) \geq 0.1$ often enough, and $\sum \frac{0.1}{k^{3/4}}$ diverges. To make this precise, note that $\cos(k^{1/3}) \geq 0.5$ if $k^{1/3} \in [2\pi n - \pi/6, 2\pi n + \pi/6]$ for a positive integer n, and this holds if $k \in \left[(2\pi n - \pi/6)^3, (2\pi n + \pi/6)^3\right]$. This allows us to make the bound

$$\sum_{k=1}^{\infty} \frac{|\cos(k^{1/3})|}{k^{3/4}} \geq \sum_{n=10}^{\infty} \sum_{k=\lceil (2\pi n-\pi/6)^3 \rceil}^{\lfloor (2\pi n+\pi/6)^3 \rfloor} \frac{|\cos(k^{1/3})|}{k^{3/4}} \geq 0.5 \sum_{n=10}^{\infty} \sum_{k=\lceil (2\pi n-\pi/6)^3 \rceil}^{\lfloor (2\pi n+\pi/6)^3 \rfloor} \frac{1}{k^{3/4}}$$

$$\geq 0.5 \sum_{n=10}^{\infty} \frac{(2\pi n + \pi/6)^3 - (2\pi n - \pi/6)^3 - 2}{(2\pi n + \pi/6)^{3 \cdot 3/4}}$$

$$\geq 0.5 \sum_{n=10^{10}}^{\infty} \frac{0.1n^2}{(10n)^{9/4}} \geq \frac{1}{10^{100}} \sum_{n=10^{10}}^{\infty} \frac{1}{n^{1/4}} = +\infty.$$

Now we tackle convergence. We claim that, unfortunately, this series converges. The intuition for this is that if we were to pretend that the sum were an integral, then

$$\int_{x=1}^{\infty} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx = \int_{1}^{\infty} \frac{3\cos(u)u^2}{u^{9/4}}\, du = 3\int_{1}^{\infty} \frac{\cos u}{u^{1/4}}\, du,$$

which converges by the Dirichlet test (alternatively, integrate by parts).

It is rather difficult to make a "u-substitution" for a sum, but what we can try is to argue that the sum and integral are quite comparable. Fix a large n and write

$$\sum_{k=1}^{n} \frac{\cos(k^{1/3})}{k^{3/4}} = \sum_{k=1}^{n} \frac{\cos(k^{1/3})}{k^{3/4}} - \int_{1}^{n+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx + \int_{1}^{n+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx.$$

$$= \sum_{k=1}^{n} \left[\frac{\cos(k^{1/3})}{k^{3/4}} - \int_{k}^{k+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx \right] + \int_{1}^{n+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx.$$

We already know that the second term converges as $n \to \infty$. So it suffices to prove that $\sum_{k=1}^{n} \left[\frac{\cos(k^{1/3})}{k^{3/4}} - \int_{k}^{k+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx \right]$ converges as $n \to \infty$. In fact, let us show that it converges absolutely. Write

$$\sum_{k=1}^{\infty} \left| \frac{\cos(k^{1/3})}{k^{3/4}} - \int_{k}^{k+1} \frac{\cos(x^{1/3})}{x^{3/4}}\, dx \right| = \sum_{k=1}^{\infty} \left| \int_{k}^{k+1} \frac{\cos(k^{1/3})}{k^{3/4}} - \frac{\cos(x^{1/3})}{x^{3/4}}\, dx \right|$$

$$\leq \sum_{k=1}^{\infty} \int_{k}^{k+1} \left| \frac{\cos(k^{1/3})}{k^{3/4}} - \frac{\cos(x^{1/3})}{x^{3/4}}\, dx \right|.$$

In the expression $\left|\frac{\cos(k^{1/3})}{k^{3/4}} - \frac{\cos(x^{1/3})}{x^{3/4}}\,dx\right|$ above, we are guaranteed that $|x-k| \le 1$ (in particular $x \in [k, k+1]$, and we may use this to attempt a good bound. For this let us use this MVT. We have

$$\frac{d}{dx}\frac{\cos(x^{1/3})}{x^{3/4}} = \frac{-\frac{1}{3}x^{-2/3}\sin(x^{1/3})x^{3/4} - \frac{3}{4}\cos(x^{1/3})x^{-1/4}}{x^{3/2}} = -\frac{\sin(x^{1/3})}{3x^{17/12}} - \frac{3\cos(x^{1/3})}{4x^{7/4}}.$$

Thus, for some ξ between k and x, we have that

$$\left|\frac{\cos(k^{1/3})}{k^{3/4}} - \frac{\cos(x^{1/3})}{x^{3/4}}\,dx\right| = |k-x|\left|-\frac{\sin(\xi^{1/3})}{3\xi^{17/12}} - \frac{3\cos(\xi^{1/3})}{4\xi^{7/4}}\right|$$

$$\le \frac{1}{3\xi^{17/12}} + \frac{3}{4\xi^{7/4}} \le \frac{10}{\xi^{17/12}} \le \frac{10}{k^{17/12}}.$$

In conclusion,

$$\sum_{k=1}^{\infty}\left|\frac{\cos(k^{1/3})}{k^{3/4}} - \int_k^{k+1}\frac{\cos(x^{1/3})}{x^{3/4}}\,dx\right| \le \sum_{k=1}^{\infty}\frac{10}{k^{17/12}} < \infty.$$

Problem 4

Intuitively the only relevant terms in the denominator are k^4 and l^2. So let us consider the following simpler problem.

Lemma: Let $p, q \in \mathbb{R}$. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^p+n^q} < \infty$$

iff $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} < 1$.

Proof. There aren't any real tricks here, we have to use our bare hands. First, it is pretty clear that $p, q > 0$ is necessary for convergence. Now, let's toss out all terms with $\max(m, n) < 100$ and observe that

$$\sum_{m=100}^{\infty}\sum_{n=100}^{\infty}\int_m^{m+1}\int_n^{n+1}\frac{1}{x^p+y^q}\,dy\,dx \le \sum_{m=100}^{\infty}\sum_{n=100}^{\infty}\frac{1}{m^p+n^q} \le \sum_{m=100}^{\infty}\sum_{n=100}^{\infty}\int_{m-1}^{m}\int_{n-1}^{n}\frac{1}{x^p+y^q}\,dy\,dx,$$

so

$$\int_{100}^{\infty}\int_{100}^{\infty}\frac{1}{x^p+y^q}\,dy\,dx \le \sum_{m=100}^{\infty}\sum_{n=100}^{\infty}\frac{1}{m^p+n^q} \le \int_{99}^{\infty}\int_{99}^{\infty}\frac{1}{x^p+y^q}\,dy\,dx.$$

Thus the sum converges iff $\int_{100}^{\infty}\int_{100}^{\infty}\frac{1}{x^p+y^q}\,dy\,dx$ converges, iff $\int_1^{\infty}\int_1^{\infty}\frac{1}{x^p+y^q}\,dy\,dx$ converges. Now make a substitution,

$$\int_1^{\infty}\int_1^{\infty}\frac{1}{x^p+y^q}\,dy\,dx = \int_1^{\infty}\frac{1}{x^p}\int_1^{\infty}\frac{1}{1+\left(\frac{y}{x^{p/q}}\right)^q}\,dy\,dx = \int_1^{\infty}\frac{x^{p/q}}{x^p}\int_{1/x^{p/q}}^{\infty}\frac{1}{1+u^q}\,du\,dx.$$

Now make the bounds

$$\int_1^{\infty}\frac{x^{p/q}}{x^p}\int_1^{\infty}\frac{1}{1+u^q}\,du\,dx \le \int_1^{\infty}\frac{x^{p/q}}{x^p}\int_{1/x^{p/q}}^{\infty}\frac{1}{1+u^q}\,du\,dx \le \int_1^{\infty}\frac{x^{p/q}}{x^p}\int_0^{\infty}\frac{1}{1+u^q}\,du\,dx.$$

We have

$$\int_1^{\infty}\frac{x^{p/q}}{x^p}\int_{0 \text{ or } 1}^{\infty}\frac{1}{1+u^q}\,du\,dx = \left(\int_1^{\infty}\frac{1}{x^{p(1-1/q)}}\,dx\right)\left(\int_{0 \text{ or } 1}^{\infty}\frac{1}{1+u^q}\,du\right).$$

The first integral converges iff $p(1-1/q) > 1$, which rearranges to $\frac{1}{p} + \frac{1}{q} < 1$, and the second integral converges iff $q > 1$, which is necessarily implied by the previous condition. This completes the proof. □

To solve the original problem, we need to massage the sum until it takes the form of the lemma we just proved. Note first that

$$kl \le \frac{k^2}{2} + \frac{l^2}{2} \le k^4 + l^2,$$

so the denominator $k^4 - kl + l^2 + 1$ is always ≥ 1. So the sum is well-defined.

In fact, we can say a bit more. We have that

$$-\frac{k^4}{2} - \frac{l^2}{2} \leq kl \leq \frac{k^4}{2} + \frac{l^2}{2},$$

so that

$$\frac{1}{2}k^4 + \frac{1}{2}l^2 \leq k^4 - kl + l^2 \leq \frac{3}{2}k^4 + \frac{3}{2}l^2.$$

Hence, provided that $\alpha \geq 0$ (which is definitely necessary for convergence),

$$\frac{1}{2}(k^4 + l^2 + 1)^\alpha \leq \frac{1}{2}(k^4 + l^2 + 2)^\alpha \leq (k^4 - kl + l^2 + 1)^\alpha \leq \frac{3}{2}(k^4 + l^2 + 2/3)^\alpha \leq \frac{3}{2}(k^4 + l^2 + 1)^\alpha.$$

Thus the original sum converges iff

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \frac{1}{(k^4 + l^2 + 1)^\alpha} < \infty.$$

Let's now turn the $\mathbb{Z}$ into $\mathbb{N}$. We have

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \frac{1}{(k^4 + l^2 + 1)^\alpha} = \frac{1}{3^\alpha} + 2\sum_{k=1}^{\infty} \frac{1}{(k^4+1)^\alpha} + 2\sum_{l=1}^{\infty} \frac{1}{(l^2+1)^\alpha} + 4\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(k^4 + l^2 + 1)^\alpha}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k^{4\alpha}} \leq \sum_{k=1}^{\infty} \frac{1}{(k^4+1)^\alpha} \leq \sum_{k=1}^{\infty} \frac{1}{(k^4+k^4)^\alpha},$$

we have that $\sum_{k=1}^{\infty} \frac{1}{(k^4+1)^\alpha} < \infty$ iff $4\alpha > 1$. Similarly $\sum_{l=1}^{\infty} \frac{1}{(l^2+1)^\alpha} < \infty$ iff $2\alpha > 1$. Finally,

$$c_\alpha \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{2k^{4\alpha} + 2l^{2\alpha}} \leq c_\alpha \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^{4\alpha} + l^{2\alpha} + 1} \leq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(k^4 + l^2 + 1)^\alpha} \leq C_\alpha \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^{4\alpha} + l^{2\alpha} + 1} \leq C_\alpha \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{k^{4\alpha} + l^{2\alpha}},$$

where c_α, C_α are constants which satisfy $c_\alpha(x+y+z)^\alpha \leq x^\alpha + y^\alpha + z^\alpha \leq C_\alpha(x+y+z)^\alpha$ for all $x, y, z > 0$. *(When $\alpha \geq 1$ these constants exist because all norms on $\mathbb{R}^3$ are equivalent. When $0 < \alpha < 1$, raise all sides to the $1/\alpha$ power and use the same reasoning, appealing to the fact that $1/\alpha > 1$.)* So by the lemma, this sum converges iff $\frac{1}{4\alpha} + \frac{1}{2\alpha} < 1$, i.e. $\alpha > \frac{3}{4}$.

The condition $\alpha > \frac{3}{4}$ is the strongest of the three necessary conditions required, thus the answer is $\boxed{\alpha > \frac{3}{4}}$.

Problem 5

Part a

Rename $(x_1, x_2, x_3) =: (x, y, z)$ because I'm a bad person. Recall the spherical coordinates,

$$\begin{cases} x = \cos\phi \\ y = \sin\phi\cos\theta \\ z = \sin\phi\sin\theta \end{cases}$$

for $0 < \phi < \pi$ and $0 < \theta < 2\pi$. This parametrizes the unit sphere $\partial B(0,1)$. To modify this to parametrize the ellipsoid Σ instead,

$$\begin{cases} x = 1 + a\cos\phi \\ y = \sqrt{2}\sin\phi\cos\theta \\ z = \sqrt{3}\sin\phi\sin\theta \end{cases}.$$

Let $g(\phi,\theta) = (1 + a\cos\phi, \sqrt{2}\sin\phi\cos\theta, \sqrt{3}\sin\phi\sin\theta)$ be the chart described above. Tangents to the point $g(\phi,\theta)$ are given by $\frac{\partial g}{\partial\phi}(\phi,\theta)$ and $\frac{\partial g}{\partial\theta}(\phi,\theta)$. Computing,

$$\frac{\partial g}{\partial\phi}(\phi,\theta) = (-a\sin\phi, \sqrt{2}\cos\phi\cos\theta, \sqrt{3}\cos\phi\sin\theta)$$

and

$$\frac{\partial g}{\partial \theta}(\phi,\theta) = (0, -\sqrt{2}\sin\phi\sin\theta, \sqrt{3}\sin\phi\cos\theta).$$

A normal at $g(\phi,\theta)$ is then given by the cross product $\frac{\partial g}{\partial \phi}(\phi,\theta) \times \frac{\partial g}{\partial \theta}(\phi,\theta)$. Computing,

$$\frac{\partial g}{\partial \phi}(\phi,\theta) \times \frac{\partial g}{\partial \theta}(\phi,\theta) = \begin{pmatrix} \sqrt{6}\sin\phi\cos\phi \\ a\sqrt{3}\sin^2\phi\cos\theta \\ a\sqrt{2}\sin^2\phi\sin\theta \end{pmatrix}.$$

Now let's try to write this in terms of x, y, z (there will be more than one way to do so). We can scale this arbitrarily, so let's start by tossing out a factor of $\sin\phi$, to get $(\sqrt{6}\cos\phi, a\sqrt{3}\sin\phi\cos\theta, a\sqrt{2}\sin\phi\sin\theta)$. Now, looking back at the change of coordinates made, we see that this can be written as $\left(\sqrt{6}\left(\frac{x-1}{a}\right), a\sqrt{3/2}y, a\sqrt{2/3}z\right)$. With some additional scaling this can be rewritten more nicely as $\left(\frac{x-1}{a^2}, \frac{y}{2}, \frac{z}{3}\right)$. It's not hard to argue that this points outward, and finally we normalize this to get the answer $\boxed{\nu(x,y,z) = \dfrac{\left(\frac{x-1}{a^2}, \frac{y}{2}, \frac{z}{3}\right)}{\sqrt{\frac{(x-1)^2}{a^4} + \frac{y^2}{4} + \frac{z^2}{9}}}}$.

Part b

Let U be the region bounded by the ellipsoid. By the divergence theorem,

$$\int_\Sigma (x,y,z)\cdot\nu\, dS = \int_U \operatorname{div}(x,y,z)\, d(x,y,z) = \int_U 3\, d(x,y,z) = 3\operatorname{Vol}(U).$$

It's not too hard to argue that $\operatorname{Vol}(U) = \operatorname{Vol}(B(0,1))\cdot(a)(\sqrt{2})(\sqrt{3}) = \frac{4}{3}\sqrt{6}a\pi$, so the answer is $\boxed{4\sqrt{6}a\pi}$.

Part c

We must distinguish between two cases: When $0 < a < 1$ and when $1 < a$.

When $0 < a < 1$, the region U does not contain the origin, so $u(\vec{x})$ is well-defined everywhere in U. So by the divergence theorem,

$$\int_\Sigma \frac{\partial u}{\partial \nu}\, dS = \int_\Sigma \nabla u\cdot\nu\, dS = \int_U \operatorname{div}\nabla u\, d\vec{x} = \int_U \Delta u\, d\vec{x}.$$

If you took a course in PDEs, you hopefully know that $\Delta u = 0$. If not, let's compute that now. We have $u(\vec{x}) = \frac{1}{|\vec{x}|} = (|\vec{x}|^2)^{-1/2}$. Since $\partial_x|\vec{x}|^2 = 2x$, we have $\partial_x u(\vec{x}) = 2x\cdot -\frac{1}{2}(|\vec{x}|^2)^{-3/2} = -x(|\vec{x}|^2)^{-3/2}$. Now

$$\partial_x^2 u(\vec{x}) = -(|\vec{x}|^2)^{-3/2} - 2x^2\cdot\frac{-3}{2}(|\vec{x}|^2)^{-5/2} = -|\vec{x}|^{-3} + 3x^2|\vec{x}|^{-5}.$$

Thus

$$(\partial_x^2+\partial_y^2+\partial_z^2)u(x,y,z) = -3(|\vec{x}|^2)^{-3/2} + 3(x^2+y^2+z^2)|\vec{x}|^{-5} = 0.$$

Cool. So the answer is 0 when $0 < a < 1$.

When $a > 1$, the divergence theorem can no longer be applied directly because of the existence of a singularity within U. To fix this, we define $V = U \setminus \overline{B(0,\varepsilon)}$ where $\varepsilon > 0$ is small enough so that $B(0,\varepsilon) \subseteq U$. We can apply divergence theorem to V. This gives us

$$\int_{\partial V} \nabla u\cdot\nu\, dS = \int_V \Delta u\, d\vec{x} = 0$$

where ν is defined to be the unit *inner* normal for the surface $B(0,\varepsilon)$. (This corresponds to a unit *outer* normal for ∂V!) On the other hand,

$$\int_{\partial V} \nabla u\cdot\nu\, dS = \int_\Sigma \nabla u\cdot\nu\, dS + \int_{B(0,\varepsilon)} \nabla u\cdot\nu\, dS,$$

so we've reduced the integral to computing $-\int_{B(0,\varepsilon)} \nabla u \cdot \nu \, dS$. Reviewing the earlier computation, we can find that $\nabla u(\vec{x}) = -|\vec{x}|^{-3}\vec{x} = -\varepsilon^{-3}\vec{x}$ on $\partial B(0,\varepsilon)$. Moreover, for $\vec{x} \in \partial B(0,\varepsilon)$, we have that the unit inner normal is $\nu(\vec{x}) = -\frac{\vec{x}}{|\vec{x}|} = -\varepsilon^{-1}\vec{x}$. So

$$-\int_{\partial B(0,\varepsilon)} \nabla u \cdot \nu \, dS = -\varepsilon^{-4} \int_{\partial B(0,\varepsilon)} -\vec{x} \cdot -vecx \, dS = -\varepsilon^{-4} \int_{\partial B(0,\varepsilon)} |\vec{c}|^2 \, dS$$

$$= -\varepsilon^{-4} \int_{\partial B(0,\varepsilon)} \varepsilon^2 \, dS = -\varepsilon^{-4} \cdot \varepsilon^2 \cdot 4\pi\varepsilon^2 = \boxed{-4\pi}.$$