Day 1

Monday, July 22, 2024 2:59 AM

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Tentative Schedule:

Day 1
- Area & Volume
- Change of Variables
  - Polar / Cylindrical
  - Spherical
- Line Integrals
- Surface Integrals
- Green's, Divergence, Stoke's

Day 2
- Convergence Tests
- Swapping Limiting Operations
- Asymptotics
- Integral Inequalities

Day 3
- Optimization and LM
- Implicit & Inverse Func Thm
- Fourier Series / Transform

Day 4
- Problems with Recursive sequences
- "Bare Hands"
- Miscellanea (?)

Other topics that could show up:

- Taylor series
- Convex functions
- (Riemann) Integrability
- Continuity, IVT
- MVT
- Hölder Ineq.
- Stone-Weierstrass
- Ascoli-Arzela
- Conservative & Irrotational vector fields
- Multivariate Differentiability
- Uniform, Lipschitz, Hölder Continuity
- "Measure theory"

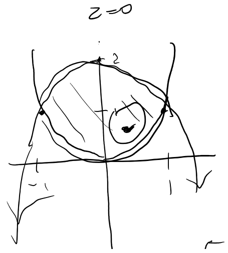
Example (GRE):

Find the volume of the region bounded by $y = x^2$, $y = 2 - x^2$, $z = 0$, and $\underbrace{z = y + 3}$.

z=0

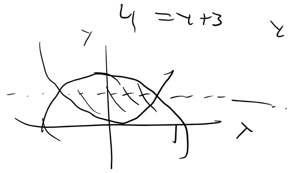
lemon

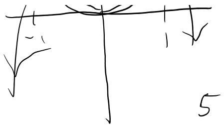
z=4

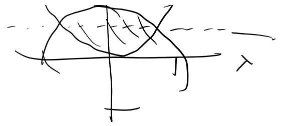


$y = x^2$, $y = 2 - x^2$

$4 = y + 3$ $\quad y = 1$







$$? \quad \int_{z=0}^{5} \int_{x=-1}^{1} (2-x^2) - (x^2) \, dx \, dz \quad ?$$

$$\int_{z=0}^{3} \qquad \int_{3}^{5}$$

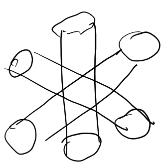
$$\int_{x=-1}^{1} \int_{y=x^2}^{2-x^2} (y+3) \quad dy \, dx$$

$$\frac{1}{2} y^2 + 3y \Big|_{x^2}^{2-x^2}$$

$32/3$

$$\int_{x=-1}^{1} \frac{1}{2}(2-x^2)^2 - \frac{1}{2}x^4 + 3(2-x^2) - 3(x^2)$$

$x > 0$
$y > 0$
$z > 0$



$$\{x^2+y^2<1\} \cap$$
$$\{x^2+z^2<1\} \cap$$
$$\{y^2+z^2<1\}$$

$$\int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 1_{x^2+y^2<1} \, 1_{x^2+z^2<1} \, 1_{y^2+z^2<1} \quad dz\,dy\,dx$$

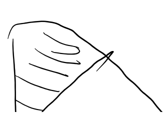
$$1_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \qquad 1_p = \begin{cases} 1, & p \text{ true} \\ 0, & p \text{ false} \end{cases}$$

$$\int_{x=0}^{1} \int_{y=0}^{1} 1_{x^2+y^2<1} \int_{z=0}^{1} \underbrace{1_{x^2+z^2<1} \, 1_{y^2+z^2<1}}_{1} \, dz\,dy\,dx$$

$z < \sqrt{1-x^2}$, $z < \sqrt{1-y^2}$

$$0 < z < \min(\sqrt{1-x^2}, \sqrt{1-y^2})$$

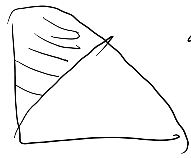
$$\iint_{(x,y)\in} \min(\sqrt{1-x^2}, \sqrt{1-y^2}) \; dy\,dx$$



$\{x<y\}$

$(1-\sqrt{2})\pi$?

$16(2-\sqrt{2})\pi$?



$\{x<y\}$

$\int x \sin(x^2)$ $\qquad u = x^2$

$f: \mathbb{R}^n \to \mathbb{R}$, $\quad x = g(y)$

$$\int_{x\in E} f(x)\,dx = \text{“} \int_{g(y)\in E} f(g(y))\; dg(y) \text{”}$$

$$= \text{“} \int_{g(y)\in E} f(g(y))\; \frac{dg(y)}{dy}\; dy \text{”}$$

$$= \int_{y\in g^{-1}(E)} f(g(y))\, \left|\det Dg(y)\right|\, dy$$

- polar coords / cylindrical
- spherical

$$\int_{E\subseteq\mathbb{R}^2} f(x,y)\; d(x,y) \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} r\, f[r, \theta]$$

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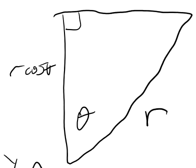
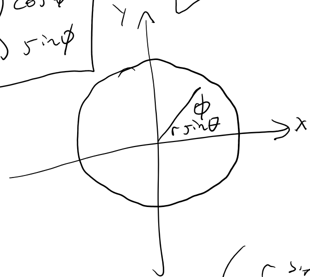
①

$$\boxed{\begin{aligned} Z &= r\cos\theta \\ x &= (r\sin\theta)\cos\phi \\ y &= (r\sin\theta)\sin\phi \end{aligned}}$$





$0 < \theta < \pi$

$$g(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

$$\left|\det Dg(r,\theta,\phi)\right| = r^2\sin\theta$$

$c : [a,b] \overset{\subset \mathbb{R}}{\longrightarrow} \mathbb{R}^n$

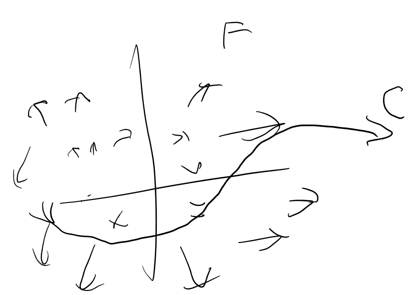
Line Integrals

- $f : \mathbb{R}^n \to \mathbb{R}$
- $F : \mathbb{R}^n \to \mathbb{R}^n$

$C, \quad c : [a,b] \to \mathbb{R}^n$

$$\int_C f\,ds = \int_a^b f(c(t))\,|c'(t)|\,dt$$

$$\int_C F\cdot ds = \int_a^b F(c(t))\cdot c'(t)\,dt$$



$F|_c = c'$

$$\int_C F\cdot ds$$

$$= \int_a^b F(c(t))\cdot c'(t)\,dt$$

$$= \int_a^b c'(t)\cdot c'(t)\,dt$$

$$= \int_a^b \|c'(t)\|\,dt$$

$$= \text{len}(C)$$

$$\int_C F\cdot ds$$

$$= \int_C F\cdot dr$$

$$= \int_C F$$

(in $\mathbb{R}^2$) $\quad F = (M, N)$

$$= \int_C M\,dx + N\,dy$$

" $= \int F \cdot ds$ "

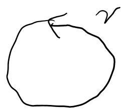
" $= \int_C F \cdot ds$ "

Example: · $F(x,y) = (xy,\ x-y)$

· γ = unit circle, CCW.

a) $\int_\gamma F \cdot ds$

b) $\int_\gamma x\, dy$



$e(t) = (\cos t,\ \sin t)$

$t \in [0, 2\pi)$

$F(e(t))$

$$\int_\gamma F \cdot ds = \int_0^{2\pi} (\cos t \sin t,\ \cos t - \sin t) \cdot (-\sin t,\ \cos t)\, dt$$

$$= \boxed{\int_0^{2\pi} -\sin^2 t \cos t + \cos^2 t - \sin t \cos t \; dt}$$

$$\int_0^{2\pi} \cos^2 t \; dt =$$

$$2\cos^2\theta - 1 = \cos(2\theta)$$

$$\cos^2\theta = \frac{\cos(2\theta) + 1}{2}$$

$$\int_0^{2\pi} \cos^2 t\, dt = \frac{1}{2}\left(\int_0^{2\pi} \sin^2 t\, dt + \int_0^{2\pi} \cos^2 t\, dt \right)$$

$$= \frac{1}{2} \int_0^{2\pi} 1 = \pi$$

$$\int_\gamma x\, dy \text{ "=" } \int_\gamma (0, x) \cdot (dx, dy) \text{ "}$$

$$= \int_\gamma (0,\ \underbrace{x}) \cdot ds$$

$\varphi(t) = (\cos t, \sin t)$

$$= \int_0^{2\pi} (0, \cos t) \cdot (-\sin t, \cos t)$$

$$= \int \cos^2 t$$

$f: \mathbb{R}^n \to \mathbb{R}$, $S \subseteq \mathbb{R}^n$ (n-1)-dim surface,

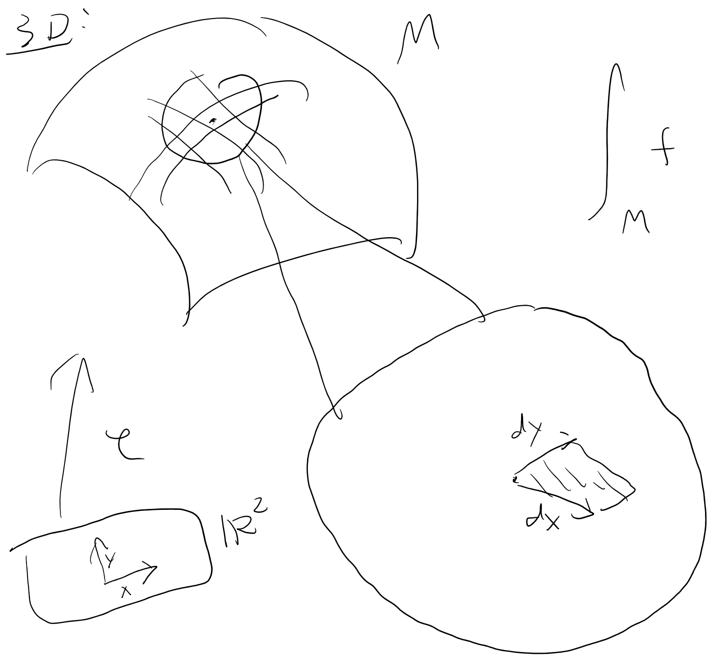
$\varphi: U \subseteq \mathbb{R}^{n-1} \to S$

$$\int_M f \, dS = \int_U f(\varphi(y)) \sqrt{\det[D\varphi(y)^T D\varphi(y)]} \, dy$$

- $\int_M f \, dA$, $\int_M f \, d\sigma$, $\int_M f \, d\Sigma$

$\int_M f \, d\mathcal{H}^{n-1}$

3D:



$$\int_M f \, dS$$

$$\left\| \frac{\partial \varphi}{d x} \times \frac{\partial \varphi}{d y} \right\|$$

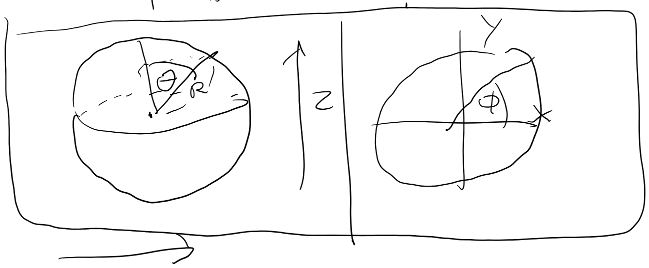
$\rightsquigarrow$ "Area of $dx \times dy$ portion"

Example Let f be nice,

Example Let f be nice

$f : \mathbb{R}^3 \to \mathbb{R}$

$$\int_{\partial B(0,R)} f \, dS$$



$\varphi :$ θ ϕ $\longrightarrow$

$$\varphi(\theta, \phi) = \left(R\sin\theta(\cos\phi),\ R\sin\theta(\sin\phi),\ R\cos\theta \right)$$

$$\left\| \frac{\partial \varphi}{\partial \theta} \times \frac{\partial \varphi}{\partial \phi} \right\| (\theta, \phi) = \left\| \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} \times \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} \right\|$$

$$= \left\| \left(R^2\sin^2\theta\cos\phi,\ -R^2\sin\theta\cos\theta\cos^2\phi,\ R^2\sin\theta\cos\theta\cos^2\phi + R^2\sin\theta\cos\theta\sin^2\phi \right) \right\|$$

$= \sim\sim\sim$

$= R^2 \sin\theta$ Yay

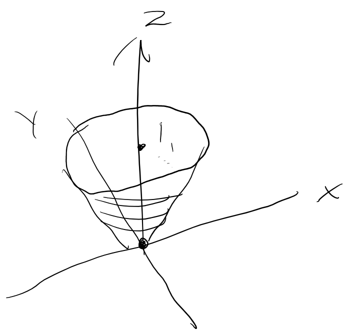
$$\boxed{\int_{\partial B(0,R)} f \, dS = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\varphi(\theta,\phi)) R^2 \sin\theta \, d\phi \, d\theta}$$

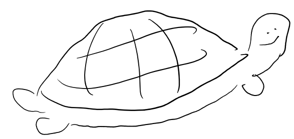
Example :

Example:

Let $f : \mathbb{R}^3 \to \mathbb{R}$, $\quad f(x,y,z) = x^2 + yz$.

Find $\displaystyle\int_C f \, dS$, $\quad C$ is the curved surface of the up-side down circular cone with base $B_2(0,1) \times \{1\}$ and vertex $(0,0,0)$.





$$\varphi(r, \phi) = (r\cos\phi, \; r\sin\phi, \; r)$$

$$\frac{\partial \varphi}{\partial r}(r,\phi) = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 1 \end{pmatrix} \qquad \frac{\partial \varphi}{\partial \phi}(r,\phi) = \begin{pmatrix} -r\sin\phi \\ r\cos\phi \\ 0 \end{pmatrix}$$

$$\left\| \frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial \phi} \right\| (r,\phi) = \sqrt{ \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix}^2 + \begin{vmatrix} \cos\phi & -r\sin\phi \\ 1 & 0 \end{vmatrix}^2 + \begin{vmatrix} \sin\phi & r\cos\phi \\ 1 & 0 \end{vmatrix}^2 }$$

$$= \sqrt{ r^2 + r^2 \sin^2\phi + r^2\cos^2\phi }$$

$$= \sqrt{2r^2} = \sqrt{2}\, r$$

$$\int_C ds = \int_{r=0}^{1} \int_{\phi=0}^{2\pi} \left(r^2 \cos^2\phi + r^2 \sin\phi \right) \sqrt{2} r \, d\phi \, dr$$

$$= \left(\int_{r=0}^{1} \sqrt{2} r^3 \, dr \right) \left(\int_{\phi=0}^{2\pi} \cos^2\phi + \sin\phi \, d\phi \right)$$

$$= \frac{\sqrt{2}}{4} \qquad\qquad = \pi$$

$$= \boxed{\frac{\sqrt{2}\,\pi}{4}}$$

Green's Theorem

Day 2

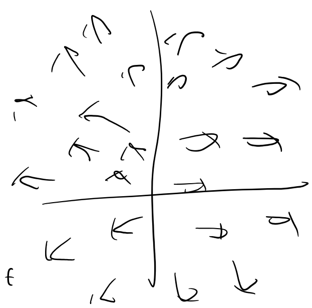
Thursday, July 25, 2024 2:39 AM

Housekeeping:

- Notes
- New resources on wiki !!!
- Exam difficulty, errors

Vector Fields

$F : \mathbb{R}^n \to \mathbb{R}^n$



$f : \mathbb{R}^n \to \mathbb{R}$

- F is conservative if

$\exists\ f : \mathbb{R}^n \to \mathbb{R}$ s.t. $\nabla f = F$

"potential"

- C is a path

$$\int_C F \cdot dr = f(b) - f(a)$$

Example 2.1 (Winter 2017 #3): Consider the integral

$$I = \int_\Gamma \frac{x}{x^2+y^2}\,dx + y\frac{1-x^2-y^2}{x^2+y^2}\,dy$$

integrated over a path Γ.

(a) Show that I does not depend on the path Γ chosen to connect two fixed points.

(b) Compute I if Γ is a path joining $A = (0,1)$ to $B = (1,1)$.

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$$F(x,y) = \left(\frac{x}{x^2+y^2} \, , \; y\frac{1-x^2-y^2}{x^2+y^2} \right)$$

f s.t. (?) $\nabla f = F$

$$\rightsquigarrow \quad \frac{\partial f}{\partial x}(x,y) = \frac{x}{x^2+y^2}$$

∂f ...

$\partial x \qquad x^2+y^2$

$$\rightsquigarrow \quad \frac{\partial f}{\partial y}(x,y) = y \cdot \frac{1-x^2-y^2}{x^2+y^2}$$

$$\int \frac{\partial f}{\partial x}(x,y)\, dx = \int \frac{x}{x^2+y^2}\, dx$$

"constant"

$$f(x,y) = \frac{1}{2}\log(x^2+y^2) + g(y)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{x^2+y^2} + g'(y)$$

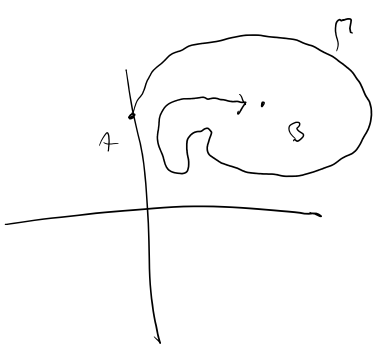
$$y \cdot \frac{1-x^2-y^2}{x^2+y^2} = \frac{y}{x^2+y^2} + g'(y)$$

$$-y = g'(y)$$

$$g(y) = -\frac{1}{2}y^2$$

$$f(x,y) = \frac{1}{2}\log(x^2+y^2) - \frac{1}{2}y^2$$

Now: Need to check f actually works. ✓



$$\int_\Gamma F \cdot dr$$

$$= \boxed{f(1,1) - f(0,1)}$$

Divergence

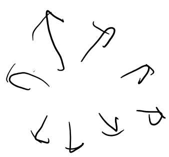
$F : \mathbb{R}^n \to \mathbb{R}^n$,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

$$: \mathbb{R}^n \to \mathbb{R}$$

$$= \nabla \cdot F \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

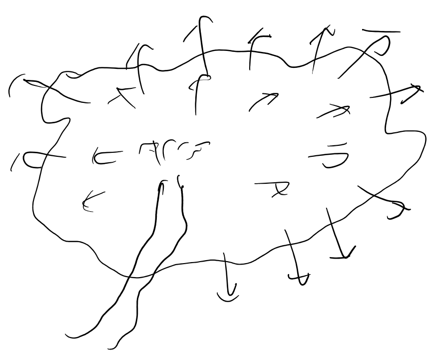
$$F = (F_1, \ldots, F_n)$$

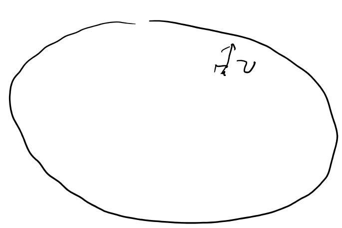


Divergence Theorem:

$U \subseteq \mathbb{R}^3$, F v.f., then

$$\int_U \operatorname{div} F = \int_{\partial U} F \cdot \nu \, dS$$





Stokes: Curl

$\cdot$ $F : \mathbb{R}^2 \to \mathbb{R}^2$,

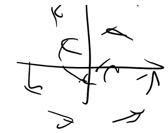
$$\operatorname{curl} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \text{"}\begin{vmatrix} \frac{\partial}{\partial x} & F_1 \\ \frac{\partial}{\partial y} & \end{vmatrix}\text{"}$$

$F(x,y) =$

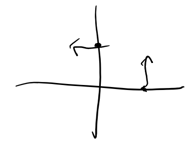
- $F: \mathbb{R}^2 \to \mathbb{R}^2$,

$$\text{curl } F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & F_1 \\ \frac{\partial}{\partial y} & F_2 \end{vmatrix}$$

$: \mathbb{R}^2 \to \mathbb{R}$

$F(x,y) =$ 

$= (-y, x)$



$\text{curl } (-y, x) = 1 - -1 = 2 > 0$

✓

- $F: \mathbb{R}^3 \to \mathbb{R}^3$,

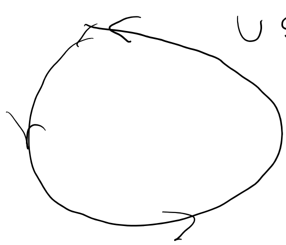
$$\text{curl } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$: \mathbb{R}^3 \to \mathbb{R}$

$$\begin{vmatrix} \frac{\partial}{\partial x} & F_1 & \hat{i} \\ \frac{\partial}{\partial y} & F_2 & \hat{j} \\ \frac{\partial}{\partial z} & F_3 & \hat{k} \end{vmatrix}$$

$\text{curl } F = \nabla \times F$

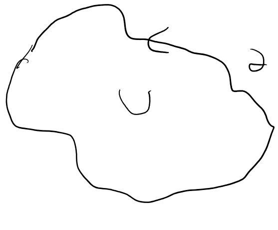
<u>Green's Thm:</u> $(\mathbb{R}^2)$



$U \subseteq \mathbb{R}^2$, $\quad F: \mathbb{R}^2 \to \mathbb{R}^2$,

∂U

$$\int_U \text{curl } F = \int_{\underbrace{\partial U}_{ccw}} F \cdot ds$$



∂U

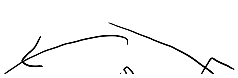
U

$F = \left(\frac{-y}{2}, \frac{x}{2} \right)$ $\quad (-y, 0)$

$(0, x)$

$\text{curl } F = \frac{1}{2} - -\frac{1}{2} = 1$

<u>Stokes:</u>



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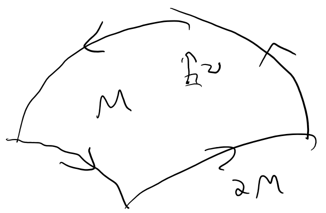
Stokes:



$S \subseteq \mathbb{R}^3$

$$\int_M \operatorname{curl} F \cdot \nu \, dS = \int_{\partial M} F \cdot ds$$

Example 2.2 (Fall 2007 #2): Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{F}(x, y, z) = (2x - y^2 - x^3, 3y - y^3, -x - z^3)$.
For a closed surface S in $\mathbb{R}^3$, consider $\int_S \mathbf{F}.\mathbf{n}dA$, the flux of $\mathbf{F}$ through S. Here $\mathbf{n}$ is chosen to be an outward normal. For what choice of S will $\int_S \mathbf{F}.\mathbf{n}dA$ be *maximal*? Explain your answer and compute $\int_S \mathbf{F}.\mathbf{n}dA$ in that case.

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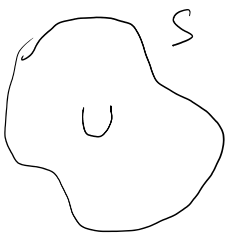

$$\int_S F . n \, dA$$

$$\boxed{\int_S F \cdot n \, dA}$$

$$= \int_U \operatorname{div} F \, dV = \int_U 2 - 3x^2 + 3 - 3y^2 - 3z^2 =$$

$$= \int_U 5 - 3x^2 - 3y^2 - 3z^2 \, d(x, y, z)$$

$$U = B\left(0, \sqrt{\tfrac{5}{3}}\right)$$

$$\int_{B(0,\sqrt{5/3})} 5 - 3(x^2+y^2+z^2) \quad dV$$

$x =$

$y =$

$z =$

$$\underbrace{4\pi \int_0^{\sqrt{5/3}} (5 - 3r^2)\, r^2\, dr}_{4\pi r^2}$$

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Example 2.4 (Winter 2009 #2): Compute

$$\oint_L (y-z)dx + (z-x)dy + (x-y)dz$$

where L is the curve given by the intersection of the two surfaces

$$\begin{cases} x^2+y^2+z^2 = a^2, a > 0 \\ x+y+z = 0 \end{cases}$$

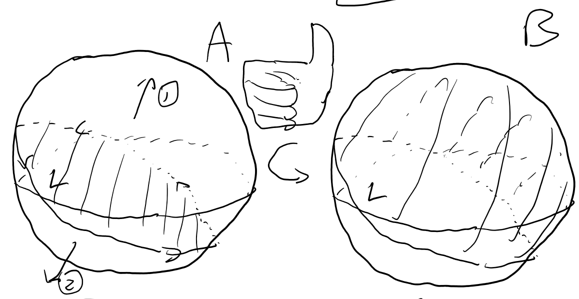
with counterclockwise orientation viewed from the positive x-axis.

$$\boxed{(-2,-2,-2)\cdot v}$$

$$F(x,y,z) = (y-z,\ \overbrace{z-x}^{\partial z},\ \overbrace{x-y}^{\partial y})$$

$$\text{curl } F = (-2,\ -2,\ -2)$$

$$\begin{vmatrix} & & \hat{i} \\ \partial_y & F_2 & \\ \partial_z & F_3 & \end{vmatrix} \quad \partial_y(x-y) - \partial_z(z-x)$$



A

B

(x,y,z)

$v = (x,y,z)$

$x+y+z=0 \quad = \{(x,y,z) : (x,y,z)\cdot(1,1,1) = 0\}$

$$v = \pm\frac{(1,1,1)}{\sqrt{3}}$$

$$\int_L F\cdot ds = \int_M \text{curl } F \cdot n \quad dS$$

$$= \int_M (-2,-2,-2)\cdot n \quad dS$$

$$= \int_M (-2,-2,-2)\cdot \frac{(1,1,1)}{\sqrt{3}}$$

$$= -2\sqrt{3} \int_M 1 \, dA$$

$$= -2\sqrt{3} \text{ Area (Ellipse)}$$

$$= \boxed{-2\sqrt{3} \cdot \pi a^2}$$

Series

$$\sum_{n=1}^{\infty} a_n \qquad \sum_{n=1}^{\infty} f_n(x)$$

- Stupid test: if $a_n \not\to 0$, $\Rightarrow$ divergence

- Direct comparison: $a_n \geq 0$, $a_n \leq b_n$

$$\underbrace{\sum a_n}_{\Rightarrow < \infty} \leq \sum b_n < \infty$$

- Ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1, \quad \Rightarrow \text{converge}$$

$$= r > 1, \quad \Rightarrow \text{divergence}$$

$$= r = 1 \quad \Rightarrow \text{know anything}$$

- Integral:

$$\sum_{n=1}^{\infty} f(n), \qquad f \geq 0, \quad f \searrow$$

$$\Downarrow \qquad \Leftrightarrow \quad \int^{\infty} f(x)\, dx$$

$\downarrow$

converge $\iff \int_1^\infty f(x)\,dx$

$a_n \geq 0$

- Alternating series: $\sum (-1)^n a_n, \quad a_n \searrow 0$

$$\sum \frac{(-1)^n}{\log n}$$

- $\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$

- Limit comparison:

$a_n, b_n \geq 0, \qquad \frac{a_n}{b_n} \to L \in (0, \infty)$

$\Rightarrow \quad \sum a_n \iff \sum b_n$

- Dirichlet test:

$$\sum_n \frac{n}{n^3 + n^2 + n} \leq \sum \frac{n}{n^3}$$

$$= \sum \frac{1}{n^2} < \infty$$

$$\sum_n \frac{n}{n^3 + n^2 + n} \times \cdot \frac{n^3}{n^3}$$

$$\asymp \sum \frac{n}{2}$$

$$\asymp \sum \frac{n}{n^3}$$

$$\sum_{10^{100}} \frac{n}{n^3 - n^2 - n} \le \sum_{10^{100}} \frac{n}{\frac{1}{2} n^3}, \quad n \ge 10^{100}$$

$$< \infty$$

$$\sum_n \frac{1}{2^{\sqrt{n}}}$$

$$\frac{1}{2^{\sqrt{n}}} \le \frac{1}{n^2} \quad n \ge 10^{100}$$

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Example 2.11 (Fall 2008 #2): For each of the following, find the range of $x \in \mathbb{R}$ for which the series converges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{n^2 + x^{2n}}$

a) $x > 1$ bad

$x < -1$ bad

$$\frac{x^n(1-x^n)}{n}$$

$[-1, 1]$ $|x| < 1$ good! ✓

$$\frac{1}{n} \le 1$$

$\Rightarrow \sum x^n$ converges

$$\sum \left| \frac{x^n(1-x^n)}{n} \right| \le \sum \frac{|x|^n \cdot 2}{n}$$

$\leq \sum |x|^n \cdot 2 < \infty$

$x=-1$

$x=1$ $\quad \sum \frac{x^n(1-x^n)}{n} \xrightarrow{x=1} \sum 0 = 0$

$x=-1 \quad \longrightarrow \quad \sum \frac{(-1)^n(1-(-1)^n)}{n}$

$= \sum \frac{(-1)^n - 1}{n}$

$= \frac{-2}{1} + \frac{0}{2} + \frac{-2}{3} + \frac{0}{4} + \frac{-2}{5}$

$= -2\underbrace{\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots\right)}_{=+\infty}$

$\boxed{x \in (-1, 1]}$

$\sum_{n=1}^{\infty} \frac{nx^n}{n^2 + x^{2n}}$

$[-1, 1]$

$|x| < 1$

$\sum_{n=1}^{\infty} \frac{n|x|^n}{n^2}$

$= \sum \frac{|x|^n}{n}$

$\leq \sum |x|^n < \infty$

$|x| > 1$

$\sum_{n=1}^{\infty} \frac{n|x|^n}{n^2 + x^{2n}} \cdot \frac{x^{2n} + n^3}{x^{2n}}$

$n^2 + x^{2n} \leq 2x^{2n} \quad n \geq$

$\sum_{n=1}^{\infty} \frac{n|x|^n}{|x|^{2n}} = \sum_{n=1}^{\infty} \frac{n}{|x|^n}$

$n \leq |x|^{n/2} \quad |x| > 1$

$\leq \frac{|x|^{n/2}}{|x|^n} = \frac{1}{\sqrt{|x|}^n}$

$$\sum \frac{n x^n}{n^2 + x^{2n}} \qquad x = -1 \qquad x = 1$$

$x=-1 \rightsquigarrow \sum \frac{n \cdot (-1)^n}{n^2+1} \qquad \frac{n}{n^2+1} \searrow$

$x=1 \rightsquigarrow \sum \frac{n}{n^2+1}$

$$\geq \sum \frac{n}{\frac{1}{2} n^2} = 2 \sum \frac{1}{n} = +\infty$$

$\boxed{x \neq 1}$

pointwise vs. uniform

$f_n : I^{\subseteq \mathbb{R}} \to \mathbb{R}$, "$f_n \to f$"

- pointwise: $\forall x \in I$, $\lim_n f_n(x) \to f(x)$
- uniform: $\sup_x |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$

$\hookrightarrow f_n \to f$ uniformly

$\Leftrightarrow$

$|f_n(x) - f(x)| \leq M_n \xrightarrow{n\to\infty} 0$

$\sum_{n=1}^{\infty} f_n$ converge uniformly

$\Leftrightarrow \sup \left| \sum_{n=N}^{\infty} f_n \right| \longrightarrow 0$

$\Uparrow \quad \not\Downarrow$

$\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$

- $\frac{\sin x}{n} \xrightarrow{\text{unif}} 0$ b/c: $|\sin x| \leq 1$

$\left| \frac{\sin x}{n} - 0 \right| \leq \frac{1}{n} \to 0$

- $\sum_{n=1}^{\infty} \frac{\sin x}{n^2}$ converges unif.

$\Uparrow$ M-test

$\left| \frac{\sin x}{n^2} \right| \leq \frac{1}{n^2}$

$\sum_n \frac{1}{n^2} < \infty$

Example 3.2 (Fall 2019 #3): For $\alpha \geq 1$, the sequence of functions $\{f_n\}$ is defined by

$$f_n(x) = x^\alpha \ln\left(x + \frac{1}{n}\right), \qquad x \in (0, \infty).$$

Show that (a) $\{f_n\}$ is uniformly convergent when $\alpha = 1$; (b) $\{f_n\}$ is not uniformly convergent when $\alpha > 1$.

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a) $f_n \to f$, $f_n(x) = x \log\left(x + \frac{1}{n}\right)$

pointwise

$f(x) := x \log x$

$|f_n(x) - f(x)| = \left| x\log(x + 1/n) - x\log x \right|$

$= x \cdot \left(\log\left(x + \frac{1}{n}\right) - \log(x) \right)$

$$= x \cdot \log\left(1 + \frac{1}{nx}\right)$$

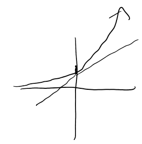
$$\leq x \cdot \frac{1}{nx}$$

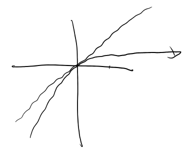
$$= \frac{1}{n} \longrightarrow 0 \quad \checkmark$$

$\log(1+y) \leq y$

$1 + x \leq e^x$

$\log(1+y) \leq y$





Swapping

- $\lim$
- $\sum_{n=1}^{\infty}$
- d/dx
- $\int$

$$\lim_n \int_E f_n \, dx \overset{?}{=} \int_E \lim_n f_n \, dx$$

$\lim \leftrightarrow \int$

$$\lim_n \int_a^b f_n(x) \, dx \overset{?}{=} \int_a^b \lim_n f_n(x) \, dx$$

$$= \int_a^b f(x) \, dx \qquad f_n \xrightarrow{\text{ptwise}} f$$

Can swap $\quad f_n \xrightarrow{\text{unif.}} f$

pf

$$\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| \overset{?}{\longrightarrow} 0$$

$$\leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq M_n} \, dx \quad , \; M_n \to 0$$

$$\leq \int_a^b M_n \, dx = (b-a) M_n \longrightarrow 0$$

$\square$

Bigger Guy: "Bounded convergence Th."

Bigger Gun: "Bounded convergence Thm"

Can swap if all the f_n's are bdd by the same constant,

$\exists M \quad |f_n(x)| \leq M \quad \forall n \quad \forall x \in [a,b]$

$\frac{d}{dx} \longleftrightarrow \int \qquad \frac{d}{dx}\int_a^b f(x,t)\,dt \overset{?}{=} \int_a^b \frac{\partial f}{\partial x}(x,t$

- MVT + BCT : $\frac{\partial f}{\partial x}(x,t)$ unif bdd over $x \in \mathbb{R}$ and $t \in [a,b]$

then can swap

$\lim \longleftrightarrow \sum \quad \Big| \quad f_n(x) \qquad x_0 \in \mathbb{R}.$

$\sum_{n=1}^{\infty} f_n$ converges uniformly $\Rightarrow \lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) \overset{?}{=} \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x)$

$$= \sum_{n=1}^{\infty} f_n(x_0)$$

$\frac{d}{dx} \longleftrightarrow \sum \quad \Big| \quad f_n(x) \qquad \sum_{n=1}^{\infty} f_n(x)$ converges ptwise

$\sum_{n=1}^{\infty} f_n'(x)$ converges ptwise

$\sum_{n=1}^{\infty} f_n'$ converges uniformly $\Rightarrow \frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x)$

Induction: $\sum f_n'$ c. unif.

$\sum f_n''$ converge uniformly

$\Rightarrow \sum f_n$ 2-times differentiable!

and $\left(\sum f_n\right)'' = \sum f_n''$

$\int \longleftrightarrow \int$

$$\int_a^b \int_c^d f(x,y)\, dy\, dx \overset{?}{=} \int_c^d \int_a^b f(x,y)\, dx\, dy$$

1) $f \geq 0$ ✓ (Tonelli)

2) f is bdd ✓ ("Budget" Fubini)

if f cts $[a,b] \times [c,d]$ ✓

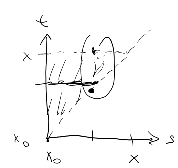
Application: Taylor series around $x = x_0$

Let f be nice.

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt \qquad (FTC)$$

$$f(x) = f(x_0) + \int_{x_0}^{x} \left(f'(x_0) + \int_{x_0}^{t} f''(s)\, ds \right) dt \qquad (FTC)$$

$$= f(x_0) + f'(x_0)\cdot(x - x_0) + \underbrace{\int_{t=x_0}^{x} \int_{s=x_0}^{t}}_{s<t} f''(s)\, ds\, dt$$



Assume f'' cts ($f \in C^2$)

$$= f(x_0) + f'(x_0)\cdot(x - x_0) + \int_{s=x_0}^{x} \int_{t=s}^{x} f''(s)\, dt\, ds$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{s=x_0}^{x} (x - s)\underbrace{f''(s)}_{f''(x_0) + \int_{x_0}^{s} f'''(r)\, dr}\, ds$$

For $f(x,y)$,

$$\frac{\partial^2 f}{\partial x \partial y} \overset{?}{=} \frac{\partial^2 f}{\partial y \partial x} \Longleftarrow f \in C^2$$

$$\frac{\partial^2 f}{\partial x \partial y} \overset{?}{=} \frac{\partial^2 f}{\partial y \partial x} \Leftarrow f \in C^2$$

(Schwarz thm)

Examples

Example 3.6 (Fall 2017 #4): Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 + x^2}$$

defines a continuously differentiable function of $x \in \mathbb{R}$.

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(1) Show $\sum_n \frac{\sin(nx)}{n^3+x^2}$ diff. on $[a,b]$ (and moreover $\frac{d}{dx} \sum_n \cdots = \sum_n \frac{d}{dx} \cdots$)

WTS $\sum_n \frac{d}{dx}\left(\frac{\sin(nx)}{n^3+x^2}\right)$ converges uniformly (by "swap d/dx and $\sum$ thm")

$$= \sum_{n=1}^{\infty} \left(\frac{n \cos(nx)}{n^3+x^2} - \frac{2x \sin(nx)}{(n^3+x^2)^2} \right)$$

Now use: Weierstrass M-test

$$\left| \frac{n \cos(nx)}{n^3+x^2} - \frac{2x \sin(nx)}{(n^3+x^2)^2} \right| \leq \left| \frac{n \cos nx}{n^3+x^2} \right| + \left| \frac{2x \sin(nx)}{(n^3+x^2)^2} \right|$$

$$\leq \left| \frac{n}{n^3} \right| + 2 \cdot \frac{|x| \leq b}{(n^3+x^2)^2}$$

$2|x| \leq$

$\leq$

$$\leq \frac{1}{n^2} + \frac{2b}{n^6} \quad \checkmark$$

M-test applies!

- $\sum \frac{d}{dx}(\ldots)$ converge uniformly (M-test)

- $\sum(\ldots)$ is differentiable ($\frac{d}{dx} \leftrightarrow \sum$ thm)

$1 + x^2$

$n^3 + x^2$

and the deriv. is: $$\sum_{n=1}^{\infty}\left(\frac{n\cos(nx)}{n^3+x^2}-\frac{2x\sin(nx)}{(n^3+x^2)^2}\right)$$

- $\frac{n\cos(nx)}{n^3+x^2}-\frac{2x\sin(nx)}{(n^3+x^2)^2}$ cts
- So suffices to show $\sum_n \frac{n\cos(nx)}{n^3+x^2}-\frac{2x\sin(nx)}{(n^3+x^2)^2}$ converges uniformly!

(unif. limit of cts is cts)

$\square$

$\boxed{\text{Lagrange Multipliers}}$

How to solve

$$\min\left(f(x) \ : \ g(x)=0\right)$$

$$\min\left(x^2+y^2 \ : \ x+y=7\right)$$

LM: $f:\mathbb{R}^n\to\mathbb{R}$

If a max/min occurs at some $\vec{x}_0\in\mathbb{R}^n$ subject $g(\vec{x})=0$, then

$$\begin{cases}\nabla f(\vec{x}_0)=\lambda\nabla g(\vec{x}_0), & \lambda\in\mathbb{R}\\ g(\vec{x}_0)=0\end{cases}$$

(, ,)

$\left(\text{----} \quad f \in C^1, \quad g \in C^1, \quad \text{and} \quad \nabla g(\vec{x}_0) \neq 0 \right)$

Example 3.10 (Winter 2013 #2): Let the nonnegative real numbers x_1, x_2, x_3, x_4 satisfy $x_1 + x_2 + x_3 + x_4 = \pi$.

(a) Show that
$$\sin x_1 \sin x_2 \sin x_3 \sin x_4 \leq \frac{1}{4}.$$

(b) Find all (x_1, x_2, x_3, x_4) that result in equality above.

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$f(\vec{x}) = f(x_1, \ldots, x_4) = \sin x_1 \sin x_2 \sin x_3 \sin x_4$

$g(\vec{x}) = g(x_1, \ldots, x_4) = x_1 + x_2 + x_3 + x_4 - \pi$

How to show $\max \left(f(\vec{x}) : \begin{array}{l} g(\vec{x}) = 0 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array} \right)$ exists?

use
- continuity
- compact!!!

$\Rightarrow$ a max exists

and is obtained at some (x_1, x_2, x_3, x_4)

$\nabla f(x_1, \ldots, x_4) = (\cos x_1 \sin x_2 \sin x_3 \sin x_4, \quad -\quad -\quad -$

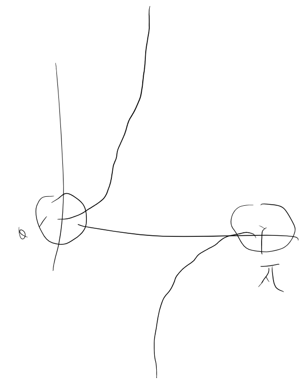
$\nabla g(x_1, \ldots, x_4) = (1, 1, 1, 1)$

$$\begin{cases} \cos x_1 \sin x_2 \sin x_3 \sin x_4 = \lambda \cdot 1 \\ \sin x_1 \cos x_2 \sin x_3 \sin x_4 = \lambda \cdot 1 \\ \vdots \\ x_1 + x_2 + x_3 + x_4 = \pi \end{cases}$$

$x_i \neq 0$

$\lambda \neq 0$

$\tan x_1 = \tan x_2$ $\quad$ $x_1 = x_2$



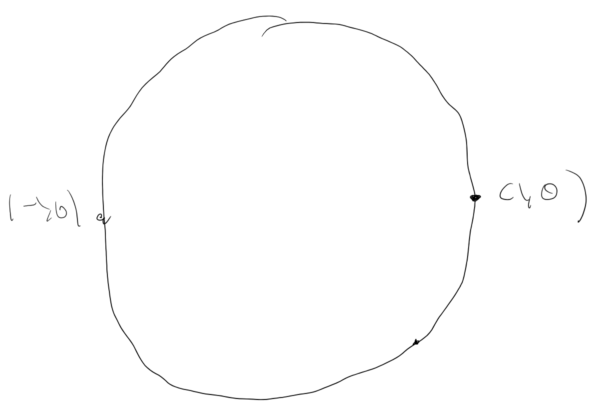
$x_1 = x_2 = x_3 = x_4$

$x_1 = x_2 = x_3 = x_4 = \frac{\pi}{4}$

$\max = \sin\frac{\pi}{4} \sin\frac{\pi}{4} \sin\frac{\pi}{4} \sin\frac{\pi}{4}$

$= \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4}$

$\boxed{\text{Implicit Function Thm}}$



$x^2 + y^2 = 1$

$\frac{\partial}{\partial y} \longrightarrow 2y$

$2y \neq 0$

$\Leftrightarrow$

$y \neq 0$

$\Leftrightarrow$

$(x, y) \neq (1,0), (-1,0)$

- $g : \mathbb{R}^3 \times \mathbb{R}$, $\quad g(x_1, x_2, x_3, y) = 0$

$\exists\ \phi$ (defined near a pt) s.t.

$g(x_1, x_2, x_3, \phi(x_1, x_2, x_3)) = 0$?

- $g \in C^\infty$ $\qquad \phi \in C^\infty$

$$\boxed{\frac{\partial g}{\partial y}(x_1, x_2, x_3) \neq 0}$$

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Example 3.12 (Fall 2011 #3): Show that $e^x - e^y + xy = 0$ defines near $(0,0)$ an implicit function $y = \phi(x)$ in C^3 and compute its expansion to order 3 at $(0,0)$.

$g(x,y) = e^x - e^y + xy$

$g(x,y) = 0$

$$\frac{\partial g}{\partial y}(0,0) = -e^y + x \Big|_{(0,0)} = -1 \neq 0 \quad \checkmark$$

IFT applies

$\exists\ \phi$ defined near $x=0$, s.t.

$g(x, \phi(x)) = 0 \quad \forall\ x$ near 0.

$\phi \in C^\infty$

How to find Taylor expansion of ϕ at $x=0$?

$g(x, \phi(x)) = 0$

$$0 = \frac{d}{dx} g(x, \phi(x)) = \frac{\partial g}{\partial x}(x, \phi(x)) + \frac{\partial g}{\partial y}(x, \phi(x))\, \phi'(x)$$

$$0 = \frac{d}{dx}\, g(x, \phi(x)) = \underbrace{\frac{\partial g}{\partial x}(x, \phi(x))}_{-1} + \underbrace{\frac{\partial g}{\partial y}(x, \phi(x))\, \phi'(x)}_{1}$$

$x = 0$

$$0 = -1 + \phi'(0) \quad \rightsquigarrow \quad \phi'(0) = 1$$

$$0 = \frac{d}{dx} \quad \frac{\partial g}{\partial x}(x, \phi(x)) + \frac{\partial g}{\partial y}(x, \phi(x))\, \phi'(x)$$

$$0 = \frac{\partial^2 g}{\partial^2 x}(\dots) + 2\frac{\partial^2 g}{\partial x \partial y}(\dots)\, \phi'(x) + \frac{\partial g}{\partial y}(\dots)\, \phi''(x)$$

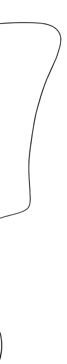
$$+ \frac{\partial^2 g}{\partial y^2}(\dots)\, \phi'(x)\, \phi'(x)$$

$$\phi''(0) = 2$$

$$\boxed{\phi(x) = 0 + 1 \cdot x + \frac{2x^2}{2} + o(x^2)}$$

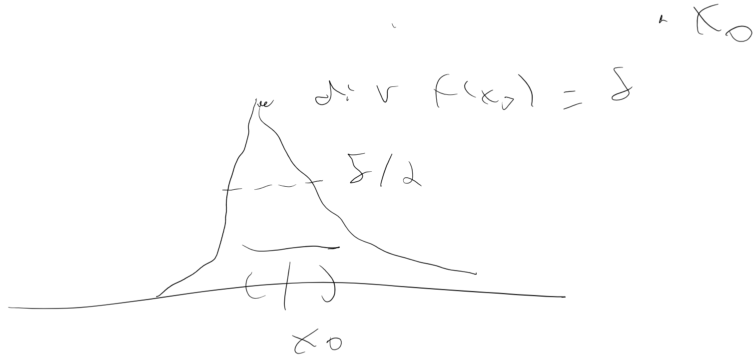
$$\| \quad O(x^3)$$

$$\int_{\mathbb{R}^3} (\mathrm{div}\, F)\, \phi = 0 \qquad \forall\, \phi \in C_c^\infty(\mathbb{R}^3)$$



$$\int_{\mathbb{R}^3} (\operatorname{div} f)\,\phi = 0 \qquad \forall\, \phi \in C_c^\infty(\mathbb{R}^3)$$

$$\operatorname{div} f\,(x_0) > 0$$



Day 4

Monday, August 12, 2024 4:21 AM

Pop Quiz

- When does <u>continuity</u> imply <u>uniform continuity</u>?
- How to find $\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{|\cos(t)|}{\sqrt{t}}\,dt$?

$$\lim_{x\to\infty} \frac{\int_1^x \frac{|\cos t|}{\sqrt{t}}\,dt}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{|\cos x|}{\sqrt{x}}}{1} = 0$$

$\boxed{\text{B are Hard}}$

Housekeeping

- Make sure to practice
- Submit solutions!
  - Holding a solution writing pizza party every Wednesday noon, @ 13th Floor

(Metric spaces) $K \subseteq \mathbb{R}^n$

$f : K \to \mathbb{R}$ cts

$\Rightarrow$ f uniformly cts

Pf: Hard $\square$

Example 4.1 (Fall 2009 #1): Let the series $\sum_{n=1}^{\infty} x_n$ be absolutely convergent. Show that, for any increasing sequence $(a_n)_1^\infty$ of positive numbers with $a_n \to \infty$, we have

$$\lim_{N\to\infty} \frac{1}{a_N}\sum_{n=1}^{N} a_n x_n = 0.$$

$\sum_{n=1}^{\infty} x_n$ abs. conv. $\left(\sum |x_n| < \infty\right)$

$0 < a_n \nearrow \infty$. $\lim_{N\to\infty} \frac{1}{a_N}\sum_{n=1}^{N} a_n x_n = 0$

$$\left|\frac{1}{a_N}\sum_{n=1}^{N} a_n x_n\right| \le \frac{1}{a_N}\sum_{n=1}^{N} a_n |x_n|$$

$$\le \sum_{n=1}^{N} |x_n| \qquad a_n \le a_N$$

Idea: N^{th} term: $\dfrac{a_1x_1 + a_2x_2 + \dots + a_{N-1}x_{N-1} + a_Nx_N}{a_N \nearrow \infty}$ $N\nearrow$ more terms

Intuition: Does $\dfrac{a_1x_1 + \dots + a_{100}x_{100}}{a_N}$ matter? $+ \dfrac{---}{a_N}$

$\varepsilon > 0$

$$\sum_{n=1}^{\infty} |x_n| < \infty \Rightarrow \exists N_\varepsilon \text{ s.t. } \sum_{n=N_\varepsilon}^{\infty} |x_n| < \varepsilon$$

$N > N_\varepsilon$, $N > M$,

1 N_ε N

$$\left|\frac{1}{a_N}\sum_{n=1}^{N} a_n x_n\right| \le \frac{1}{a_N}\sum_{n=1}^{N} a_n|x_n| \overset{?}{<} 2\varepsilon$$

$$= \underbrace{\frac{1}{a_N}\sum_{n=1}^{N_\varepsilon - 1} a_n|x_n|}_{} + \underbrace{\frac{1}{a_N}\sum_{n=N_\varepsilon}^{N} a_n|x_n|}_{}$$

$\dfrac{\text{(finite)}}{a_N} \to 0$ ($< \varepsilon$ $N > M$)

$a_n \le a_N$

$\le \sum_{n=N_\varepsilon}^{N} |x_n| < \varepsilon$ yay

$$\left|\frac{1}{a_N}\sum_{n=1}^{N} a_n x_n\right| \le \begin{pmatrix}\text{something}\\ \to 0\\ \text{in } N\end{pmatrix} + \varepsilon$$

$$\limsup_{N\to\infty} |\dots| \le \varepsilon \quad \forall \varepsilon \quad \varepsilon \to 0$$

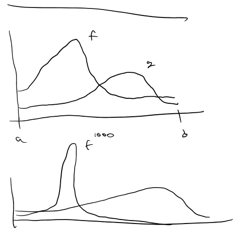
$\square$

Example 4.3 (Winter 2014 #2): Assume f and g are two continuous positive functions on $[a,b]$. Determine

$$\lim_{n\to\infty}\left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n}.$$

$f \in L^p(X)$, $\mu(X) < \infty$

$\|f\|_{L^p(X)} \xrightarrow{p\to\infty} \|f\|_{L^\infty(X)}$



Guess: limit $= M$

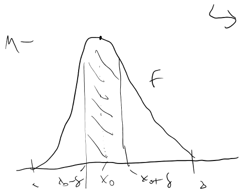
$M := \max_{[a,b]} f$

$\lim_n (\ldots) \sim \lim_n \left(\int_a^b M^n\,dx\right)^{1/n} = M$

$$\sim M \overset{?}{\le} \left(\int_a^b f(x)^n g(x)\,dx\right)^{1/n} \overset{?}{\le} \sim M$$

$$\hookrightarrow \le \left(\int_a^b M^n g(x)\,dx\right)^{1/n}$$

$$= M \underbrace{\left(\int_a^b g\,dx\right)^{1/n}}_{\to 1} \to M$$



$$(M-\varepsilon)\left(\underset{\to 1}{n\to\infty}\right) \le \left(\int_a^b f(x)^n g(x)\,dx\right)^{1/n} \le M$$

$$M-\varepsilon \le \liminf_n (\ldots) \le \limsup_n (\ldots) \le M \quad \forall \varepsilon$$

$\varepsilon \to 0^+$.

$\square$

$\varepsilon > 0$

$f \ge M-\varepsilon \quad \forall |x-x_0| < \delta$

$$\left(\int_a^b f(x)^n g(x)\,dx\right)^{1/n} \ge \left(\int_{x_0-\delta}^{x_0+\delta} f(x)^n g(x)\,dx\right)^{1/n} \ge \left(\int_{x_0-\delta}^{x_0+\delta} (M-\varepsilon)^n g(x)\,dx\right)^{1/n}$$

$$= (M-\varepsilon)\left(\int_{x_0-\delta}^{x_0+\delta} g(x)\,dx\right)^{1/n} \xrightarrow{n\to\infty} M-\varepsilon$$

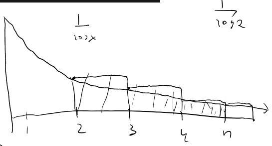
Example 4.10 (Winter 2009 #1): Show that

$$\lim_{n\to\infty}\frac{\log n}{n}\sum_{k=2}^{n}\frac{1}{\log k} = \lim_{n\to\infty}\frac{\log n}{n}\int_2^n \frac{1}{\log x}\,dx = 1.$$

$\frac{\log n}{n}\sum_{k=2}^{n-1}\frac{1}{\log k} \quad \left(\frac{\log n}{n}\frac{1}{\log n} \to 0\right)$

$\frac{1}{\log 2}$

Riemann Sum

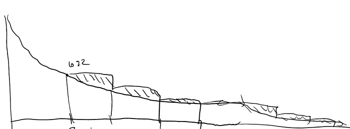


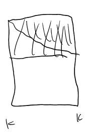
$\lim_n \frac{\log n}{n}(\text{Sum} - \text{integral}) = 0$

$\frac{\log n}{n}(\text{Error}_n) \to 0$

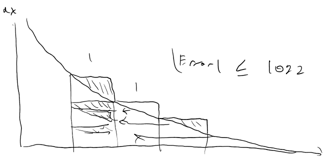
$\downarrow$

0





$\frac{1}{\log k}$ vs. $\int_k^{k+1}\frac{1}{\log x}\,dx$



$|\text{Error}| \le \log 2$

$\frac{1}{\log k} - \int_k^{k+1}\frac{1}{\log x}\,dx$

$\overset{?}{\le} \frac{1}{\log(k)} - \frac{1}{\log(k+1)}$

on $[k, k+1]$,

$\frac{1}{\log x} \ge \frac{1}{\log(k+1)}$

$\le \frac{1}{\log k} - \left(\int_k^{k+1}\frac{1}{\log(k+1)}\,dx\right)$

$$\sum_{k=2}^{n-1}\frac{1}{\log k} - \int_2^n \frac{1}{\log x}\,dx = \sum_{k=2}^{n-1}\left(\frac{1}{\log k} - \int_k^{k+1}\ldots\right)$$

$$\sum_{k=2}^{n-1} \frac{1}{\log k} - \int_2^n \frac{1}{\log x}\,dx = \sum_{k=2}^{n-1}\left(\frac{1}{\log k} - \int_k^{k+1} \cdots\right)$$

$$\leq \sum_{k=2}^{n-1}\left(\frac{1}{\log k} - \frac{1}{\log(k+1)}\right)$$

$$= \frac{1}{\log 2} - \frac{1}{\log n} \leq \frac{1}{\log 2}$$

$$\frac{\log n}{n}\left(\leq \right) \leq \frac{\log n}{n}\left(\frac{1}{\log 2}\right) \to 0$$

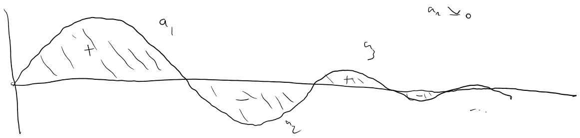
$\square$

Example 4.14 (Fall 2016 #1): Prove that the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

$\int_0^\infty \frac{\sin x}{x}\,dx$ cond. convergent.

① $\underline{\lim_{T\to\infty}} \int_0^T \frac{\sin x}{x}\,dx$ converges

"$\sum (-1)^n a_n$" $a_n \searrow 0$



$$a_n = \int_{\pi n}^{\pi(n+1)} \frac{|\sin x|}{x}\,dx, \quad n = 0, 1, \ldots$$

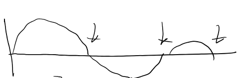
is $a_n \searrow$? ? is $a_n \to 0$?

$$|a_n| \leq \frac{1}{\pi n}(\pi) = \frac{1}{n} \to 0$$

$\sum (-1)^n a_n$ converges!

$$\lim_{n\to\infty} \int_0^{\pi n} \frac{\sin x}{x}\,dx = \lim_n \sum_{n=0}^{\infty} (-1)^n a_n \quad \text{converges}$$

$$\lim_{T\to\infty} \int_0^T \frac{\sin x}{x}\,dx$$



- $\varepsilon > 0$, find N_ε $\sum_{n=0}^{N_\varepsilon} (-1)^n a_n = \int_0^{\pi N_\varepsilon} \frac{\sin}{x}\,dx$ less than
- $T > N_\varepsilon$ $\int_0^T$ in between $\int_0^{\pi n}$ $\int_0^{\pi(n+1)}$ ε away from limit

So $\int_0^T$ ε-away from limit

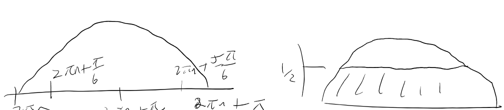
② $\underline{\int_0^\infty \frac{|\sin x|}{x}\,dx = +\infty}$

$$\int_1^\infty \frac{1}{x} = +\infty$$

$\sim \frac{1}{n}$ $\quad \sum_n \frac{1}{n} = +\infty$!

$x = 2\pi n + \frac{\pi}{2}$

$$= \sum_{n=0}^{\infty} \int_{2\pi n}^{2\pi(n+1)} \frac{|\sin x|}{x}\,dx \geq \sum_{n=0}^{\infty} \int_{2\pi n + \frac{\pi}{6}}^{2\pi n + \frac{5\pi}{6}} \frac{|\sin x|}{x} \geq \sum_{n=0}^{\infty} \frac{1}{2} \int_{2\pi n + \frac{\pi}{6}}^{2\pi n + \frac{5\pi}{6}} \frac{1}{x}\,dx$$



$$\geq \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2\pi n + \pi}$$

$$= \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}}$$

$$\geq \frac{1}{4\pi} \sum^{\infty} \frac{1}{n+1} = +\infty$$

Example 4.16 (Fall 2014 #3): Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $f'(x)$ is strictly increasing, with $\lim_{x\to-\infty} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = \infty$, and $f(0) \neq 0$.

(a) Prove that $\forall \xi \neq 0$, there exists an η such that $f(\xi + \eta) = f(\xi) + f(\eta)$.

(b) Prove that through this point (ξ, η) there is a solution $y = \phi(x)$ of $f(x + y) = f(x) + f(y)$ which is unique in a neighborhood of (ξ, η).

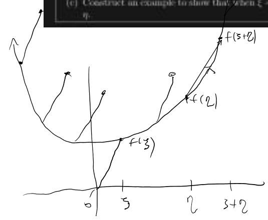
(c) Construct an example to show that when $\xi = 0$ there may be no corresponding η.

$f(\xi+\eta)$

a) $\xi \neq 0$

(b) Prove that through this point (ξ, η) there is a solution $y = \phi(x)$ of $f(x+y) = f(x) + f(y)$ which is unique in a neighborhood of (ξ, η).

(c) Construct an example to show that when $\xi = 0$ there may be no corresponding η.



a) $\xi \neq 0$

want η s.t.

$f(\xi) = \underbrace{f(\xi+\eta) - f(\eta)}$

$-\infty \xleftarrow{\eta \to -\infty}$ $\quad \xrightarrow{\eta \to \infty} +\infty$

$\Rightarrow$ IVT ✓

2

$\geq \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n+1} = +\infty$

$\forall \eta$,

$f(\xi+\eta) - f(\eta) = (\xi+\eta-\eta) f'(c_\eta)$,

$\xi > 0$ $\qquad c_\eta \in (\eta, \xi+\eta)$

$= \xi f'(c_\eta) > \xi f'(\eta) \xrightarrow{\eta \to \infty} \infty$

$f(\xi+\eta) - f(\eta) = \int_\eta^{\xi+\eta} f'(x)\,dx$

b) (ξ, η) $\qquad g(x,y) = f(x+y) - f(x) - f(y)$

$g(x,y) = 0$ $\qquad \cdot\ g(\xi, \eta) = 0$

$Y = \phi(x)$ $\qquad \cdot\ \frac{\partial g}{\partial y}(\xi, \eta) \neq 0$

$\hookrightarrow = f'(\xi+\eta) - f'(\eta)$

$\xi \neq 0$, $f' \nearrow$.

$\square$

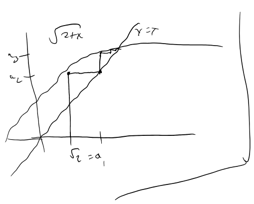
Example 4.21 (Winter 2015 #1 Part (b)): Let a_n be a sequence defined as follows: $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that a_n is bounded, nondecreasing and that the limit of a_n exists. Find $\lim_{n\to\infty} a_n$.

$a_1 = \sqrt{2}$, $\qquad a_{n+1} = \sqrt{2 + a_n}$

WTS: $\cdot\ a_n$ bdd ✓

$\cdot\ a_n \nearrow$ ✓

$\cdot\ \exists \lim a_n$



Let's show $a_n \nearrow$. ✓

$a_{n+1} \geq a_n$

$\sqrt{2+a_n} \overset{?}{\geq} a_n$

$\Leftrightarrow$ $2 + a_n \overset{?}{\geq} a_n^2$ ✓

$\Leftrightarrow$ $a_n^2 - a_n - 2 \leq 0$

$\Leftrightarrow$ $(a_n - 2)(a_n + 1) \leq 0$

$\Leftrightarrow$ $a_n \in [-1, 2]$ $\quad \forall n$. ✓

$a_n \geq -1$

$a_n \leq 2$ $\qquad a_{n+1} = \sqrt{2+a_n} \leq \sqrt{2+2} = 2$ ✓

Let $a_n \to L$.

$a_{n+1} = \sqrt{2 + a_n}$

$\downarrow$ $n\to\infty$ $\downarrow$

$L = \sqrt{2 + L}$

$L = 2, -1$

$\boxed{L = 2}$ $\square$

Example 4.7 (Fall 2018 #2): Suppose $f_n : \mathbb{R} \to \mathbb{R}$, $n \geq 0$, is a sequence of continuous functions converging to f uniformly on every compact set. Consider any continuous function ϕ on $\mathbb{R}$. Show that $\phi \circ f_n$ converges to $\phi \circ f$ uniformly on every compact set, too. What if the f_n are not necessarily continuous?

$\cdot$ $f_n : \mathbb{R} \to \mathbb{R}$ cts, $\quad f_n \to f$ unif on every compact set.

$\cdot$ ϕ cts.

$\cdot$ WTS $\phi \circ f_n \xrightarrow{\text{unif}} \phi \circ f$ (on every cpt set)

$\varepsilon > 0$, pick $n \geq$

(fix K)

$|\phi(f_n(x)) - \phi(f(x))| \overset{?}{<} \varepsilon$ $\qquad x \in K$

$f_n \to f$, $\qquad f_n, f$ "close",

$\phi(f_n)$, $\phi(f)$ "close"

$n \geq N_\varepsilon$, $f_n \to f$ uniformly on K,

$\underbrace{|f_n(x) - f(x)| < \delta}$ $\forall x \in K$

$|\phi(\dots) - \phi(---)|$ $|\dots - ---| <$ small

$\phi|_{[-M, M]}$ is uniformly cts. $M > 0$

$f_n(x), f(x), \in [-M, M]$

$\forall n$ $\forall x \in K$

ϕ is u.c., $\exists \delta > 0$ s.t. if $|s - t| < \delta$

$\Rightarrow$ $|\phi(s) - \phi(t)| < \varepsilon$

$n \geq N_\varepsilon$, $|f(x) - f_n(x)| < \delta$

s, t

$\shortparallel$ $\shortparallel$

$f(x)$ $f_n(x)$

$|\phi(f(x)) - \phi(f_n(x))| < \varepsilon$

$\forall x \in K$ $\forall n \geq N_\varepsilon$.

$f(K)$ compact

$\subseteq [-M, M]$

$f_n(K) \subseteq [-M-1, M+1]$

$|f(x) - f_n(x)| < 1$ $\forall n \geq N_\varepsilon$

$1 \leq n \leq N_\varepsilon$

$M' = \max_{n \leq N_\varepsilon} \sup_{x \in K} |f_{\dots}$

$f_n, f \subseteq [-\hat{M}, \hat{M}]$

$\hat{M} = \max(M+1, M')$