NYU Honors Analysis I Recitation

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Fall 2025

Contents

Remarks

These notes are an abbreviated take on what was done in recitation.

Notational conventions:

- The naturals start at 1.
- Increasing sequences need not be "strictly" increasing.
- The sequence $a_1, a_2, \dots$ is denoted as $\{a_n\}_n$. If the starting index must be clarified, I may write $\{a_n\}_{n=1}^{\infty}$.
- $\sup\limits_{x \in E} f(x)$ is the same as $\sup\{f(x) : x \in E\}$.
- $\sup\limits_{x} f(x)$ is the same as $\sup\{f(x) : x\}$ where x is taken over the set over which $f(x)$ is defined. For example $\sup\limits_{n} a_n = \sup\{a_n : n \in \mathbb{N}\}$.
- $\lim\limits_{n} a_n$ is a shorthand for $\lim\limits_{n \to \infty} a_n$. Ditto for $\limsup\limits_{n} a_n$.
- log is the natural log.

1 Sup, Inf, and Friends!

Welcome to real analysis! Real analysis is the *study of real numbers*. It's important because we actually don't understand real numbers very well. They are very unintuitive creatures! For example, many a middle schooler may think that

$$0.\overline{9} < 1.$$

But by now you probably know better! A good understanding of the real numbers is crucial for doing calculus correctly.

1.1 Exercises with sup

You should think of the "sup" as a "max". Of course, the sup isn't always obtained. For example, the supremum of the interval $[0, 1)$ is 1, but 1 isn't in the set. Thus I like to think of sup as "the max, even if there isn't a max".

To work with sup, you need to be using the raw definition: It is the least upper bound.

Example 1.1: For non-empty sets $S, T \subseteq \mathbb{R}$, define

$$S + T := \{x + y : x \in S, y \in T\}.$$

How do $\sup(S + T)$ and $\sup S + \sup T$ compare? Assume that S and T are bounded from above.

If we think of "sup" as "max" here, it's intuitive that they are equal. Let's try to prove it.

Proof. To prove an equality, we want to show $\leq$ and $\geq$.

Proof of $\leq$: How to upper bound $\sup(S + T)$? Let's just start with an element z of $S + T$. Then $z = x + y$ for $x \in S$ and $y \in T$. But

$$x \leq \sup S \qquad y \leq \sup T$$

so $z \leq \sup S + \sup T$.

This is true for all $z \in S+T$, so $\sup S+\sup T$ is an upper bound on $S+T$. So it must be at least the *least* such upper bound, which is $\sup(S+T)$. Therefore $\sup(S+T) \leq \sup S+\sup T$.

Proof of $\geq$: Now we need to upper bound $\sup S + \sup T$. Well, let's start with an $x \in S$ and a $y \in T$. I know that $x + y$ is in $S + T$, so

$$x + y \leq \sup(S + T).$$

But now how to get sups on the left side? Here is the trick: If we move the y over, then

$$x \leq \sup(S+T) - y.$$

For a fixed y, this is true for all $x \in S$. So $\sup(S+T) - y$ is an upper bound on S, and moreover needs to be at least the *least* such upper bound. So

$$\sup S \leq \sup(S+T) - y.$$

Now move the y to the other side!

$$y \leq \sup(S+T) - \sup S$$

This holds for all $y \in T$, so $\sup(S+T) - \sup S$ is an upper bound on T, and is greater than or equal to the *least* such upper bound. We conclude that

$$\sup T \leq \sup(S+T) - \sup S$$

which is what we wanted. $\square$

Remarks:

- In general, if you know that $x \leq M$ for all $x \in S$, you can "take the sup on the left" to get $\sup S \leq M$. We did this three times in the above proof.
- Similarly, if $x \geq m$ for all $x \in S$, we can "take the inf on the left" to get $\inf S \geq m$.
- Another way to do this is to make use the following characterization of sup: $M = \sup S$ iff (1) M is an upper bound on S, and (2) for all $\varepsilon > 0$, the intersection $(M - \varepsilon, M] \cap S$ is non-empty. (This means that there are elements of S that are arbitrarily close to M, which intuitively should mean that M is the least upper bound. As an exercise you can try to prove that this characterization of supremum is equivalent to the definition of supremum.)

Let's try another example.

Example 1.2: Let a_n and b_n be two sequences, both bounded from above. How do $\sup_n(a_n + b_n)$ and $\sup_n a_n + \sup_n b_n$ compare?

This might look like the same problem, but actually no: We only get $\leq$. A counterexample which shows why $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$ may not necessarily hold is given by $a_n = (-1)^n$ and $b_n = -(-1)^n$.

Let's prove that $\sup_n(a_n + b_n) \leq \sup_n a_n + \sup_n b_n$.

Proof. Let's start with some $a_n + b_n$. Then

$$a_n \leq \sup_k a_k$$

and

$$b_n \le \sup_k b_k$$

so

$$a_n + b_n \le \sup_k a_k + \sup_k b_k.$$

This means that $\sup_k a_k + \sup_k b_k$ is an upper bound on $a_n + b_n$, so by "taking the sup" on the left we conclude that

$$\sup_n (a_n + b_n) \le \sup_k a_k + \sup_k b_k$$

as needed. □

Remark: Notice how I wrote $a_n \le \sup_k a_k$ instead of $a_n \le \sup_n a_n$. Using different letters helps a lot to prevent confusion.

1.2 The Intuition of Limsup

You should think of limsup a_n as:

- the "best upper bound on the asymptotic behavior of a_n as $N \to \infty$"
- the "upper bound on a_n **near** $n = \infty$"
- the best upper bound on the **tail** of a_n

Whereas lim is about the exact asymptotic, $\limsup$ is only an upper bound on the asymptotic (when the limit doesn't exist), whereas $\liminf$ is only a lower bound.

Limsup and liminf are really useful when we want to discuss the asymptotic behavior of a function or sequence, but the limit doesn't actually exist!

You can think of $f(x) = \sin x$ as a prototypical example for limsup and liminf. $\lim_{x\to\infty} \sin x$ does not exist. But "in the limit it's between -1 and 1", in other words,

$$-1 \le \liminf_{x\to\infty} \sin x \le \limsup_{x\to\infty} \sin x \le 1.$$

In fact $\liminf_{x\to\infty} \sin x = -1$ and $\limsup_{x\to\infty} \sin x = 1$, and these are "obtained" because $\sin x = -1$ infinitely often for large x and $\sin x = 1$ infinitely often for large x.

1.2.1 Definitions

A priori the limsup is defined as

$$\limsup_{n\to\infty} a_n = \inf_n \sup_{k>n} a_k.$$

However, you should know that it is equivalent to write

$$\lim_{n\to\infty} \sup_{k>n} a_k$$

because $\{\sup_{k>n} a_k\}_n$ is monotone decreasing in n! This monotonicity is important to keep in mind.

Limsup can be defined in other contexts where I think it's easier to think about. For example, for functions,

$$\limsup_{x\to\infty} f(x) = \lim_{N\to\infty} \sup_{x>N} f(x),$$

and

$$\limsup_{x\to x_0} f(x) = \lim_{\delta\to 0^+} \sup_{0<|x-x_0|<\delta} f(x).$$

See the following Desmos visualizations:

- https://www.desmos.com/calculator/tiwrgpoa0x
- https://www.desmos.com/calculator/va3cdionyv

These visualizations should really help show you what limsup is intuitively.

1.2.2 Examples of why limsup is useful

Let's try to prove the squeeze rule!

Example 1.3: Suppose that $g_1(x) \le f(x) \le g_2(x)$, and both $\lim_{x\to x_0} g_1(x)$ and $\lim_{x\to x_0} g_2(x)$ exist and are equal to L. Prove that $\lim_{x\to x_0} f(x) = L$.

Proof. [WRONG PROOF] Just take $g_1(x) \le f(x) \le g_2(x)$ and take the limit of all three parts, to get $L \le \lim_{x\to x_0} f(x) \le L$, so $\lim_{x\to x_0} f(x) = L$. Tada? □

This is **very very very very very very very very wrong** because I don't actually know that $\lim_{x\to x_0} f(x)$ exists in the first place! So this is very bad and horrible and terrible.

...

But I *do* know that $\limsup_{x\to x_0} f(x)$ and $\liminf_{x\to x_0} f(x)$ exist. Because they *always* exist.

Proof. [Actual Proof] Taking limsup on both sides of the right inequality and liminf on both side of the left inequality, we get the following for free:

$$\liminf_{x\to x_0} g_1(x) \le \liminf_{x\to x_0} f(x) \le \limsup_{x\to x_0} f(x) \le \limsup_{x\to x_0} g_2(x)$$

Ok but, $\lim_{x\to x_0} g_1(x)$ exists, so $\liminf_{x\to x_0} g_1(x) = \lim_{x\to x_0} g_1(x) = L$... and similarly, we know that $\limsup_{x\to x_0} g_2(x) = \lim_{x\to x_0} g_2(x) = L$. So actually this is just saying that:

$$L \le \liminf_{x\to x_0} f(x) \le \limsup_{x\to x_0} f(x) \le L$$

So the liminf and limsup of f were equal, and in fact both are equal to L, so the limit exists and is L. Yay! $\square$

Motto of the above proof: "The asymptotics of f are bounded from above by L and bounded from below by L, so the limit exists."

Here's another application. The ratio test says that if

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

then $\sum_{n=1}^{\infty} a_n$ converges. At first this seems quite intimidating if you're still getting used to limsup. But intuitively this statement is actually quite simple! In English, all it's saying is this: "If a series **eventually goes down faster than a geometric series**, then it converges!" The "eventually" part is the "lim", and the "faster" is the "sup".

In Layman's terms, a series which (eventually) converges faster than a geometric series must be convergent. That's all! The limsup formalizes this statement.

1.3 A limsup exercise

Example 1.4: Let a_n and b_n be sequences. Show that

$$\limsup_n (a_n + b_n) \le \limsup_n a_n + \limsup_n b_n$$

provided that both sides exist and are finite.

Proof. Since the limsup is the lim of a sup, let's start by working with the "inner-most operation", which is sup. Can we compare these two quantities?

$$\sup_{k>n}(a_k + b_k), \qquad \sup_{k>n} a_k + \sup_{k>n} b_k$$

It turns out we can! We proved that the correct relationship is $\leq$ in the first section. So

$$\sup_{k>n}(a_k + b_k) \leq \sup_{k>n} a_k + \sup_{k>n} b_k$$

for all n. Now we want to "send $n \to \infty$ on each side". This is safe because all the limits exist. (More precisely, we want to appeal to the fact that if $x_n \to K$ and $y_n \to L$ and $x_n \leq y_n$ for all n, then $K \leq L$. If this wasn't done in lecture, try to prove it!) So

$$\lim_{n\to\infty} \sup_{k>n}(a_k + b_k) \leq \lim_{n\to\infty} \left(\sup_{k>n} a_k + \sup_{k>n} b_k\right)$$

$$\leq \lim_{n\to\infty} \sup_{k>n} a_k + \lim_{n\to\infty} \sup_{k>n} b_k.$$

That's exactly what we wanted to prove!

□

1.4 Rigorous Write-up for the Quiz

Not done in recitation but I thought I should include this so you have a good sense of what is expected.

Theorem 1.1 (1D Cantor Intersection)

Let $\{[a_n, b_n]\}_n$ be a sequence of closed intervals such that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \geq 1$. Then the intersection

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is non-empty. Moreover, if $\lim_{n\to\infty} b_n - a_n = 0$, then the intersection has a single element.

Proof. Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \geq 1$, we have that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

for all $n \geq 1$. In particular, $a_n \leq b_n \leq b_1$ for all n, which entails that b_1 is an upper bound on $\{a_n : n \in \mathbb{N}\}$. Thus the supremum $M := \sup_n a_n$ exists.

We claim that $M \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, which will prove that the intersection is non-empty. It suffices to prove that for all n, $a_n \leq M \leq b_n$.

Fix n. Since $M = \sup_k a_k$, M is an upper bound on $\{a_k : k \in \mathbb{N}\}$, so $M \geq a_n$.

On the other hand, $a_k \leq b_n$ for all k (if $k \leq n$ then $a_k \leq a_n \leq b_n$, and if $k > n$ then $a_k \leq b_k \leq b_n$), so b_n is an upper bound on $\{a_k : k \in \mathbb{N}\}$. So b_n is $\geq$ the least such upper bound (the supremum), which is M. That is, $b_n \geq M$. This proves the claim.

Now assume that $\lim_{n\to\infty} b_n - a_n = 0$. To prove that $\bigcap_{n=1}^{\infty}[a_n, b_n]$ has exactly one element, it is sufficient to show that if $x, y \in \bigcap_{n=1}^{\infty}[a_n, b_n]$, then $x = y$.

Take such an x and y, and assume without loss of generality that $x \leq y$. Note that $a_n \leq x$ for all n, and $y \leq b_n$ for all n. So

$$|x - y| = y - x \leq b_n - a_n$$

for all n. Sending $n \to \infty$, and using the hypothesis that $b_n - a_n \to 0$, we conclude by the squeeze theorem (or by fixing $\varepsilon > 0$ or by using $\liminf$, etc.) that $|x - y| = 0$. That is, $x = y$. $\square$

2 Series and Stuff

2.1 Comparing series and integrals

Draw a picture! (I'm too tired to reproduce a diagram in these notes, sorry! The key idea is that the series is basically a Riemann sum so draw that and the area under the curve to compare them.)

> **Example 2.1:** Consider the series $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Get a decent lower and upper bound on s_n. In particular can we show that $s_n \leq 2$?

Solution. By drawing a picture, we can reason that

$$\int_k^{k+1} \frac{1}{x^2}\,dx \leq \frac{1}{k^2} \leq \int_{k-1}^{k} \frac{1}{x^2}\,dx$$

for all k. Now let's sum this starting from $k = 2$ (we're skipping $k = 1$ because otherwise the integral on the right side explodes). This gives

$$1 + \int_2^{n+1} \frac{1}{x^2}\,dx \leq s_n \leq 1 + \int_1^{n} \frac{1}{x^2}\,dx.$$

Evaluating the integrals,

$$1.5 - \frac{1}{n+1} \leq s_n \leq 2 - \frac{1}{n}.$$

That seems like a decent bound! ■

2.2 Estimate for the Factorial

We can use the integral technique to obtain some pretty nice bounds for $n!$. First we need to turn this into a sum, so we'll instead estimate its log:

$$\log(n!) = \sum_{k=1}^{n} \log k$$

Now log is increasing and so for each k we have the estimates

$$\int_{k-1}^{k} \log x\,dx \leq \log k \leq \int_k^{k+1} \log x\,dx.$$

We want to sum from $k = 1$ to $k = n$, but the $k = 1$ term makes the left bound problematic. Instead we sum from $k = 2$, which is just as good because the $k = 1$ term is $\log 1 = 0$.

$$\int_1^{n} \log x\,dx \leq \sum_{k=2}^{n} \log k \leq \int_2^{n+1} \log x\,dx$$

Evaluating the integrals,

$$n \log n - n + 1 \leq \log(n!) \leq (n+1)\log(n+1) - 2\log 2 - n + 1.$$

Exponentiating, we end up with

$$n^n e^{-n+1} \leq n! \leq \frac{1}{4}(n+1)^{n+1} e^{-n+1}.$$

This suggests that $n!$ grows roughly like $n^n e^{-n}$. It turns out that the correct asymptotic is $n! \sim C n^{n+\frac{1}{2}} e^{-n}$ where the constant C is $\sqrt{2\pi}$. So, we got pretty close with a relatively elementary method!

2.3 A few exercises with series tests

Example 2.2: Find all $x \in \mathbb{R}$ for which

$$\sum_{n=1}^{\infty} \frac{nx^n}{n^2 + x^{2n}}$$

converges.

Solution. We claim that it converges for all $x \neq 1$.

If $|x| > 1$, we have that

$$\sum_{n=1}^{\infty} \frac{n|x|^n}{n^2 + x^{2n}} \leq \sum_{n=1}^{\infty} \frac{n|x|^n}{x^{2n}} = \sum_{n=1}^{\infty} \frac{n}{|x|^n}$$

which converges by the ratio test, so by comparison, the series converges absolutely, and thus converges.

If $|x| < 1$, we instead write

$$\sum_{n=1}^{\infty} \frac{n|x|^n}{n^2 + x^{2n}} \leq \sum_{n=1}^{\infty} \frac{n|x|^n}{n^2} = \sum_{n=1}^{\infty} \frac{|x|^n}{n} \leq \sum_{n=1}^{\infty} |x|^n < \infty.$$

If $x = -1$, the series converges by the alternating series test.

If $x = 1$, then the series is

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}.$$

This is comparable to $1/n$ in the limit so this should diverge by limit comparison. Alternatively one can write something like

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \geq \sum_{n=1}^{\infty} \frac{n}{n^2+n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

■

Example 2.3: Does $\sum_{n=1}^{\infty} \sin(1/n^2)$ converge?

Solution. Yes, by using $\sin x \leq x$ (for $x > 0$) and comparison. ■

Example 2.4: Does $\sum_{n=1}^{\infty} \frac{1}{n\log(n)^2}$ converge?

Solution. Yes by the integral test. If we want to be a bit more precise, we can write, for $n > 1000$,

$$\frac{1}{n\log(n)^2} \leq \int_{n-1}^{n} \frac{1}{x\log(x)^2}\,dx.$$

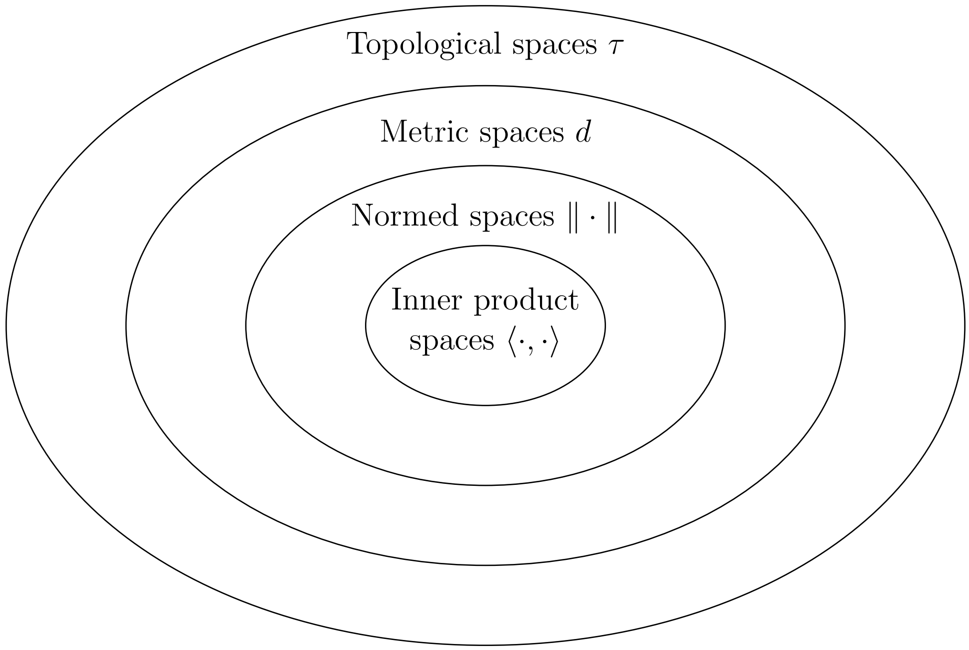
So

$$\sum_{n=10000}^{\infty} \frac{1}{n\log(n)^2} \leq \int_{9999}^{\infty} \frac{1}{x\log(x)^2}\,dx.$$

By doing calculus (u-sub with $u = \log(x)$), the integral converges, hence so does the sum. ■

3 The Analysis Hierarchy

The bulk of (undergraduate) analysis lives in the following hierarchy of spaces.



3.1 Inner Product Spaces

These are the most specialized spaces, and are quite uncommon. Some very nice things happen in inner product spaces, but I don't think we'll dive very deeply into it. You saw a bunch of it in a class called linear algebra.

Definition 3.1 (Inner Product Space)

An *inner product space* is a **vector space** X equipped with an **inner product** $\langle\cdot,\cdot\rangle$ satisfying a bunch of properties, such as $\langle tx, y\rangle = t\langle x, y\rangle$ for all $x, y \in X$ and $t \in \mathbb{R}$. (there are more conditions of course but whatever you can look it up.)

Intuitively you can think of an inner product space as "a space with a notion of **angles**". The inner product $\langle x, y\rangle$ measures how much x and y "agree".

Examples:

- Euclidean space (a.k.a. $\mathbb{R}^N$) equipped with the standard inner product, $\langle x, y\rangle := x \cdot y = \sum_j x_j y_j$.

- The sequence space l^2, consisting of all sequence of real numbers $\{x_n\}_n$ such that $\sum_n |x_n|^2 < \infty$, equipped with the inner product $\langle \{x_n\}_n, \{y_n\}_n \rangle_{l^2} := \sum_n x_n y_n$. Some work needs to be done to show that this is valid.

- The space $L^2(\mathbb{R})$ of all square-integrable "functions" $f : \mathbb{R} \to \mathbb{R}$, equipped with the inner product
$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}} f(x)g(x)\, dx.$$
(This is a lie, hence the quotes.)

3.2 Normed Spaces

Normed spaces are much more common. A lot of analysis happens here.

Definition 3.2 (Normed Space)

A *normed space* is a **vector space** X equipped with a **norm** $\|\cdot\|$ satisfying the following conditions:

- $\|x\| \geq 0$ always, with $\|x\| = 0$ if and only if $x = 0$. (also its always finite.)
- You can take out scalars: $\|tx\| = |t| \cdot \|x\|$ for all $t \in \mathbb{R}$ and $x \in X$.
- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

Simply stated: A normed space is a space that has a notion of **size**.

Every inner product space is a normed space. This is because any inner product $\langle \cdot, \cdot \rangle$ induces a norm, given by
$$\|x\| := \sqrt{\langle x, x \rangle}.$$
Compare this with the identity $\|x\| = \sqrt{x \cdot x}$ in Euclidean space.

Examples

- On $\mathbb{R}$, the absolute value $|\cdot|$ is a norm.
- On $\mathbb{R}^N$, the usual norm is $\|x\| := \sqrt{\sum_j x_j^2}$. Of course, this is just given by the usual inner product as $\sqrt{x \cdot x}$.
- For $p \geq 1$, there is the L^p space, $L^p(\mathbb{R})$, of all "functions" $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$\int_{\mathbb{R}} |f(x)|^p\, dx < \infty,$$

and the norm is given by $\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}$.

Questions: (1) Why do we need to raise to the $1/p$ *power? (2) Why is this a lie as written?*

- There are other norms on $\mathbb{R}^N$. These are not the ones induced by the usual inner product, but they are still norms and can be useful.
  - The *taxicab norm*, $\|x\|_1 := \sum_j |x_j|$.
  - The l^∞ *norm*, $\|x\|_\infty := \max_j |x_j|$.
  - In general, for $1 \leq p < \infty$, we have the p-norm, $\|x\|_p := \left(\sum_j |x_j|^p\right)^{1/p}$. The case $p = 2$ is the standard Euclidean norm.

3.3 Metric Spaces

Metric spaces are *very* common! They arise whenever there is some notion of "distance".

Definition 3.3 (Metric Space)

A *metric space* is a **set** X equipped with a **metric** d satisfying the following conditions:

- $0 \leq d(x, y) < \infty$, with $d(x, y) = 0$ if and only if $x = y$
- Symmetry: $d(x, y) = d(y, x)$
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Note that the underlying set X no longer needs to have a vector space structure for it to qualify as a metric space. You just need "distance".

Every normed space is a metric space. If you have a norm $\|\cdot\|$, then the "distance" between x and y is given by $\|x - y\|$. You can verify that this satisfies the properties of a metric, so $d(x, y) := \|x - y\|$ is a metric for any norm $\|\cdot\|$.

Examples:

- X = the tiles of a Civilization 6 board, $d(x, y)$ = number of moves it takes to go from x to y.
- X = Warren Weaver Hall, $d(x, y)$ = time it takes to walk from x to y.
- $\mathbb{R}^N$ is a metric space induced by the standard norm (or really, any of its norms).
- X = the set of all English words, $d(x, y)$ = the number of single-letter edits you need to turn x into y.

3.3.1 Balls and Open Sets

We play with metric spaces a lot, so we've got a bunch of constructs in them that we study (which, of course, also exist in normed spaces). The first is the **ball**,

$$B(x,r) := \{y : d(x,y) < r\}.$$

Balls are important because we think of them as "neighborhoods" which surround a point x. They're also just a very convenient notation in general.

With balls, we obtain **open sets**: Sets $U \subseteq X$ such that for every $x \in U$ there is a small $r > 0$ such that $B(x,r) \subseteq U$. In layman's terms, opens sets are the sets with "wiggle room" everywhere. This intuition is why we like them: It allows us to make small changes in any direction when we're inside it.

We also get **closed sets**, which are defined to be the complements of open sets. (Closed does NOT mean "not open"!)

Then we get **compact sets**, which are defined as those sets for which every *open cover* has a finite subcover. In $\mathbb{R}^N$, these happen to be the closed and bounded sets, and therefore we can think of them as sets which are *restrictive*, and *prevent too much "change"*. An instance of this is the extreme value theorem: Any continuous function on a closed and bounded interval must have a max and min. The closed and bounded interval here is a compact set, and we see that it is preventing the continuous function on it from "exploding". More on compact sets in future recitations.

The most important types of sets are the open sets and the compact sets, for the reasons described.

3.3.2 Properties of open sets

- The union of open sets is always open (no matter how many open sets are used!).
- The **finite** intersection of open sets is open.
- $\{\}$ is open.
- The whole space is open.

3.3.3 The Shit Metric

I define the *shit metric space* as follows: It is $(\mathbb{R}^2, d_{\text{shit}})$, where

$$d_{\text{shit}}(x,y) := \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

Verify that this is a metric!

The actual name for this is the discrete metric, but whatever. This is an important "extreme" case to keep in mind, as it can serve as a convenient counterexample to test any conjectures you have about metric spaces. We'll talk more about that in future weeks, probably.

Verify the following:

- $B_{\text{shit}}((0,0), 0.75) = \{(0,0)\}$
- $B_{\text{shit}}((0,0), 1.2) = \mathbb{R}^2$
- Every set is open.
- Every set is closed.
- Every set is bounded.
- The only compact sets are finite sets.

3.4 Topologies

Now what if there is no notion of distance? There is an even weaker structure called a *topology*: Instead of a distance, we specify what the open sets are.

Definition 3.4 (Topological space)

A *topological space* is a set X equipped with a *topology*, τ, which is a collection of sets (the "open sets") satisfying the properties listed in the *Properties of open sets* section.

It's quite abstract and it's normal to find it hard to imagine what kind of structure a topology could bring to the table. Here are some intuitions:

- Topologies specify the "neighborhoods", which therefore give some loose notion of which points are "close" to each other.
- A topology can be thought of as specifying a *notion of convergence*. For example, the topology on $\mathbb{R}$ "generated" by the intervals of the form $(a, b]$ corresponds to "convergence from below", and in some sense formalizes the concept of the left-sided limit $\lim_{x \to a^-} f(x)$.

And of course, every metric space is a topological space.

4 Limits and Stuff

4.1 Be suspicious!

If your proof isn't relying on the definitions and/or theorems, that's probably a bad sign!

Problem 1: Let A and B be compact subsets of $\mathbb{R}^n$. Show that $A + B := \{a + b : a \in A, b \in B\}$ is compact.

Fake Proof: Since A and B are bounded, $A + B$ is also bounded. Since A and B are closed, $A + B$ is also closed. So by Heine-Borel, $A + B$ is compact. □

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that $f'(a) > 0$ for some $a \in \mathbb{R}$, and $f'(b) < 0$ for some $b \in \mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

Fake Proof: f is differentiable everywhere, so its derivative f' is continuous. So by the Intermediate Value Theorem, f' has a zero. □

Problem 3: Consider the normed vector space $C_b(\mathbb{R})$ of all bounded continuous functions on the real line, equipped with the sup norm. Then the function $F : C_b(\mathbb{R}) \to \mathbb{R}$ defined by

$$F(f) := f(5)$$

is continuous.

Fake Proof: Note that for any $f, g \in C_b(\mathbb{R})$, we have $F(f+g) = F(f)+F(g)$. Moreover, for any $t \in \mathbb{R}$ and $f \in C_b(\mathbb{R})$, we have that $F(tf) = tF(f)$. Therefore, $F : C_b(\mathbb{R}) \to \mathbb{R}$ is linear. So F is continuous. □

4.2 Limits and Continuity

Example 4.1:

$$\lim_{x\to 5} x^2$$

Solution. We claim that the limit is 25.

Fix $\varepsilon > 0$. And we'll pick $\delta =$ (TBD). Now, if $|x-5| < \delta$, we shall show that $|x^2-25| < \varepsilon$.

Well, we can write

$$|x^2 - 25| = |(x - 5)(x + 5)| < \delta \cdot |x + 5|.$$

That δ will be pretty small so we just need the $|x+5|$ to be pretty small. It should be about

10, right? Indeed,

$$|x+5| = |x-5+10| \leq |x-5| + |10| < \delta + 10.$$

So provided $\boxed{\delta \leq 7}$, we have that $|x+5| < 17$. Now,

$$|x^2 - 25| < \delta \cdot |x+5| < 7\delta,$$

which is $\leq \varepsilon$ provided that $\boxed{\delta \leq \varepsilon/7}$. Ok, so since we chose $\boxed{\delta = \min(7, \varepsilon/7)}$, both of those hold and so we're good. ■

Example 4.2:

$$\lim_{x \to 42} 1_{\mathbb{Q}}(x)$$

Solution. We claim that the limit does not exist.

Suppose the limit were L. Take $\varepsilon = 1/4$ or something. We want to show that for any $\delta > 0$, there is some x with $0 < |x - 9001| < \delta$ such that $|1_{\mathbb{Q}}(x) - L| \geq \varepsilon$.

To wit, take any $\delta > 0$. There are two cases.

- If $L \geq 1/2$, we use density of irrationals to pick $x \in (9001, 9001 + \delta)$ irrational. This gives $1_{\mathbb{Q}}(x) = 0$ so $|1_{\mathbb{Q}}(x) - L| \geq 1/2 \geq \varepsilon$.
- If $L < 1/2$, use density of rationals instead.

So the limit does not exist. ■

Example 4.3: Do Problem 3, but correctly.

Solution. We recall $F : C_b(\mathbb{R}) \to \mathbb{R}$ with $F(f) := f(5)$. Fix $f_0 \in C_b(\mathbb{R})$. We show that F is continuous at f_0.

Fix $\varepsilon > 0$. Then, for all $f \in C_b(\mathbb{R})$ with $\|f - f_0\|_\infty < \delta$, where δ shall be chosen later, we will show that $|F(f) - F(f_0)| < \varepsilon$.

Well, we want to show that $|f(5) - f_0(5)| < \varepsilon$. But

$$\delta > \|f - f_0\|_\infty = \sup_{x \in \mathbb{R}} |f(x) - f_0(x)| \geq |f(5) - f_0(5)|.$$

So we may take $\delta = \varepsilon$. ■

Example 4.4: Is $C_b(\mathbb{R})$ complete?

4.3 Exercises with Metric Spaces

Example 4.5: Let (X, d) be a metric space, and $C \subseteq K \subseteq X$ with C closed and K compact. Prove that C is compact.

This can be done using either sequences or open covers.

Example 4.6: Let (X, d) be a metric space, and $E \subseteq X$. Define the *boundary* of E as follows:

$$\partial E := \{x \in X : B(x, r) \cap E \neq \emptyset \text{ and } B(x, r) \cap E^c \neq \emptyset \ \forall r > 0\}$$

Show that ∂E is closed.

If $x \notin \partial E$ then there is $r > 0$ so that either $B(x, r) \cap E = \emptyset$ or $B(x, r) \cap E^c = \emptyset$. Without loss of generality let's suppose the former happened. Then $B(x, r) \subseteq E^c$. We claim $B(x, r) \subseteq (\partial E)^c$. Indeed, take any $y \in B(x, r)$. Then $B(y, \min(d(x, y), r - d(x, y))) \subseteq B(x, r) \subseteq E^c$ (or alternatively $B(x, r)$ is open so there's gotta be some r' so that $B(y, r') \subseteq B(x, r)$), so $B(y, \min(d(x, y), r - d(x, y))) \cap E = \emptyset$, so by definition of ∂E, $y \notin \partial E$.

Example 4.7: Let (X, d) be a metric space, and $E \subseteq X$. Show that, for $F \subseteq E$, the following two conditions are equivalent:

1. There exists an open set $U \subseteq X$ such that $F = E \cap U$.
2. For every $x \in F$ there exists $r > 0$ such that $B(x, r) \cap E \subseteq F$.

*(If either holds, we say that F is **relatively open** in E, or just "open in E". Morally speaking, we are interpreting E as its own metric space, whose topology is inherited from that of X by taking all the open sets in X and intersecting them with E to form a new "restricted" topology. Think about both the conditions (1) and (2) and see which one you find more intuitive.)*

Solution. $(1 \implies 2)$ easy

$(2 \implies 1)$ for each $x \in F$ find r_x so that $B(x, r_x) \cap E \subseteq F$. Now take $U = \bigcup_{x \in F} B(x, r_x)$. This is open and you can show that $F = E \cap U$. ■