1 Homework 1

1. We claim x_n monotone decreasing. Indeed, $x_{n+1} \le x_n$ iff $2 + x_n \le x_n^2$ iff $(x_n - 2)(x_n + 1) \ge 0$ iff $x_n \in (-\infty, -1] \cup [2, \infty)$. So it suffices to show $x_n \ge 2$ for all n. Indeed, $x_{n+1} \ge 2$ iff $2 + x_n \ge 4$ iff $x_n \ge 2$. Since $x_0 \ge 2$ this is true by induction. So x_n decreasing. Moreover we just showed 2 is a lower bound so the limit exists. By continuity of square root we may send $n \to \infty$ in the recurrence to find $L = \sqrt{2 + L}$ and so $L \in \{-1, 2\}$. The case $L = -1$ is absurd (why?) so $L = 2$.

2. It converges to 0. We claim by induction that $|x_n| \le \frac{1000}{\sqrt{n}}$ for all $n \ge 1$, which would be sufficient. Clearly it is true for $n = 1$. Now if it is true for some n then

$$|x_{n+1}| \le \frac{10^6}{2n} + \frac{1}{2n} \le \frac{100}{n},$$

so we are done provided that $\frac{100}{n} \le \frac{1000}{\sqrt{n+1}}$. This is equivalent to $n + 1 \le 100n^2$ which is true for all n.

3. If $|a| > 1$ then can show $|x_n| = |a|^n \to \infty$. If $a = -1$ or $a = 1$ then $x_n = 1$ for all large enough n. Otherwise if $|a| < 1$ can show $|x_n| = |a|^n \to 0$. The possible limits are 0 and 1.

4. Any limit L must satisfy $L = L^2 - 1.5L + 1.5$ which factors as $(L - 1.5)(L - 1) = 0$. So $L = 1, 1.5$ are the only possible limits. To obtain them, take the constant sequences $x_n = 1 \forall n$ and $x_n = 1.5 \forall n$.

5. I feel like this problem has a typo in it but you basically do something similar to problem 1.

6. see recitation notes / brightspace announcement

7. ditto

8. take some n, we claim that $\sup_{k>n}(a_k b_k) \le \sup_{k>n} a_k \cdot \sup_{k>n} b_k$. this is because for a particular k we have $a_k b_k \le \sup_{j>n} a_j \cdot \sup_{j>n} b_j$ and now we may sup over k. now send $n \to \infty$. the assumptions in the problem ensure that the limit of the RHS exists in $[0, \infty]$ (allowing $+\infty$ as a sensible limit).

9. (a) the monotone behavior follows by working backwards from the desired statement $x_{n+1} \le x_n$ using algebra. to argue the limit exists, argue that $x_n \ge \sqrt{\alpha}$ for all n. to compute the limit, solve $L = (L + \alpha/L)/2$. (b) use induction

10. see recitation notes

11. let x_n be a bounded sequence. then there exists a bounded closed interval I containing all x_n. split I in the middle into two closed intervals $I_{1,1}$ and $I_{1,2}$ of half the size of I whose union is I. one of these intervals contains infinitely many terms of x_n, call it

I_1. Split I_1 into two halves, as before, and one will have infinitely many terms, call it I_2. repeating this forever we obtain a sequence $I \supseteq I_1 \supseteq I_2 \supseteq \dots$ of nested closed intervals. by the previous problem their intersection is nonempty, moreover their sizes decrease at the rate $c/2^n \to 0$ so their intersection has exactly one element x. we claim x is a subsequential limit of x_n. indeed, define a subsequence as follows. take $x_{n_1} \in I_1$. take $x_{n_2} \in I_2$ such that $n_2 > n_1$, which must be possible since I_2 contains infinitely many terms. and so on. by induction $|x_{n_k} - x| \leq \text{length}(I)/2^k \to 0$.

12. if the sequence x_n is unbounded, say from above, then we can recursively take x_{n_k} so that $n_k > n_{k-1}$ and $x_{n_k} > x_{n_{k-1}}$. (if this is impossible then there has to be an upper bound...) so we may presume the sequence is bounded. so it has a limit point x. if x_n contains x infinitely often then the problem is boring so assume this is not the case. then either $(-\infty, x)$ or (x, ∞) contains infinitely many points of x_n. wlog the former. then we may recursively define x_{n_k} so that $n_k > n_{k-1}$ and $x_{n_k} \in (x_{n_{k-1}}, x)$.

13. the assumptions show that the limsup exists and is finite. for each k we have by definition of limit that there exists N_K so large that $\left|\sup_{j \geq n} a_j - \limsup(a)\right| < 1/k$ for all $n \geq N_k$; in particular it's true for $n = N_k$. by properties of sup there exists some $j \geq N_k$ so that $|a_j - \sup_{j' \geq N_k} a_{j'}| < 1/k$. set j_k to be this j. then by triangle inequality $|a_{j_k} - \limsup(a)| < 1/k$ and so $a_{j_k} \to \limsup(a)$. (j_k may not be a strictly increasing sequence of indices, to fix this we can inductively enforce that $N_k > j_{k-1}$.)

14. if a_n cauchy then for all ε there is N_ε so that
$$|a_m - a_n| < \varepsilon$$
for all $m, n \geq N_\varepsilon$. in particular
$$a_n - \varepsilon < a_m < a_n + \varepsilon$$
for all such m, n. for fixed m, the above is true for all n large enough, so can send $n \to \infty$ by taking limsup and liminf, to find that
$$-\varepsilon + \limsup_n a_n \leq a_m \leq \varepsilon + \liminf_n a_n.$$
taking the two ends of this inequality shows that the limsup and liminf are within 3ε. as ε was arbitrary we deduce they are equal.

15. AFSOC x_n does not converge to x_*. then there exists $\varepsilon > 0$ such that infinitely many x_n's satisfy $|x_n - x_*| > \varepsilon$. take a subsequence from those x_n's. clearly it cannot converge to x_*, contradiction.

2 Homework 2

1. the trick is that
$$a_n = \prod_{k=1}^{n} \frac{a_k}{a_{k-1}}.$$

(where we let "$a_0 = 1$") let $L = \limsup(a_{n+1}/a_n)$. fix $\varepsilon > 0$, then there is N_ε so that for all $n > N_\varepsilon$, we have $a_{n+1}/a_n \le L + \varepsilon$. (this is a property of limsup.) now for $n > N_\varepsilon$,

$$a_n = \prod_{k=1}^{N_\varepsilon} \frac{a_k}{a_{k-1}} \prod_{k=N_\varepsilon}^{n} \frac{a_k}{a_{k-1}} \le \prod_{k=1}^{N_\varepsilon} \frac{a_k}{a_{k-1}} \cdot (L+\varepsilon)^{n-N_\varepsilon}.$$

some indices may be off by 1 but i don't care. we get that

$$a_n^{1/n} \le C_\varepsilon^{1/n} \cdot (L+\varepsilon)^{1-N_\varepsilon/n}$$

where C_ε is a constant depending only on ε (and the original sequence but whatever). taking limsup,

$$\limsup_n (a_n^{1/n}) \le L + \varepsilon.$$

but ε was arbitrary.

2. (1) omitted (2) use previous problem

3. see rubric

4. note that $\lim_n \sum_{k=n}^{\infty} |a_{k+1} - a_k| = 0$. using this, combined with the inequality

$$|a_m - a_n| \le \sum_{k=m}^{n-1} |a_{k+1} - a_k|, \qquad n > m,$$

you can show that it's cauchy, so it converges. converse is not true by taking $a_k = (-1)^k/k$.

5. let that sum be S_n. assume for now that the limit p exists. by integration

$$\int_1^{n+1} 1/\sqrt{x} \le S_n \le \int_0^n 1/\sqrt{x}$$

or

$$2\sqrt{n+1} - 2 \le S_n \le 2\sqrt{n}.$$

so

$$0 \le 2\sqrt{n} - S_n \le 2 - 2(\sqrt{n+1} - \sqrt{n}).$$

so $0 \le p \le 2$ (send $n \to \infty$). for the lower bound just show that a_n is increasing and note that $a_1 = 1$. im too lazy to tighten the upper bound to < 2. ok now let's go back and show the limit exists. this is because $a_n = a_{n-1} + \int_{n-1}^{n} 1/\sqrt{x}\,dx - 1/\sqrt{n}$. show that $\int_{n-1}^{n} 1/\sqrt{x}\,dx - 1/\sqrt{n}$ is positive (using that $1/\sqrt{x}$ is decreasing) and you win by monotonicity, combined with the upper bound of 2.

6. let that sum be H_n (this is a common notation). then for each k we have
$$\int_k^{k+1} 1/x \le \frac{1}{k} \le \int_{k-1}^{k} 1/x \qquad (*)$$
and so, if $a_n = H_n - \log n$, then by the LHS of $(*)$,
$$\begin{aligned} a_{n+1} &= a_n + 1/n + \log n - \log(n+1) \\ &= a_n 1/n - \int_n^{n+1} 1/x\,dx \ge a_n. \end{aligned}$$
so a_n increasing. it remains to get an upper bound. summing $(*)$ from $k = 2$ to $k = n$,
$$\int_2^{n+1} 1/x \le H_n - 1 \le \int_1^n 1/x$$
and so
$$\log((n+1)/2) \le H_n - 1 \le \log n.$$
so $H_n - \log n \le 1$. (rmk: γ is the euler-mascheroni constant)

7. you can show $1 + x \le e^x$ by showing the minimum of $e^x - x - 1$ is 0. (calculus 1 allowed) if we apply this then
$$\prod_{n=1}^{N}(1 + a_n) \le \exp\left(\sum_{n=1}^{N} a_n\right).$$
if $\sum a_n$ converges then the RHS converges as $N \to \infty$ (here we appeal to continuity or monotonicity of e^x). so the LHS has an upper bound. but the LHS increases in N so its limit as $N \to \infty$ exists.

8. by induction
$$a_n \le \left(\frac{1}{n+1}\right)^p a_0.$$
(this is one of those times where it's convenient to pretend that there's a zeroeth term but you don't have to do it like that if u want) we win by comparison and p-test.

9. we can actually show that $1 - px \le (1 - x)^p$ for all $x \in \mathbb{R}$. this is due to three ingredients. (1) we have $d^2/dx^2\ (1-x)^p = d/dx\ -p(1-x)^{p-1} = p(p-1)(1-x)^{p-2} \ge 0$ for all x (there may be some edge cases for, say, $p = 1$, but it is still true), so $(1 - x)^p$ is convex. (2) at $x = 0$, we have that $(1-x)^p\,|_{x=0} = 1$ and $\frac{d}{dx}(1-x)^p\,|_{x=0} = -p$, so the tangent line to $(1 - x)^p$ at $x = 0$ is given by $1 - px$. (3) a convex function is always $\ge$ a tangent line to it, everywhere. this proves the inequality. (this isn't the only way to prove it.) now suppose that $|a_{n+1}| \le (1 - p/n)|a_n|$ and $p > 1$. then
$$|a_{n+1}| \le [1 - p(1/n)] \cdot |a_n| \le (1 - 1/n)^p |a_n| = \left(\frac{n-1}{n}\right)^p |a_n|.$$
now you can apply problem 8 (the slight mismatch in indices is not a problem. why?)

10. first we need to handle an annoying problem: the a_n's may have different signs which will certain make manipulations and theorems illegal. let's see if we can show that this isn't really the case. since the limit exists, we have for all large enough n (say, $n \geq N_1$) that

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \leq L + 10.$$

so

$$1 - \frac{a_{n+1}}{a_n} \leq \frac{L+10}{n}$$

and

$$1 - \frac{L+10}{n} \leq \frac{a_{n+1}}{a_n}$$

for all $n \geq N_1$. now note that the left side goes to 1 as $n \to \infty$. so, we must have

$$0.5 \leq \frac{a_{n+1}}{a_n}$$

for all large enough n; say, $n \geq N_2$. since 0.5 is positive, what this implies is that for all $n \geq N_2$, the sign of a_n is either all positive or all negative. (because if a_n and a_{n+1} are ever of different signs, then their quotient is negative!) flipping all signs does not change whether or not the series converges, so we can assume without loss of generality that $a_n > 0$ for all large enough n (i.e. $n \geq N_2$).

now we can actually tackle the problem. suppose that $L > 1$. let $\varepsilon = (L-1)/2$. then for all large enough n (say $n \geq N' \geq N$),

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geq L - \varepsilon = \frac{L+1}{2} = 1 + \frac{L-1}{2} = 1 + \varepsilon.$$

so

$$1 - \frac{a_{n+1}}{a_n} \geq \frac{1+\varepsilon}{n}$$

so

$$1 - \frac{1+\varepsilon}{n} \geq \frac{a_{n+1}}{a_n}$$

so (since a_n's are positive)

$$a_{n+1} \leq \left(1 - \frac{1+\varepsilon}{n}\right) a_n$$

so (by the previous problem and the fact that $1 + \varepsilon > 1$) we have convergence!

now suppose that $L < 1$. then

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geq 1$$

for all large enough n; say, all $n \geq N \geq N_2$. (elaboration: take $\varepsilon = 1 - L > 0$, then by definition of limit, there is some N for which $n\left(1 - \frac{a_{n+1}}{a_n}\right) \leq L + \varepsilon$ for all $n \geq N$. oh wait, $L + \varepsilon = 1$.) this implies

$$1 - \frac{a_{n+1}}{a_n} \leq \frac{1}{n}$$

so

$$\frac{n-1}{n} \leq \frac{a_{n+1}}{a_n}$$

for all $n \geq N$. now for all $n > N$,

$$a_n = a_N \prod_{k=N+1}^{n} \frac{a_k}{a_{k-1}} \geq a_N \prod_{k=N+1}^{n} \frac{k-2}{k-1} = \frac{N-1}{n-1} a_N = \frac{N-1}{a_N} \cdot \frac{1}{n-1}.$$

so by comparison $\sum a_n$ diverges because eventually it dominates the harmonic series.

lastly suppose $L = 1$. we'll come up with a sequence a_n which causes convergence and another sequence which causes divergence.

a natural guess for divergence is $a_n = 1/n$. indeed for this a_n,

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = n\left(1 - \frac{n}{n+1}\right) = \frac{n}{n+1} \to 1$$

so this satisfies the condition $L = 1$.

getting a convergent example is trickier. we'd like to choose something like $a_n = 1/n^{1.1}$, but this probably "converges too fast" to get $L = 1$. so let's try a series which converges slower than that. we can take

$$a_n = \frac{1}{n \log(n)^2},$$

which converges by integral test. now let's plug it into the limit and hope we get $L = 1$.

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = n\left(1 - \frac{n\log^2(n)}{(n+1)\log^2(n+1)}\right) = \frac{n}{n+1}\left(n + 1 - \frac{n\log^2(n)}{\log^2(n+1)}\right)$$

$$= \frac{n}{n+1}\left(1 - n\left(1 - \frac{\log^2 n}{\log^2(n+1)}\right)\right)$$

alright, we know $\frac{n}{n+1} \to 1$, so we want to show that $\left(1 - n\left(1 - \frac{\log^2 n}{\log^2(n+1)}\right)\right) \to 1$. this means we want to show that $n\left(1 - \frac{\log^2 n}{\log^2(n+1)}\right) \to 0$.

let's show that. we have

$$1 - \frac{\log^2 n}{\log^2(n+1)} = \frac{\log^2(n+1) - \log^2 n}{\log^2(n+1)}$$

$$= \frac{[\log(n+1) - \log(n)] \cdot [\log(n+1) + \log(n)]}{\log^2(n+1)}$$

$$= \log\left(1 + \frac{1}{n}\right) \cdot \frac{\log(n+1) + \log n}{\log^2(n+1)}$$

$$\leq \log\left(1 + \frac{1}{n}\right) \cdot \frac{\log(n+1) + \log(n+1)}{\log^2(n+1)}$$

$$\leq \log\left(1 + \frac{1}{n}\right) \cdot \frac{2}{\log(n+1)}$$

by the way there's this inequality $\log(1+x) \leq x$, try to prove it (hint: $1 + x \leq e^x$).

$$\leq \frac{1}{n} \cdot \frac{2}{\log(n+1)}$$

therefore

$$n\left(1 - \frac{\log^2 n}{\log^2(n+1)}\right) \leq \frac{2}{\log(n+1)} \to 0.$$

tada. easy problem. literally trivial.

3 Homework 3

1. by partial geometric series formula (with $r = e^{ix}$),

$$S_n = \sum_{k=1}^{n} e^{ikx} = \frac{e^{ix}(1 - e^{inx})}{1 - e^{ix}}.$$

now you have to do stuff.

$$= \frac{e^{ix/2}(1 - e^{inx})}{e^{-ix/2} - e^{ix/2}}$$

$$= e^{ix/2} \frac{1 - e^{inx}}{-2i \sin(x/2)}$$

$$= e^{i(n+1)x/2} \frac{e^{-inx/2} - e^{inx/2}}{-2i \sin(x/2)}$$

$$= e^{i(n+1)x/2} \frac{-2i \sin(nx/2)}{-2i \sin(x/2)}$$

so we're basically done with that.

now

$$|S_n| \leq \frac{|\sin(nx/2)|}{|\sin(x/2)|} \leq \frac{1}{|\sin(x/2)|}$$

which is nice. in particular,

$$\left|\sum_{k=1}^{n} \sin k\right| = |\operatorname{Im} S_n| \leq |S_n| \leq 1/\sin(1/2).$$

in other words the sum of sines is bounded (!!). so that's nice.

to conclude that $\sum \sin(n)/n$ converges, first use summation by parts:

$$\sum^N \sin(n)/n \sim \frac{1}{N}\left(\sum_{n=1}^N \sin n\right) - \sum_{n=1}^N \left(\sum_{j=1}^n \sin j\right)(1/n - 1/(n+1))$$

(i write $\sim$ because i almost definitely screwed up the indices, but i dont care)

the first term $\to 0$ because sum of sines is bounded and $1/N \to 0$. the second term converges absolutely because $\sum_{n=1}^N |\ldots| \le (1/\sin(1/2)) \sum_{n=1}^N 1/n - 1/(n+1) < \infty$.

this shows that it converges. remains to show it does not converge absolutely. to do this, pick a positive integer $n_k \in [k\pi + \pi/4, (k+1)\pi - \pi/4]$ for all k. then $|\sin(n_k)| \ge \sqrt{2}/2$. so $\sum |\sin(n)|/n \ge \sum_k |\sin n_k|/n_k \ge \sum_k \sqrt{2}/(2n_k) \ge 0.01 \sum_k \frac{1}{(k+1)\pi} = +\infty$.

2. this problem seemed silly to me
3. ditto
4. $a_n = 1$ if n odd, $a_n = 2$ if n even.
5. use the AM-GM inequality $\sqrt{|ab|} \le \frac{1}{2}(|a| + |b|)$, combined with comparison.
6. (1) true. take $y_n = (-1)^n$, $x_n = (-1)^n/n$. (2) true...? unless im missing something.
7. (1) conditionally; alternating series test; to see it doesn't converge absolutely try showing that the terms are $\ge 1/n^{0.6}$ for all n large enough. (2) conditionally; use summation by parts as in problem 1 (3) integral test (4) conditionally; alternating series test; to show doesn't converge absolutely try using the bound $\sin(1/n) \ge \frac{1}{100n}$ and compare with an integral.
8. see rubric
9. (a) for fixed m, $a_{l,m}$ is zero for all $l > m$ so the limit better be 0. other limit is similar. (b) first two limits exist and $= 0$ by part (a). the second one is also zero because for all ε you can show that if $l, m \ge 100/\varepsilon$ then $|a_{l,m}| < \varepsilon$.
10. (a) you need $|r| < 1$ (b) ditto (c) i think it's fine? (d) once again want $|r| < 1$ (e) seems legit (f) it's true (g) there's no way those sums converge, $\cos(nx)$ should get pretty close to both 1 and -1 infinitely often. not entirely sure what the point of this problem was, i guess it's like a "fake proof" that that sum is 1/2? so many issues come from the fact that the correct identities only are kosher for $|r| < 1$ though.