Problem 1

Part a

It's clear that $f_n \to 0$ pointwise. As for g_n, note that $g_n(0) \to 0$ whereas for $x \neq 0$, $g_n(x) \to \frac{x}{x^2} = \frac{1}{x}$. So $g_n(x) \to \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$.

Part b

The convergence of g_n cannot be uniform. If it were, then since g_n is continuous, it follows that its pointwise limit is continuous, but it is not.

The convergence of f_n is uniform because

$$\frac{nx^2}{1+n^2x^2} \leq \frac{nx^2}{n^2x^2} = \frac{1}{n} \to 0.$$

Problem 2

We have that for all n,

$$\mathrm{Var}_{[-1,1]} f(x) \geq \sum_{k=1}^{n} \left| f\left(\frac{1}{2k}\right) - f\left(\frac{1}{2(k+1)}\right) \right| = \sum_{k=1}^{n} \left| \frac{\pm 1}{(2k)^{1/5}} - \frac{\mp 1}{(2k+2)^{1/5}} \right|$$

$$= \frac{1}{2^{1/5}} \sum_{k=1}^{n} \frac{1}{k^{1/5}} + \frac{1}{(k+1)^{1/5}} \geq \frac{1}{2^{1/5}} \sum_{k=1}^{n} \frac{1}{k^{1/5}},$$

which goes to $+\infty$ when we send $n \to +\infty$. Thus $f \notin \mathrm{BPV}([-1,1])$.

Problem 3

On one hand,

$$\limsup_{n\to\infty} \left(\int_a^b f(x)^n \, dx \right)^{1/n} \leq \limsup_{n\to\infty} (b-a)^{1/n} M = M.$$

Now we get the other bound. Fix $\varepsilon > 0$. Suppose the maximum of f is achieved at x_0. WLOG we may assume $x_0 \neq b$. Then there exists $\delta > 0$ such that $M - \varepsilon \leq f(x) \leq M$ for all x with $x_0 \leq x \leq x + \delta$. It follows that

$$\left(\int_a^b f(x)^n \, dx \right)^{1/n} \geq \left(\int_{x_0}^{x_0+\delta} f(x)^n \, dx \right)^{1/n} \geq \left(\int_{x_0}^{x_0+\delta} (M-\varepsilon)^n \, dx \right)^{1/n}$$

$$= \delta^{1/n}(M - \varepsilon),$$

so

$$\liminf_{n\to\infty} \left(\int_a^b f(x)^n \, dx \right)^{1/n} \geq \liminf_{n\to\infty} \delta^{1/n}(M-\varepsilon) = M - \varepsilon.$$

But $\varepsilon > 0$ was arbitrary so the liminf is $\geq M$.

Problem 4

We can directly apply the divergence theorem. We have

$$\operatorname{div} F(x,y,z) = y^2 - z + z + x^2 + 0 = x^2 + y^2.$$

So

$$\int_S F\cdot\nu\,dS = \int_{(x,y)\in B(0,2)}\int_{z=x^2+y^2}^{4} x^2+y^2\,dz\,d(x,y)$$

$$= \int_{(x,y)\in B(0,2)} (4-x^2-y^2)(x^2+y^2)\,d(x,y) = 2\pi\int_0^2 (4-r^2)r^2\cdot r\,dr$$

$$= 2\pi\int_0^2 4r^3-r^4\,dr = 2\pi\left(16-\frac{32}{5}\right) = \frac{96\pi}{5}.$$

Problem 5

Part a

We can assume that $a,b>0$ because if $a=0$ or $b=0$ then this is stupid.

It suffices to find the minimum and maximum of $xv-uy$ and square them to find the maximum of $(xv-uy)^2$. Let $f(x,y,u,v)=xv-uy$, $g(x,y,u,v)=x^2+y^2-a^2$, and $h(x,y,u,v)=u^2+v^2-b^2$. The admissible set $E=\{g=0\}\cap\{h=0\}$ is closed and bounded, hence it is compact, so f attains a maximum and a minimum over E. Now compute

$$\nabla g(x,y,u,v) = \begin{bmatrix}2x\\2y\\0\\0\end{bmatrix},$$

$$\nabla h(x,y,u,v) = \begin{bmatrix}0\\0\\2u\\2v\end{bmatrix}.$$

Let an extremum be (x,y,u,v). To apply Lagrange multipliers we must show that $\{\nabla g(x,y,u,v),\nabla h(x,y,u,v)\}$ is linearly independent. Indeed, since $x^2+y^2=a^2>0$, one of x,y must be non-zero, and similarly one of u,v must be non-zero, which is enough.

Thus Lagrange multipliers applies: There exist $\lambda/2,\mu/2\in\mathbb{R}$ such that $\nabla f=(\lambda/2)\nabla g+(\mu/2)\nabla h$ at the extremum (the factor of 2 is for convenience). That is,

$$\begin{cases} v=\lambda x\\ -u=\lambda y\\ -y=\mu u\\ x=\mu v\\ x^2+y^2=a^2\\ u^2+v^2=b^2\end{cases}.$$

Plug in the first and second equations into the sixth equation. This gives

$$\lambda^2y^2+\lambda^2x^2=b^2.$$

Since $x^2+y^2=a^2$, this tells us $a^2\lambda^2=b^2$. Hence $\lambda=\pm\frac{a}{b}$. (Similarly we find that $\mu=\pm\frac{b}{a}$ but we don't need it.) Now,

$$f(x,y,u,v)=xv-uy=x(\lambda x)-(-\lambda y)y=\lambda(x^2+y^2)=\lambda a^2=\pm ab.$$

From this, we can say that the only possible values for the extrema are ab and $-ab$. It's simple to see that these can be obtained (take $x\pm=a$, $y=0$, $u=0$, $v=b$). So these ab and $-ab$ are the max and min respectively. We conclude that the maximum of $(xv-uy)^2$ must be a^2b^2.

Part b

lol i dont actually know why lagrange multipliers works