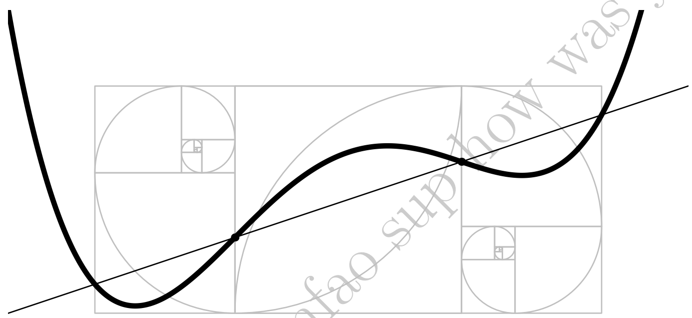
CMUMC

PROBLEM OF THE DAY

Thomas Lam

The Carnegie Mellon University Math Club Problem of the Day



Curated by Thomas Lam

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Printed nowhere because this is intended to be an ebook but you can print it if you really want to.

Written with LaTeX and way too much Asymptote.

This is a version of the book that removes the problems and hints, leaving only the solutions.

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Solution to the Grand Finale

CHAPTER 1

Solutions

Solution 1

The cups started with the same volume. Since the cups end with the same volume, the net amount of milk transferred to the tea must equal the net amount of tea transferred to the milk. That is, the contamination is equal.

■

Remark: We did not need to use the fact that the cups were stirred. Indeed, the problem still holds true if we do not stir, by the same logic.

Source: This is a classic. I first heard this in an old book of puzzles more than a decade ago.

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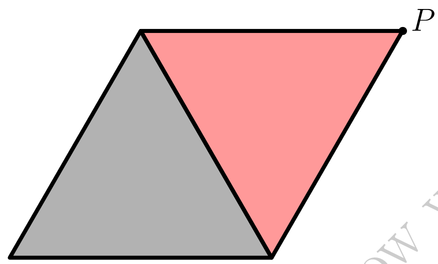
Solution 2

We claim that the answer is $\frac{2}{3}$.

For each triangle we place, we associate it with some empty space near it. If we can show that the area of this empty space is at least half the area of the triangle, then we have the at-least-2/3 ratio.

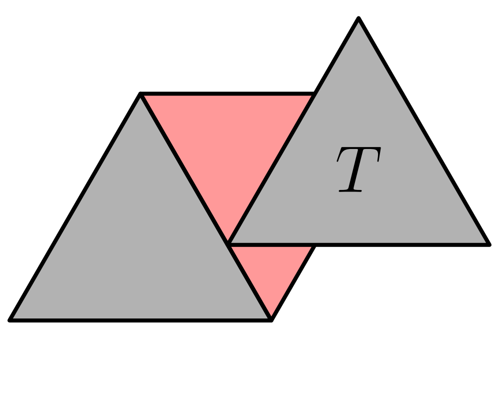


Suppose we place down a triangle such as the gray one in the above image. We claim that at least $\frac{1}{2}$ of the red area is blank space. This would solve the problem because this red area cannot be associated to any other triangle that we place in this way. That is, we're guaranteed that we're not "double-counting any blank space".

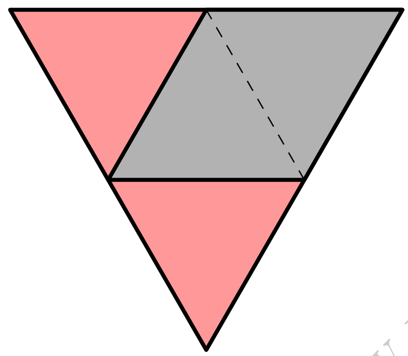
First, observe that at most one triangle can intersect the red region. This is because any triangle that does so must contain the point P in its interior. Call the triangle that intersects the red region (if any) T.

Next, we observe that to maximize that fraction of the red region's area covered by T, it must be the case that T's bottom-left vertex lies on the bottom-left edge of the region. This is not difficult to argue: If this were not the case, then the area covered will increase when T is "pushed" down and/or to the left.

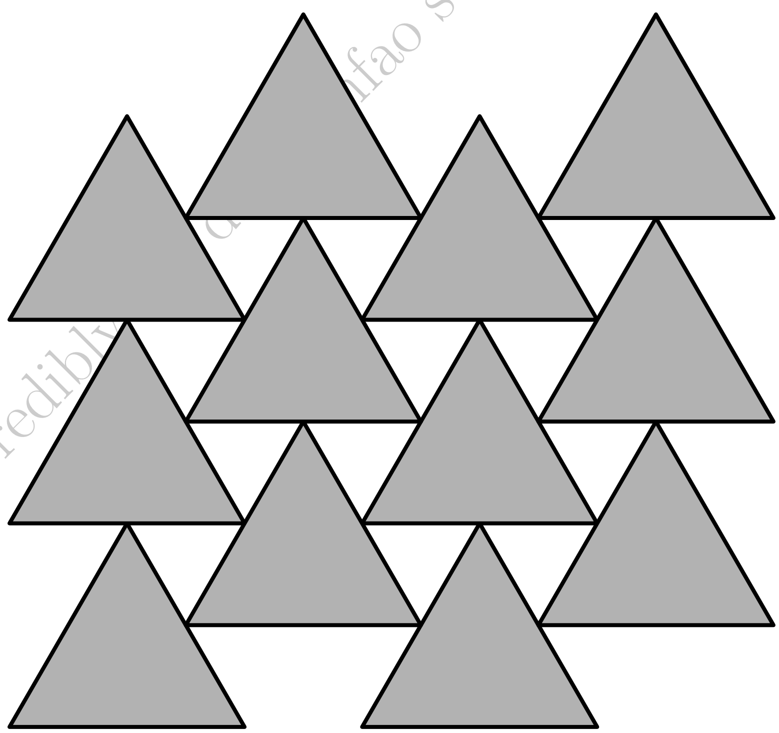
Thus the case of maximal area coverage must look like the diagram below.



We see that there are two uncovered "red triangles". It remains to show that the sum of their areas is at most half the area of the original red region. Indeed, if their side lengths are x and y, then the sum $x + y$ is fixed. Their areas are proportional to $x^2 + y^2$, so we must minimize the quantity $x^2 + y^2$. By the QM-AM inequality, the minimum exists and occurs when $x = y$. It is not hard to see that when this holds, exactly half the red region's area is covered.



The ratio 2/3 is achieved below.



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Solution 3

We claim the answer is 4.

Construction

Draw a tetrahedron around the sun. For each side of the tetrahedron, place a large planet that contains that entire side but does not intersect the sun.

Since the boundary of the tetrahedron is fully contained in the union of the planets, no ray of light will escape the sun.

The issue is that the planets may intersect. To fix this, take a planet and apply a homothety or dilation on it, centered at the sun. This enlarges the planet, but in return we can sent it as far away from the other planets as we want. By applying homotheties to each planet, we can ensure that no two planets intersect. Moreover, any ray of light blocked by a planet will still be blocked upon homothety. This finishes the construction.

Minimality

We want to show that 3 is impossible.

Associate each possible direction for a ray of light with a point on the surface of a sphere S centered at the sun. Observe that if we have a planet centered at some point P, and P' is the intersection of S with the light ray that goes though P, then the set of all light rays blocked by this planet may be viewed as a strict subset of the open hemisphere on S centered at P'.

Thus, it is sufficient to show that 3 open hemispheres cannot cover the surface of S.

Suppose we could, and call the hemispheres H_1, H_2, and H_3. The boundary of H_1 is a great circle C_1 that is not covered. Likewise, the boundary of H_2 is another great circle C_2 that must intersect C_1 at two diametrically opposite points A and B. However, the third hemisphere H_3 cannot cover both of the uncovered points A and B, contradiction.

■

Remarks: The problem is slightly more challenging if the sun were instead a sphere whose surface emitted rays of light. (Note that such rays need not be collinear with the sun's center.) I leave this as an exercise.

In general, $n+1$ is the least number of n-dimensional planets required to shield a source of light in n dimensions.

A natural question follow-up is as follows: Among all configurations of four planets that satisfy the problem, what is the minimum possible value of

$$\frac{\text{Largest Planet Radius}}{\text{Smallest Planet Radius}}$$

(or rather, the *infimum* of this quantity)? The following explicit construction gives an upper bound of $(5+\sqrt{24})^3 \approx 970$:

- A planet of radius $\sqrt{\frac{8}{3}}$ centered at $(5+\sqrt{24}) \cdot (1,1,1)$
- A planet of radius $\sqrt{\frac{8}{3}} \cdot (5+\sqrt{24})$ centered at $(5+\sqrt{24}) \cdot (-1,-1,1)$
- A planet of radius $\sqrt{\frac{8}{3}} \cdot (5+\sqrt{24})^2$ centered at $(5+\sqrt{24})^2 \cdot (1,-1,-1)$
- A planet of radius $\sqrt{\frac{8}{3}} \cdot (5+\sqrt{24})^3$ centered at $(5+\sqrt{24})^3 \cdot (-1,1,-1)$

You can find a picture of this construction on the front cover of this book.

Source: I have no idea, I first heard it at AMSP

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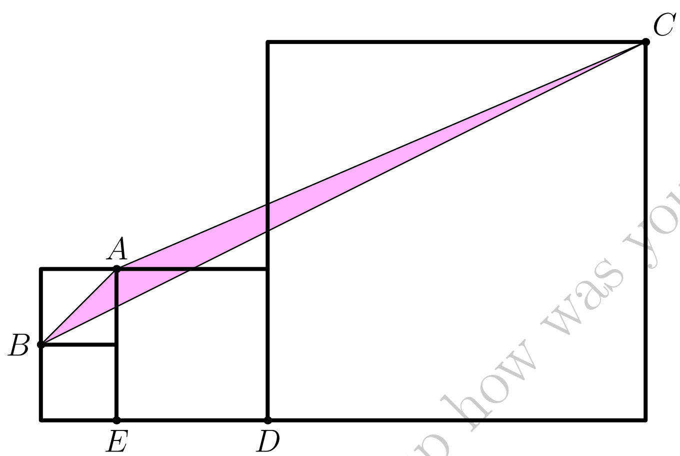
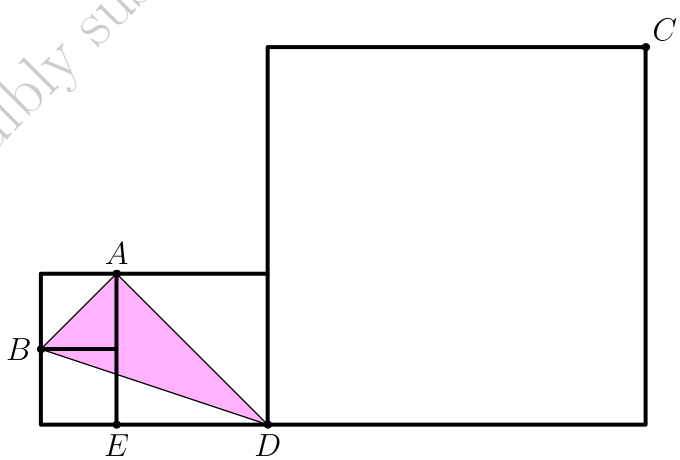
Solution 4

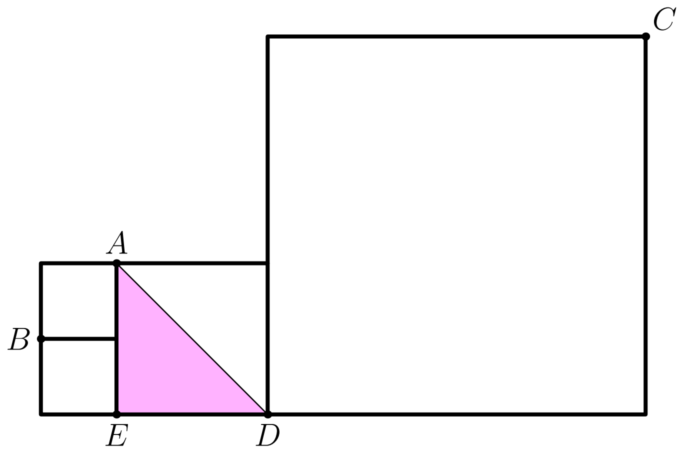
Let us apply the principle given in the hint. Label some points as shown.



First we slide the vertex at C towards D, which is a direction that is parallel to $\overline{AB}$. This does not change the triangle's area by the principle.



Next, we apply the principle again by sliding the vertex at B to E. This direction is parallel to $\overline{AD}$.



We conclude that the area of the original triangle, $\triangle ABC$, is exactly the area of $\triangle AED$, which is clearly $\boxed{2}$.

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Source: Catriona Agg

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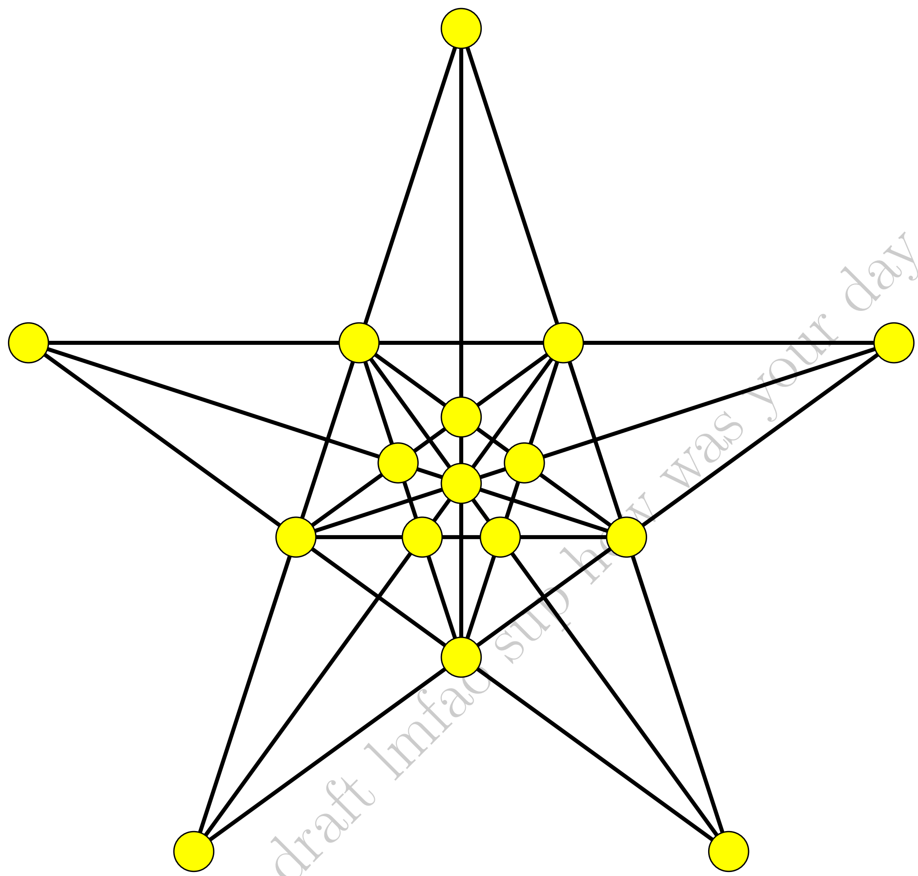
Solution 5



Remark: This may or may not be a true story. Whether Kaz took my quarters or not is omitted and left as an exercise.

Source: I saw this in a puzzle book a long time ago.

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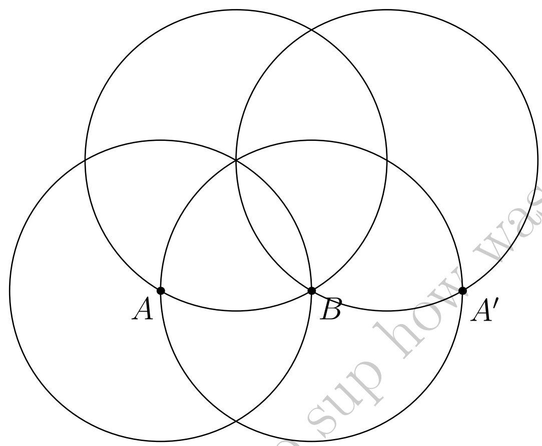
Solution 6

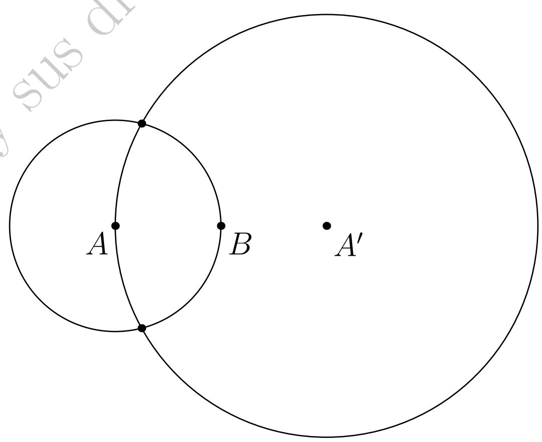
Alice can pass whereas Bob is doomed to fail.

Proof that Alice can pass

First she constructs the reflection of A over B by drawing the following four circles.

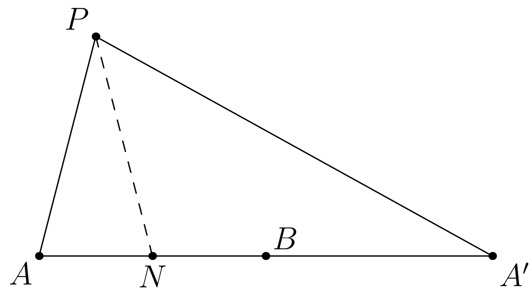


Calling the reflection A', she then draws the circle centered at A' that passes through A, and marks its intersections with the circle centered at A.



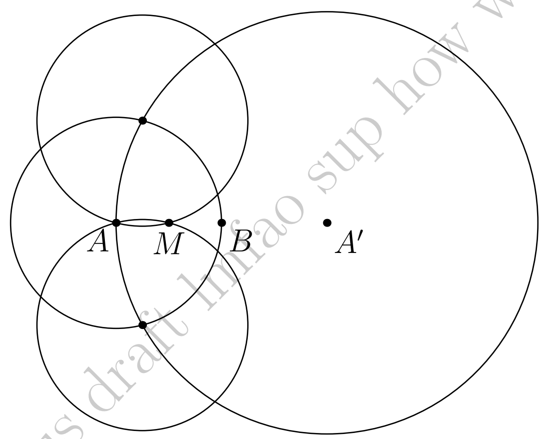
We claim that these intersections both lie on the perpendicular bisector of M, where M is the midpoint of AB!

This can be shown easily with coordinate geometry, but there is also a clean approach. Without loss of generality let us assume that $AB = 1$. Call one of the intersections P, and construct N on $\overline{AA'}$ for which $\triangle PAN \sim \triangle A'NA$.



We know that $\frac{AP}{A'P} = \frac{1}{2}$, so by the similarity, $\frac{AN}{AP} = \frac{1}{2}$. But $AP = 1$, so $AN = \frac{1}{2}$, which implies that N is the midpoint of segment AB, which has length 1.

With the claim proven, Alice may finish constructing M by drawing a circle at each intersection that passes through A.



The claim implies that the intersection of these two circles must be M.

Proof that Bob will fail

To show that Bob will fail, we will use the idea behind *projective transformations*. If you are familiar with projective geometry, then the proof is summarized in one line: "There exists a projective transformation that fixes A and B but moves the constructed midpoint M, and projective transformations send lines to lines". Otherwise, fear not. There is a perfectly elementary way to visualize the reasoning.

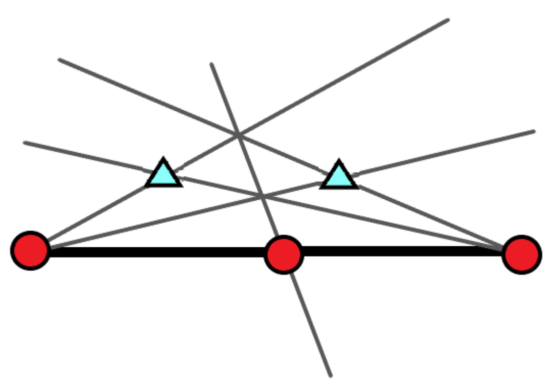
Let us assume for contradiction that Bob can construct the midpoint using only a straight-edge. Then Bob can find a finite algorithm for constructing the midpoint that consists of a combination of the following two steps:

- Pick an arbitrary point in the plane.
- Take two marked points or intersections and draw the line that connects them.

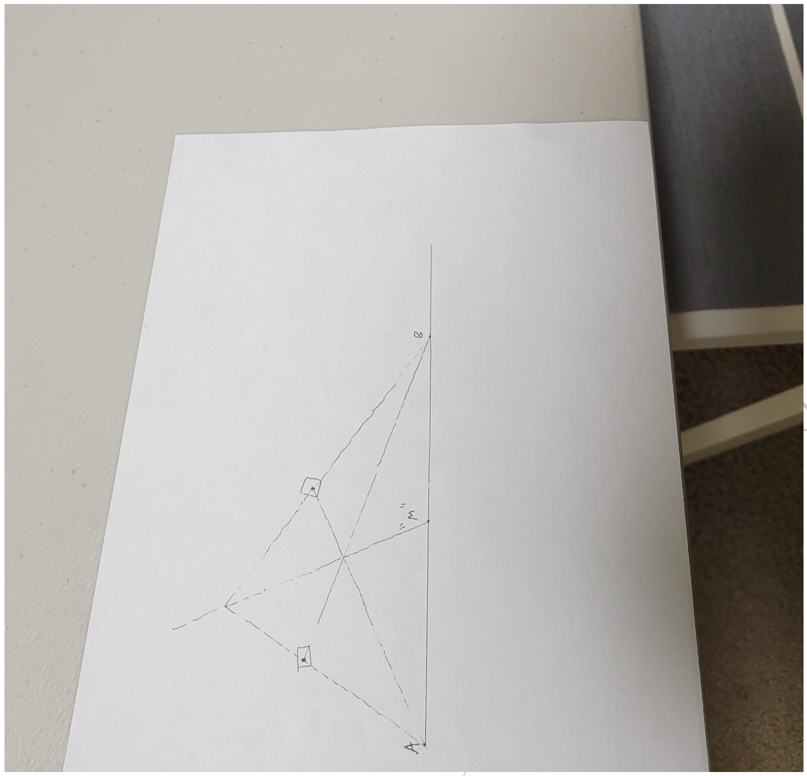
Bob wins if he can mark the midpoint as the point of intersection of two of the lines he draws. But crucially, we moreover must have that this marked point will remain the midpoint even if we perturb the arbitrary points that Bob picked. Otherwise, there would be no guarantee that Bob's scheme will construct the midpoint.

Of course, Bob *must* pick an arbitrary point in the plane at some point in his construction, else there is only one line he could ever draw: The one between A and B. This is the vulnerability we will exploit. To be precise, we will show that we can move the arbitrary points that Bob chooses such that the supposed midpoint also moves.

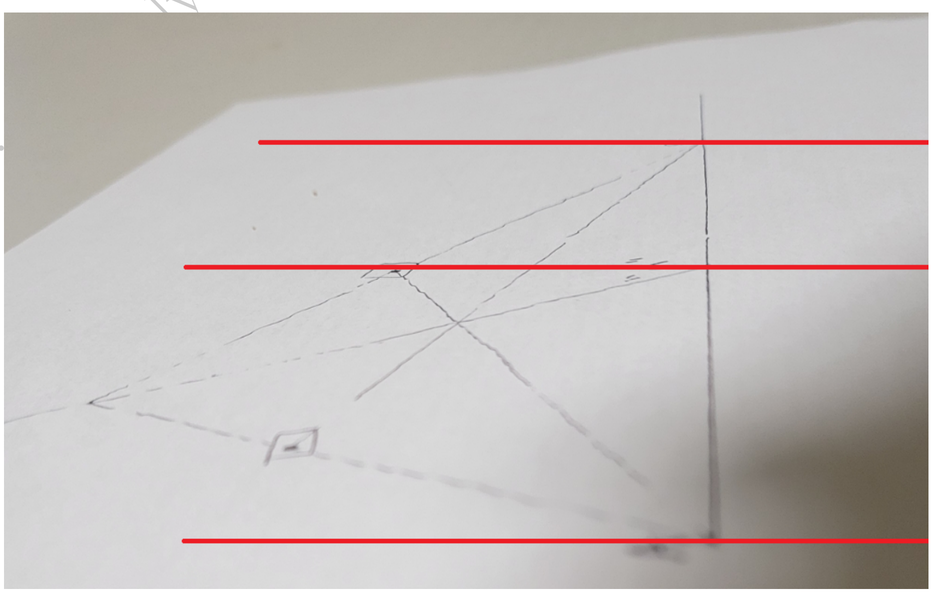
Let us say, for instance, that Bob's construction is as shown below, with the blue triangles representing points that were selected arbitrarily.



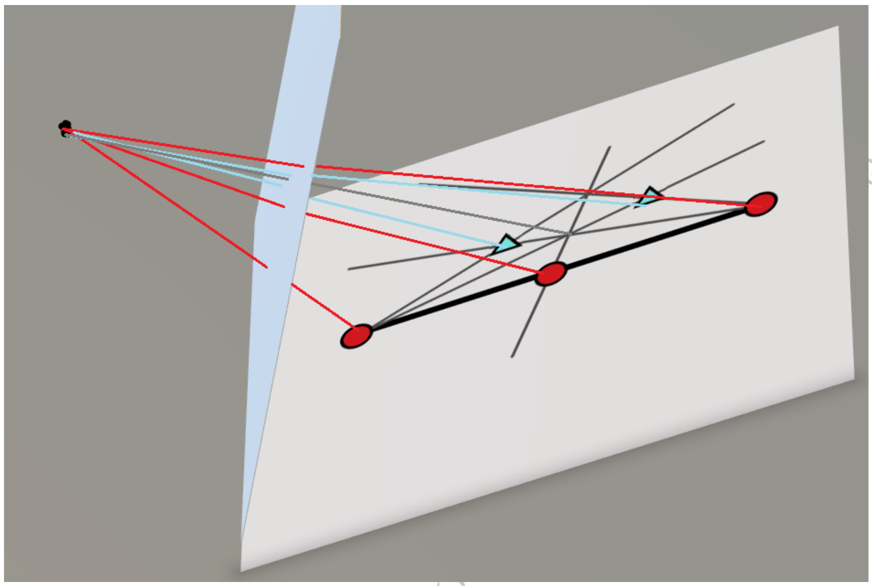
To help illustrate the point, I have also drawn this diagram on a real-life table.



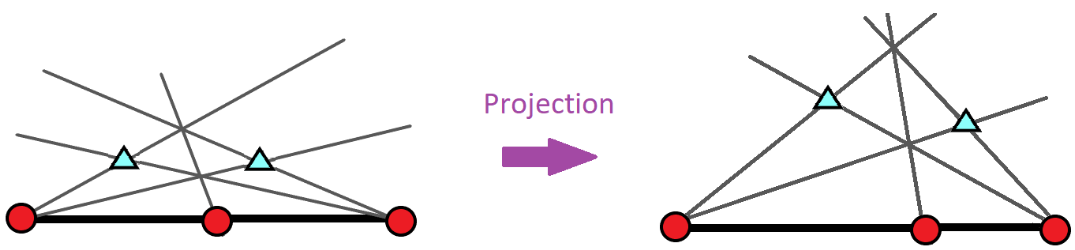
The intuitive idea is that an observer that perceives this construction from a different angle (in 3D space) will still see a construction that uses straight lines, but the relative lengths of the perceived segments may not be preserved. Alternatively, one can think of this routine as taking a picture of the construction from an angled camera. If the angle is chosen appropriately, then Bob's purported "midpoint" could fail to be the midpoint in the picture captured by the angled camera. This is demonstrated below.



Formally speaking, we are executing a *central projection* in which we select a point F in space (the "eye", where we locate ourselves to view the construction), select an angled plane P (the "lens", where the angled version of the construction is drawn), and then map every point X in the plane of the construction to the intersection of line $\overline{FX}$ with plane P.



When all points in lines in Bob's construction are transformed to the new plane P in this way, we can obtain a picture in which the purported midpoint is not the midpoint of AB.



More precisely, the images of the arbitrarily-selected points have moved to a set of new locations for which Bob's construction will construct a point that is not the midpoint, showing that his construction could have failed.

■

Source: Famous

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Solution 7

The trick is to write $9N = 10N - N$. If $N = \overline{d_1 d_2 \ldots d_n}$ with $d_1 < d_2 < \ldots < d_n$, then $9N$ can be though of as the result of the following stacked subtraction.

$$\begin{array}{rcccccc} & d_1 & d_2 & d_3 & \ldots & d_{n-1} & d_n & 0 \\ - & & d_1 & d_2 & \ldots & d_{n-2} & d_{n-1} & d_n \\ \hline & ? & ? & ? & \ldots & ? & ? & ? \end{array}$$

The first $n-1$ digits of the answer are clear: They are d_1, $d_2 - d_1$, $d_3 - d_2$, $\ldots$, $d_{n-1} - d_{n-2}$, which is valid because the digits are increasing. As for the last two columns, they are instead $(d_n - 1) - d_{n-1}$ and $10 - d_n$, which is valid because the digits are *strictly* increasing.

From here, the digit sum is evidently

$$d_1 + (d_2 - d_1) + \ldots + (d_{n-1} - d_{n-2}) + (d_{n-1} - 1 - d_{n-1}) + (10 - d_n) = -1 + 10 = \boxed{9}.$$

Source: Kvant Magazine?

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Solution 8

We present four different solutions. If you must read only one solution, I highly recommend the fourth one.

First Solution

Let $y = \sqrt{5-x} = 5 - x^2$. Then we have:

$$x^2 + y = 5$$

$$y^2 + x = 5$$

Subtracting, we get that $(x+y-1)(x-y) = 0$, so either $x = y$ or $x + y = 1$. Thus, either $x^2 + x - 5 = 0$ or $x^2 - x - 4 = 0$. This gives the four possibilities:

$$x = \frac{-1 \pm \sqrt{21}}{2}, \frac{1 \pm \sqrt{17}}{2}$$

Two of these solutions happen to be extraneous, and so the solutions are

$$\boxed{x = \frac{-1 + \sqrt{21}}{2}, \frac{1 - \sqrt{17}}{2}}.$$

■

Second Solution

First, after squaring both sides, we obtain

$$5 - x = x^4 - 10x^2 + 25$$

which we can rearrange as

$$P(x) := x^4 - 10x^2 + x + 20 = 0.$$

We would like to factor the polynomial $P(x0)$. To find a suitable factor, observe that $f(x) := \sqrt{5-x}$ and $g(x) := 5 - x^2$ are function inverses, and so the graph of $g(x)$ is obtained by reflecting the graph for $f(x)$ over the line $y = x$. In particular, the graph of $f(x)$ intersects the line $y = x$ exactly where $g(x)$ does (and it is simple to argue that such intersections must exist by the Intermediate Value Theorem), so one class of solutions to $f(x) = g(x)$ are those values of x for which $f(x) = x$. That is, $x = \sqrt{5-x}$. Hence we should expect that $x^2 + x - 5$ is a factor of the polynomial $P(x)$. Indeed it is, and we will find that

$$P(x) = (x^2 + x - 5)(x^2 - x - 4).$$

Setting each factor to 0, we can then proceed as in the first solution.

■

Third Solution

Let $g(x) := 5 - x^2$. We may write the given equation as

$$x = 5 - (5 - x^2)^2,$$

or $x = g(g(x))$. Observe that if x satisfies $x = g(x)$, then $g(g(x)) = g(x) = x$, so one class of solutions to the original equation is given by those x satisfying $x = g(x) = 5 - x^2$. So we expect $x^2 + x - 5$ to be a factor of the polynomial $P(x)$ as defined in the second solution. We proceed as in the second solution.

■

Fourth Solution

Eyeing the equation obtained after squaring both sides,

$$5 - x = x^4 - 10x^2 + 25,$$

we make the insane leap that this is a quadratic. That is, a quadratic in 5.

$$5^2 - (1 + 2x^2)5 + (x^4 + x) = 0$$

We now may apply the quadratic formula to solve for 5. This gives

$$\begin{aligned} 5 &= \frac{(1 + 2x^2) \pm \sqrt{(1 + 2x^2)^2 - 4(x^4 + x)}}{2} \\ &= \frac{(1 + 2x^2) \pm \sqrt{(1 + 4x^2 + 4x^4 - 4x^4 - 4x}}{2} \\ &= \frac{(1 + 2x^2) \pm \sqrt{(1 - 4x + 4x^2}}{2} \\ &= \frac{(1 + 2x^2) \pm (1 - 2x)}{2} \end{aligned}$$

Casing on the sign of $(1 - 2x)$, we get two possible quadratics which admit four possible solutions in total. We get the desired answer after testing them all.

■

Source: Probably Titu Andreescu.

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Solution 9

We first recall that
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
for all x. Taking $x = -1$ gives
$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!},$$
and so
$$\frac{n!}{e} = \sum_{k=0}^{\infty} (-1)^k \frac{n!}{k!}.$$
The quantity $A := \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$ is clearly an integer. Note that when $0 \le k \le n-2$, we have that $\frac{n!}{k!}$ is necessarily even, so the parity of A is determined by the last two terms in the sum, which add to $n-1$ (up to a sign). Hence, if n is even then A is odd, and when n is odd then A is even.

Let $f := \sum_{k=n+1}^{\infty} (-1)^k \frac{n!}{k!}$ be the rest of the sum. It is not hard to show that $|f| < 1$ (by, say, writing a geometric series as an upper bound). Now case on the parity of n.

- If n is even, then $f < 0$ because the first and largest term, $(-1)^{n+1} \frac{1}{n+1}$, is negative, and the remaining terms decrease too rapidly to change the sign of the partial sum. (Rigorously, you can bound the rest of the sum via a geometric series.) It follows that
$$\left\lfloor \frac{n!}{e} \right\rfloor = \lfloor A + f \rfloor = A - 1,$$
which is even because A is odd.
- If n is odd, then $f > 0$ by a similar argument, and so
$$\left\lfloor \frac{n!}{e} \right\rfloor = \lfloor A + f \rfloor = A,$$
which is even because A is even.

So $\left\lfloor \frac{n!}{e} \right\rfloor$ is always even.

■

If you wish to see the details on some of the claimed bounds, this is for you. To see that $|f| < 1$, write
$$|f| \le \sum_{k=n+1}^{\infty} \frac{n!}{k!}.$$

When $k \geq n+1$, we have that

$$\frac{n!}{k!} = \frac{1}{(n+1)(n+2)\cdots(k)} \leq \frac{1}{(n+1)^{k-n}},$$

so

$$|f| \leq \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}} = \frac{1}{n} < 1.$$

The other bound I used without proof was that

$$\left|\sum_{k=n+2}^{\infty} (-1)^k \frac{n!}{k!}\right| < \frac{1}{n+1}.$$

For this, we simply use the result we just proved — that $\sum_{k=n+1}^{\infty} \frac{n!}{k!} < \frac{1}{n}$ — but replace n with $n+1$. This gives $\sum_{k=n+2}^{\infty} \frac{(n+1)!}{k!} < \frac{1}{n+1}$. Thus

$$\left|\sum_{k=n+2}^{\infty} (-1)^k \frac{n!}{k!}\right| \leq \sum_{k=n+2}^{\infty} \frac{n!}{k!} = \frac{1}{n+1} \sum_{k=n+2}^{\infty} \frac{(n+1)!}{k!} < \frac{1}{(n+1)^2} < \frac{1}{n+1}.$$

This shows the bound and that it is, in fact, quite loose.

Source: I saw this on Math Stack Exchange.

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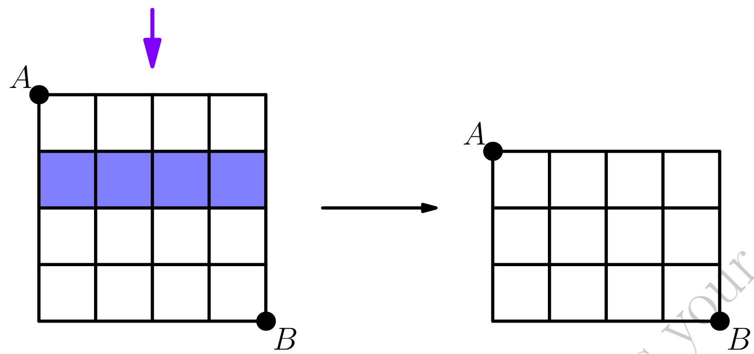
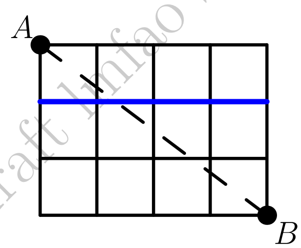
Solution 10

Become God and delete the river by slamming the two landmasses together.

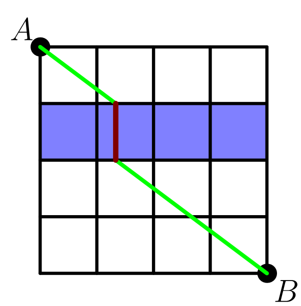


This decreases the length of the shortest path from A to B by 1 (why?). Since the shortest distance is now clearly 5, the original minimum distance was $\boxed{6}$.

To determine where to place the bridge, we draw the straight-line path from A to B in the transformed problem.



Then, for this path to correspond naturally to a valid path in the original problem with the river, we must place the vertical bridge where this path intersects the blue segment (where the river used to be).



To be precise, the bridge must be placed $\frac{4}{3}$ miles east of Town A.

■

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Solution 11

The line from $(0, 1-t)$ to $(t, 0)$ is given by $y = \frac{t-1}{t}x + 1 - t = x + 1 - \left(t + \frac{x}{t}\right)$. For some fixed $x \in [0,1]$, we are interested in finding the line (i.e. the value of t) that obtains the maximum possible y value at that x. By AM-GM

$$x + 1 - \left(t + \frac{x}{t}\right) \leq x + 1 - 2\sqrt{t \cdot \frac{x}{t}} = x + 1 - 2\sqrt{x} = (1-\sqrt{x})^2$$

with equality obtained at $t = \sqrt{x}$. So the equation of the curve is given by $y = (1-\sqrt{x})^2$. This magically rearranges to $\boxed{\sqrt{x} + \sqrt{y} = 1}$.

■

Source: Back in middle school, I had graph paper and I was really bored.

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Solution 12

In this solution, the curly brackets $\{\cdot\}$ denote *multisets* instead of sets.

As per the hint, we must find a positive integer S such that there is exactly one ordered pair of positive integers (P, n) for which there exist two distinct ways to partition S into a sum of n positive integers whose product is P.

We claim that the only possible value for S is $S = 12$.

Claim 1: $S = 12$ **is a valid solution to the problem.**

Various mathematical acquaintances and I have tried various approaches for demonstrating this without an extreme amount of casework. Alas, although there were a few successes, they were just not as elegant or short as getting our hands dirty and listing out (essentially) every possible partition of 12. So let's just do it.

We can skip partitions of 12 into a sum of 1 or 2 integers. That is, it is impossible for $n = 1$ or $n = 2$. For $n = 1$, there is obviously only one way for an integer to equal 12, so we need not consider $n = 1$. As for $n = 2$, you will have a difficult time finding two distinct solutions to the system $a + b = S$ and $ab = P$.

Thus we may start checking partitions of 12 from $n = 3$.

The chart begins on the next page.

n	Partition of $S = 12$ into n terms	Product (P)
3	1,1,10	10
3	1,2,9	18
3	1,3,8	24
3	1,4,7	28
3	1,5,6	30
3	2,2,8	32
3	2,3,7	42
3	2,4,6	48
3	2,5,5	50
3	3,3,6	54
3	3,4,5	60
3	4,4,4	64
4	1,1,1,9	9
4	1,1,2,8	16
4	1,1,3,7	21
4	1,1,4,6	24
4	1,1,5,5	25
4	1,2,2,7	28
4	1,2,3,6	36
4	1,2,4,5	40
4	1,3,3,5	45
4	**1,3,4,4**	**48**
4	**2,2,2,6**	**48**
4	2,2,3,5	60
4	2,2,4,4	64
4	2,3,3,4	72
4	3,3,3,3	81

(Continued on next page)

n	Partition of $S = 12$ into n terms	Product (P)
5	1,1,1,1,8	8
5	1,1,1,2,7	14
5	1,1,1,3,6	18
5	1,1,1,4,5	20
5	1,1,2,2,6	24
5	1,1,2,3,5	30
5	1,1,2,4,4	32
5	1,2,2,2,5	40
5	1,2,2,3,4	48
5	2,2,2,2,4	64
5	2,2,2,3,3	72
6	1,1,1,1,1,7	7
6	1,1,1,1,2,6	12
6	1,1,1,1,3,5	15
6	1,1,1,1,4,4	16
6	1,1,1,2,2,5	20
6	1,1,1,2,3,4	24
6	1,1,2,2,2,4	32
6	1,1,2,2,3,3	36
6	1,2,2,2,2,3	48
6	2,2,2,2,2,2	64
7	1,1,1,1,1,1,6	6
7	1,1,1,1,1,2,5	10
7	1,1,1,1,1,3,4	12
7	1,1,1,1,2,2,4	16
7	1,1,1,1,2,3,3	18
7	1,1,1,2,2,2,3	24
7	1,1,2,2,2,2,2	32
8	1,1,1,1,1,1,1,5	5
8	1,1,1,1,1,1,2,4	8
8	1,1,1,1,1,1,3,3	9
8	1,1,1,1,1,2,2,3	12
8	1,1,1,1,2,2,2,2	16
9	1,1,1,1,1,1,1,1,4	4
9	1,1,1,1,1,1,1,2,3	6
9	1,1,1,1,1,1,2,2,2	8
10	1,1,1,1,1,1,1,1,1,3	3
10	1,1,1,1,1,1,1,1,2,2	4
11	1,1,1,1,1,1,1,1,1,1,2	2
12	1,1,1,1,1,1,1,1,1,1,1,1	1

We see that there is exactly one pair of rows that have the same value of n and product P.

Thus $S = 12$ is a valid solution.

It is important to note the details of this solution: The value of n is 4 and the two possible multisets of ages are $\{1, 3, 4, 4\}$ and $\{2, 2, 2, 6\}$.

Claim 2: No value of S less than 12 can be a valid solution.

Suppose some $S < 12$ is a valid solution. Then there exists n, P, and two distinct multisets of ages $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ that both have sum S and product P. (The two multisets need to be unique, but we won't need this.)

But now the two multisets $\{12 - S, x_1, x_2, \cdots, x_n\}$ and $\{12 - S, y_1, y_2, \cdots, y_n\}$ contain positive integers, have the same number of elements, have the same product, and each sum to 12. This pair of multisets is a different one from the one we found before — $\{1, 3, 4, 4\}$ and $\{2, 2, 2, 6\}$ — because they share a common element $(12 - S)$, whereas the pair of multisets we found before does not. This contradicts the validity of $S = 12$ as a solution.

Claim 3: No value of S greater than 12 can be a valid solution.

To rule out $S = 13$, note that there are at least two pairs of multisets with the same number of elements, same product, and sum 13:

- $\{1, 1, 3, 4, 4\}$ and $\{1, 2, 2, 2, 6\}$
- $\{2, 2, 9\}$ and $\{1, 6, 6\}$

To rule out any $S \geq 14$, we have the following general construction of two such pairs:

- $\{1, 3, 4, 4, 15 - S\}$ and $\{2, 2, 2, 6, 15 - S\}$
- $\{2, 2, 9, 15 - S\}$ and $\{1, 6, 6, 15 - S\}$

In conclusion, there is exactly one value of S that works: $S = 12$. That is:

- Beth's favorite number is 12.
- Alice's favorite number is the product, $\boxed{48}$, and is the number that must fill the blank.
- The two possible multisets of stuffed animal turtle ages are $\{1, 3, 4, 4\}$ and $\{2, 2, 2, 6\}$.

■

Remarks: As is apparent, the proof is not pretty, and I do not know of a more elegant proof. But what would a *solution* look like? That is, how would one go about finding the answer of $S = 12$?

For sure, the easiest way to do this is to tell the computer to do it. But if we are stuck on an island, then an intuitive line of reasoning could begin by observing that there are two forces at play. The first is one that we have identified in the proof above: If S is too large, then there are "too many partitions of S", and we can expect that there will be multiple pairs of partitions with the same cardinality and product. In contrast, if S is too small, then there will be no such pairs.

To more carefully formalize this, we may begin by classifying the possible values of S into three categories.

- Let us say that $S = k$ has *multiplicity* m if there are exactly m distinct pairs of partitions of k with the same number of elements and product.
- $S = k$ is a *non-solution* if it has multiplicity 0.
- $S = k$ is a *strong solution* if it has multiplicity 1.
- $S = k$ is a *weak solution* if it has multiplicity at least 1.

Our goal is to seek a value of S that is a strong solution. We can show that small values of S are non-solutions and large values of S are weak solutions, so that a strong solution must be a carefully-selected in-between value which strikes some sort of balance.

Indeed, we can observe that the sequence of multiplicities for the values $S = 1, 2, 3, \cdots$ must be increasing. This is because if $S = k$ has multiplicity m, then we may append the element 1 to each of the $2m$ partitions of k to obtain a collection of m pairs of partitions of $k+1$ satisfying the required conditions. Hence the multiplicity of $S = k+1$ must be at least m, the multiplicity of $S = k$.

It follows that if there exists a strong solution, then the smallest weak solution must be a strong solution. Thus, for simplicity, one can focus on finding the smallest weak solution, i.e. the smallest value of S which admits two distinct partitions $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$ with the same product. The advantage of this assumption is that for S to be minimal, it cannot be the case that the two partitions have an element in common. Otherwise, we could simply remove said common element from both partitions. The products of the new multisets will still be equal, as will the sum, which will have decreased.

From this assumption, we can eliminate quite a few possible values for the product P when S is minimal. For example, we now know that at least one of the partitions of S in question cannot have a 1, and this fact can be used to show that P cannot be a product of three primes. (*Sketch: If it were, then, letting the primes be p, q, r, one of the two partitions* *must be* $\{p, q, r\}$*, since this is the only one that does not use 1's. The other partition is either* $\{1, 1, pqr\}$ *or* $\{1, p, qr\}$ *up to reoredering, so we either have* $p + q + r = 1 + 1 + pqr$ *or* $p+q+r = 1+p+qr$*, neither of which are possible because you can show that* $p+q+r < 2+pqr$ *and* $p + q + r < 1 + p + qr$*.)*

This fact is extremely helpful, as now some of the first few possible values for P are $16, 24, 32, 36, 48, 54, 56, 60, \cdots$. The "correct" value, $P = 48$, is not too far down this list. So, while this methodology is far from concrete, a hopeful solver could reasonably come up with $P = 48$ provided that there is an efficient way to eliminate values of P that do not correspond to weak solutions for S.

One way to approach this is by finding a simple upper bound for S. It is plausible to stumble upon an upper bound of 13 by observing that $S = 13$ is a weak solution, witnessed by the pair of partitions $\{1, 6, 6\}$ and $\{2, 2, 9\}$. Though, finding this requires a knack for intelligent guesswork that I do not possess. Is there another approach? This, I leave to the reader. I hope the ideas presented here were interesting nonetheless.

Source: John Conway

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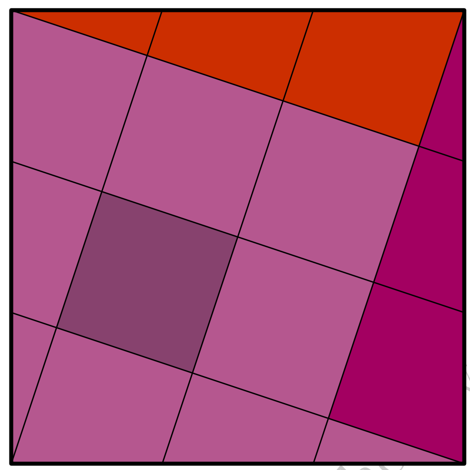
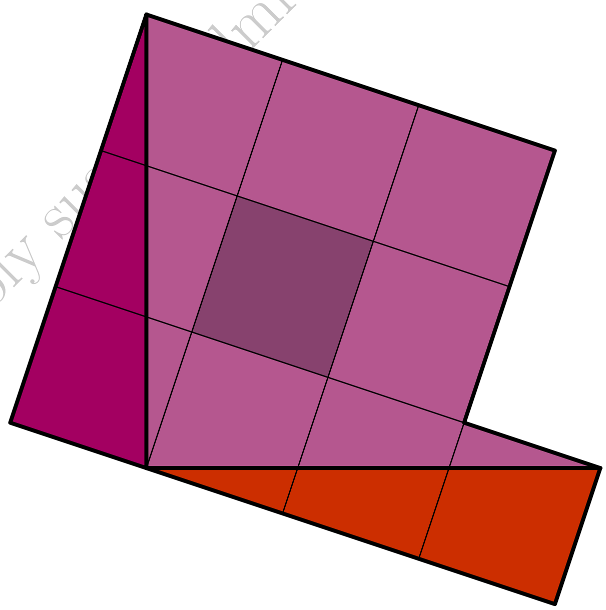
Solution 13

We divide the 3×3 square into the following three pieces.



We now rearrange the pieces by moving the top triangle to the bottom and the right triangle to the left side.



From this, it is clear that the shaded region occupies $1/10$ of the area of this figure. Since the area of the original square was 9, the shaded area is $\boxed{9/10}$

■.

Source: Adapted from the 2002 Lomonosov Tournament

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Solution 14

Let $y = 2^x$. Then $x = \log_2 y = \frac{\log y}{\log 2}$, so we are solving:

$$y + 3^{\frac{\log 2}{\log y}} = 5$$

Move the y over and take the log:

$$\begin{aligned} 3^{\frac{\log 2}{\log y}} &= 5 - y \\ \log\left(3^{\frac{\log 2}{\log y}}\right) &= \log(5-y) \\ \frac{\log 2}{\log y} \cdot \log 3 &= \log(5-y) \\ \log 2 \cdot \log 3 &= \log(5-y)\log(y) \end{aligned}$$

Now it is clear that $y = 2, 3$ are solutions. This corresponds to $x = 1, \log_2 3$, respectively. So the other solution is $\boxed{\log_2 3}$.

■

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Solution 15

Draw a "finish line" on the hiking trail, so that we may view the game as occurring in "laps". The fireman's strategy is amusingly simple: On every lap, extinguish every tree until the pyromaniac sets a tree on fire (and do nothing for the rest of the lap). That's it!

To show that this works, consider the binary number N formed by viewing each burning tree as a 1, each extinguished tree as a 0, and reading the digits from the end to the start. So the first tree is worth 1, the second tree is worth 2, etc. until the last tree, which is worth 2^{2021}.

If the pyromaniac doesn't set any trees on fire during a lap, then the fireman's strategy extinguishes everything, meaning we're already done. So we can assume that the pyromaniac tries to set something on fire on every lap.

In this case, we have on every lap that the last tree whose state is changed is some tree that the pyromaniac sets on fire (because the fireman does nothing after some tree is set on fire). Since this tree has the highest place value in the binary representation of N among all trees whose states were changed, we deduce that N must have increased during the lap.

N cannot increase forever because it is capped by $2^{2022} - 1$. So eventually it must necessarily hit $2^{2022} - 1$, meaning that all trees are on fire at the end of the lap. The fireman can then extinguish all the trees in one lap.

■

Remarks: There is also an inductive approach. Credits to "InductionEnjoyer" for spotting this. We strengthen the problem to showing that for any tree T, the firefighter can always reach the state where the firefighter and pyromaniac are leaving tree T, and all trees are extinguished.

Clearly this is true if there is only 1 tree. Assume that the firefighter can complete their goal at any particular tree if there are n trees. Now suppose there are $n + 1$ trees.

The firefighter applies the inductive hypothesis to the n trees other than T, so that all trees other than T are extinguished and the firefighter and pyromaniac are leaving the tree before T, so that the next tree they arrive at is T. There are now two cases.

- T is on fire at this time. Then the firefighter simply extinguishes it and they move on from T, completing the induction.
- T is not on fire at this time. Then the only way to prevent the firefighter's goal from being completed is for the pyromaniac to set T on fire. Since the pyromaniac can't extinguish T, the firefighter then repeats their strategy so that they once again reach the state where they reach T with all other trees extinguished. Now the firefighter

extinguishes T and moves on from T, completing the induction.

It turns out that the strategy generated by this induction is the same as the explicit strategy that we constructed from before.

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Solution 16

If you are like me and unable to pull a construction out of thin air, here is a cute approach. Suppose there were indeed a decomposition $f(x) = e(x) + o(x)$ for some even $e(x)$ and an odd $o(x)$. This equation holds for all x, so if I replace x with $-x$ then it should still be true:

$$f(-x) = e(-x) + o(-x)$$

This simplifies to

$$f(-x) = e(x) - o(x).$$

But now we have

$$\begin{cases} f(x) = e(x) + o(x) \\ f(-x) = e(x) - o(x) \end{cases},$$

which is a system of equations in the two "variables" $e(x)$ and $o(x)$! Solving, we get:

$$e(x) = \frac{f(x) + f(-x)}{2}$$

$$o(x) = \frac{f(x) - f(-x)}{2}$$

This is what $e(x)$ and $o(x)$ would have to be if the described decomposition existed. It remains to verify that these are indeed even and odd respectively. Fortunately, they are!

■

Source: Folklore

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Solution 17

Let the speeds of the Red October and USS Dallas be v_1 and v_2, respectively. So, $v_2 > v_1 > 0$. Without loss of generality, we may say that the Red October was detected at the origin $(0, 0)$.

The locus of points that the Red October can be at some time t is a circle centered at $(0, 0)$ whose radius is expanding at a rate of v_1. Tracking down the Red October means that we must eliminate every possible angle that it could have taken by traversing this expanding circle a full 2π radians. Evidently we must first make contact with the expanding circle to begin traversing its circumference, so the first step of our strategy is to send our ship straight towards $(0, 0)$ until we hit the imaginary expanding circle. We say that this occurs at time $t = 0$ and that, without loss of generality, we make contact with the circle at $(R_0, 0)$.

We now derive parametric equations in polar form that describe our path from here on out. We will follow the path of the expanding circle counter-clockwise, and this necessitates that our distance from $(0, 0)$ is increasing at a rate of v_1. Thus the equation for $r(t)$ is:

$$r(t) = v_1 t + R_0$$

It remains to find $\theta(t)$, i.e. our angular velocity.

We obtain this by using the fact that we are traveling at speed v_2. We can quickly derive a formula for our speed in parametric form: We know that $x = r\cos\theta$ hence $x' = r'\cos\theta - r\theta'\sin\theta$. Similarly $y' = r'\sin\theta + r\theta'\cos\theta$. Squaring these equations and adding, we obtain $v_2^2 = (r')^2 + r^2(\theta')^2$. Clearly $r'(t) = v_1$ and $r(t)^2 = (v_1 t + R_0)^2$. Hence:

$$v_2^2 = v_1^2 + (v_1 t + R_0)^2 \theta'(t)^2$$

$$\theta'(t) = \frac{\sqrt{v_2^2 - v_1^2}}{v_1 t + R_0}$$

Here it is clear why we needed $v_2 > v_1$.

Integrating:

$$\theta(t) = \theta(0) + \int_0^t \frac{\sqrt{v_2^2 - v_1^2}}{v_1 s + R_0}, ds = \sqrt{\left(\frac{v_2}{v_1}\right)^2 - 1}\log\left(\frac{v_1}{R_0}t + 1\right)$$

It remains to verify that we obtain $\theta(t) = 2\pi$ in finite time. Indeed we do, since log increases to ∞ (albeit at a turtle's pace)! Thus we indeed eventually traverse the entire circle, thereby checking every possible angle. Ergo, our strategy eventually lets us crash into the Red October.

■

Source: I saw this on the Data Genetics blog.

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Solution 18

At $t = 0$, let P start at C and Q start at A. We then let P move toward D at speed $|CD|$ and let move Q move toward B at speed $|AB|$. Thus, P and Q end up at D and B respectively at time $t = 1$.

It suffices to prove that the problem statement holds for all time t.

CLAIM: $[ABP]$ and $[CQD]$ change linearly with time.

Proof. View AB as the base of $\triangle PAB$, so that the height is the altitude from P. Note then that the height is changing linearly with time because P is moving along a line. Thus so is the area. The logic for $\triangle QCD$ is the same.

CLAIM: We are done.

Proof. $[APB] + [CQD] = [ABCD]$ holds at the beginning of time and at the end of time. But by the previous claim, the quantity $[APB] + [CQD]$ changes linearly with time. We deduce that in fact, this quantity must be constant with time, because it takes the same value $[ABCD]$ at two distinct times.

Thus $[APB] + [CQD] = [ABCD]$ for all time, which is what we wanted.

■

Source: I saw this in some random corner of AoPS.

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Solution 19

Yes we can.

- Begin by setting three fires at once: Two on both ends of the 1-minute rope, and a third on one end of the 2-minute rope.
- After **30 seconds** have elapsed, the 1-minute rope will be burnt up, and there will be 90 seconds left on the 2-minute rope. At this point in time, set the other end of the 2-minute rope on fire.
- After **45 seconds** have elapsed, the remaining 90 seconds of the 2-minute rope will have burnt up.

In sum, this procedure achieves **30 + 45 = 75** seconds of burning time, as needed. ■

Source: Classic

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Solution 20

We claim that Sydney cannot reach $(1, 0)$.

Call a point (a, b) *even* iff a and b have the same parity. We claim Sydney can only reach even points.

To see this, assume Sydney moves from an even to an odd point. Without loss of generality let us suppose that the even point was $(0, 0)$, and that she moves to an odd point (a, b). Then the equation of the perpendicular bisector, which can be written as $2(ax + by) = a^2 + b^2$, must pass through an integer point (x, y).

But then $2(ax + by)$ would be even, so $a^2 + b^2$ would have to be even, so $a + b$ must be even, so a and b have the same parity. This contradicts the premise that (a, b) is odd.

Since $(1, 0)$ is an odd point, Sydney cannot reach it.

■

Source: Thomas Lam, USAMTS

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Solution 21

It is still impossible.

Credits to "axcaea" for this approach.

Let us begin by proving some well-known lemmas. The interested reader should note that their proofs may be simplified using the fact that $\mathbb{Z}[i]$ is a UFD.

Lemma 1

Suppose that x and y can each be written as the sum of two squares. Then the same is true of their product.

Proof. Write $x = a^2 + b^2 = |a + bi|^2$ and $y = c^2 + d^2 = |c + di|^2$. Then

$$xy = |(a + bi)(c + di)|^2 = |ac - bd + (ad + bc)i|^2 = (ac - bd)^2 + (ad + bc)^2,$$

which completes the argument. □

Lemma 2

Suppose that x is even and can be written as the sum of two squares. Then the same is true of $\frac{x}{2}$

Proof. Write $x = a^2 + b^2$. If a and b are both even then we may simply divide each side by 4 to see that $\frac{x}{4} = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$ is a sum of two squares, thus $(x/4)(1^2 + 1^2)$ is a sum of two squares by the previous lemma. If a and b are both odd, then write

$$x = a^2 + b^2 = |a + bi|^2 = \frac{1}{2}|(1 + i)(a + bi)|^2 = \frac{1}{2}|a - b + (a + b)i|^2 = \frac{(a - b)^2 + (a + b)^2}{2},$$

so $x/2 = \left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2$. □

Back to the problem. We claim that

$$S := \left\{\left(\frac{p}{d}, \frac{q}{d}\right) : d \text{ is an odd sum of two coprime squares, and } p \equiv q \pmod 2\right\}$$

is the set of all points reachable by Sydney. Of course, it is not necessary for us to show that all points in S are indeed reachable. For the sake of the problem, it is sufficient to show that Sydney cannot escape S, since $(1, 0) \notin S$. So we shall just prove this direction and leave the reachability as an exercise to the curious reader.

Let $(p/d, q/d) \in S$. Suppose $ax + by + c = 0$ is the equation of a line that connects two lattice point. It is sufficient to show that the reflection of $(p/d, q/d)$ over this line will remain in S.

Since $ax + by + c = 0$ has an integer solution for x and y, it must be the case that $\gcd(a, b) \mid c$. By dividing out by the GCD, we may assume without loss of generality that $\gcd(a, b) = 1$.

We now leave it to the reader to check that the coordinates of the reflection in question are:

$$\left(\frac{p(a^2 - b^2) - 2b(aq + c)}{d(a^2 + b^2)}, \frac{-q(a^2 - b^2) - 2a(bp + c)}{d(a^2 + b^2)}\right)$$

There are now two cases to check.

1. If $2 \mid ab$, then since $\gcd(a, b) = 1$, exactly one of a, b is even and the other is odd. So $d(a^2 + b^2)$ is odd. But d is a sum of two squares, so by the first lemma we may deduce that $d(a^2 + b^2)$ is also a sum of two squares.

   As for the numerators, the fact that $a^2 - b^2$ is odd implies that the first numerator has the same parity as p, and the second numerator has the same parity as q. These have the same parity by the assumption that $p \equiv q \pmod 2$. We conclude that the reflection is in S in this case.

2. If $2 \nmid ab$, then $d(a^2 + b^2)$ is even, but a quick mod 4 argument reveals that it has only one factor of two. The numerator also has a factor of 2, since $a^2 - b^2$ must be even. So we may reduce each fraction by dividing by 2 to write the coordinates as

   $$\left(\frac{p(a^2 - b^2)/2 - b(aq + c)}{d(a^2 + b^2)/2}, \frac{-q(a^2 - b^2)/2 - a(bp + c)}{d(a^2 + b^2)/2}\right).$$

   Now, $d(a^2 + b^2)/2$ must be odd, and since $d(a^2 + b^2)$ is a sum of two squares, the same is true for $d(a^2 + b^2)/2$ by the second lemma.

   As for the numerators, $a^2 - b^2$ is divisible by 4, so both $p(a^2 - b^2)/2$ and $q(a^2 - b^2)/2$ are even. And, since $a, b \equiv 1 \pmod 2$, we have that

   $$b(aq + c) \equiv aq + c \equiv q + c \equiv p + c \equiv bp + c \equiv a(bp + c) \pmod 2.$$

   So again, the numerators have the same parity, and thus the reflection is in S.

This completes the proof.

■

Remarks: This problem has a funny history. My original solution to this, claiming the answer of "yes", was incorrect as it had fallen for the trap mentioned by the first hint. After outsourcing the problem, we found at least two more proofs claiming an answer of "yes" and at least two more proofs claiming an answer of "no", all of which were wrong.

Source: Me!

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Solution 22

The shaded area is obtained by simply adding up the areas of the first and third circles and subtracting out the second circle! So the answer is $51^2\pi + 48^2\pi - 42^2\pi = \boxed{3141\pi}$.

■

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Solution 23

The key observation is that $(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)=1$. So if we multiply both sides by $(\sqrt{x^2+1}-x)$, we get

$$y+\sqrt{y^2+1}=\sqrt{x^2+1}-x.$$

Likewise, if we instead multiplied both sides by $(\sqrt{y^2+1}-y)$, we'd get

$$x+\sqrt{x^2+1}=\sqrt{y^2+1}-y.$$

Adding these two equations together, we conclude that $x+y=0$.

■

Source: The earliest source I could find was the 1985 Norway Math Olympiad

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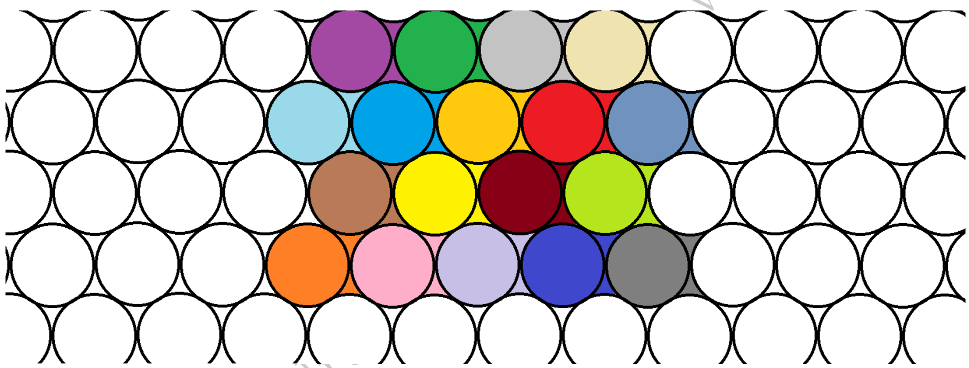
Solution 24

The idea is as follows: Tile $\mathbb{R}^2$ with a triangular packing of radius-10 circles. We show that we can translate this tiling so that each crewmate is in a circle. This is done by picking a *random* translation of the tiling, and proving that it covers more than 9 crewmates in expectation. We then select those circles containing crewmates to be the buoys, and we are guaranteed that we use no more than 10 buoys because there are only 10 crewmates.

To wit, if we pick a random translation of the tiling (by, e.g. choosing one circle's center uniformly at random from some fundamental domain of the tiling, and extending this to a full tiling with a consistent orientation), then the probability that a particular crewmate is saved is given by the efficiency of the tiling, i.e. the "ratio" of the plane taken up by the circles' interiors.



To find this "ratio", we take a nice fundamental domain such as the shape of the colored regions above. We now determine what fraction of this domain is taken up by the circle.

Using underhanded tricks, this can be done very quickly. Specifically, the total area can be found by dividing the circle into 6 sectors and rearrarranging them so that the region becomes two equilateral triangles. From this argument, we may find the ratio to be $\frac{\pi}{2\sqrt{3}} \approx 0.9069$.

To finish, we denote by A_n the event that crewmate n is saved. By linearity of expectation, we deduce that the expected number of crewmates saved upon picking a random arrangement of buoys is given by:

$$\mathbb{E}\sum_{n=1}^{10} 1_{A_n} = \sum_{n=1}^{10} \mathbb{E}1_{A_n} = \sum_{n=1}^{10} \mathbb{P}(A_n) \approx 10(0.907) \approx 9.07 > 9$$

So there must exist an arrangement that saves more than 9 crewmates. That's equivalent to saving *all* the crewmates, hence we have proven that it is possible.

■

Remark: It is possible to do better than 10! If n is the most number of crewmates that can be saved if they all fall into the water, then according to the paper http://2012.cccg.ca/papers/paper13.pdf, we have that $12 \leq n \leq 44$.

Source: Naoki Inaba

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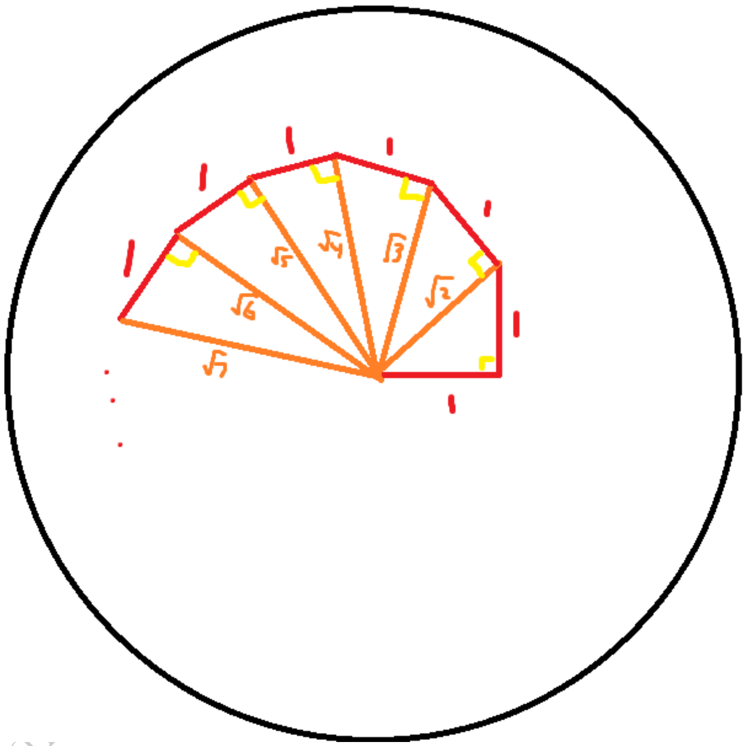
Solution 25

Minsung can escape within 100 seconds.

In the optimal strategy, we always make Minsung move in the direction that is "most angled toward the center/spawnpoint". Hence, Minsung's optimal strategy is to actually move at a "right angle" from the center. That is, he always faces in a direction v such that v is perpendicular to the line segment connecting the center and Minsung.



Inductively, by using the Pythagorean theorem, we deduce that after t seconds, Minsung will be $\sqrt{t}$ feet away from the center. Solving the equation $\sqrt{t} = 10$ for t, we get that Minsung might murder us all after $t = 100$ seconds.

■

Source: Classic.

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Solution 26

Suppose the B coins I use to make the A cents are worth $c_1, c_2, \cdots, c_B$ in cents. Then:

$$c_1 + c_2 + \ldots + c_B = A$$

The A coins we will use to make B dollars are as follows:

- c_1 coins worth $\frac{100}{c_1}$
- c_2 coins worth $\frac{100}{c_2}$
- $\cdots$
- c_B coins worth $\frac{100}{c_B}$

Indeed, there are $c_1 + c_2 + \ldots + c_B = A$ coins here. Moreover, their total worth is

$$c_1 \left(\frac{100}{c_1} \right) + c_2 \left(\frac{100}{c_2} \right) + \ldots + c_B \left(\frac{100}{c_B} \right) = 100B$$

which is B dollars.

■

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Solution 27

The answer is no. The identity function on $\mathbb{R}$ is the only ring endomorphism on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a ring endomorphism. We will show that $f(x) = x$ for all $x \in \mathbb{R}$.

Step 1: We show that f is the identity on rationals.

Clearly $f(2) = f(1) + f(1) = 2$. Inductively we see that $f(n) = n$ for all naturals n. Moreover $f(0 + 0) = f(0) + f(0)$, so $f(0) = 0$, and from here we see that $f(n + -n) = f(n) + f(-n)$ so that $f(-n) = -f(n)$. Thus f is the identity on integers. Lastly, for any rational m/n, where m is an integer and n is natural, we may write

$$f(m/n) + f(m/n) + \ldots + f(m/n) = f(m/n + m/n + \ldots + m/n) = f(m),$$

where the "…" represents continuing on for n terms. Here we may deduce that $f(m/n) = f(m)/n = m/n$, so indeed f is the identity on rationals.

Step 2: We show that f sends positive reals to non-negative reals.

If $x > 0$, then $x = y^2$ for some y. Hence $f(x) = f(y^2) = f(y)f(y) \geq 0$.

Step 3: Now we may finish.

Take any real x, and suppose $f(x) \neq x$. Then, appealing to the symmetry $f(-x) = -f(x)$, we may suppose without loss of generality that $f(x) < x$. Now find a rational q with $f(x) < q < x$.

Since $q < x$, we have $x - q > 0$, so by Claim 2 we have that $f(x - q) \geq 0$. So $f(x) \geq f(q) = q$ by Claim 1. But $f(x) < q$, contradiction.

■

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Solution 28

We claim that the numbers are actually equal.

For each lit triangle of perimeter 2019, add 1 to each of its sides. Then:

1. It is still a triangle, because if $a + b > c$, then $(a + 1) + (b + 1) > (c + 1)$
2. The perimeter is now 2022.

So it becomes a lit triangle of perimeter 2022.

We claim that every lit triangle of perimeter 2022 can be obtained in this way! This would show that there is a one-to-one correspondence, so that the number of lit triangles of perimeters 2019 and 2022 respectively are equal.

To see this, suppose otherwise. Then there is a lit triangle of perimeter 2022 that cannot be obtained using the above procedure. This entails that if we subtract 1 from each side, then we obtain an invalid triangle.

Take such a triangle with sides a, b, and c, so that $a + b + c = 2022$. Without loss of generality, let us assume that c is the longest side. Then $a + b > c$, and moreover c would be the longest side of the hypothetical triangle with sides $(a - 1)$, $(b - 1)$, and $(c - 1)$. This would be a valid triangle if and only if $(a - 1) + (b - 1) > (c - 1)$. We are assuming that it is not valid, hence $(a - 1) + (b - 1) \leq (c - 1)$, or $a + b \leq c + 1$. But $a + b > c$, so $c + 1 \leq a + b \leq c + 1$. We deduce that $a + b = c + 1$.

Adding c to both sides, we obtain $2022 = a+b+c = 2c+1$, thus 2022 is odd, contradiction. ∎

Source: 2022 ICMC, Constantinos Papachristoforou

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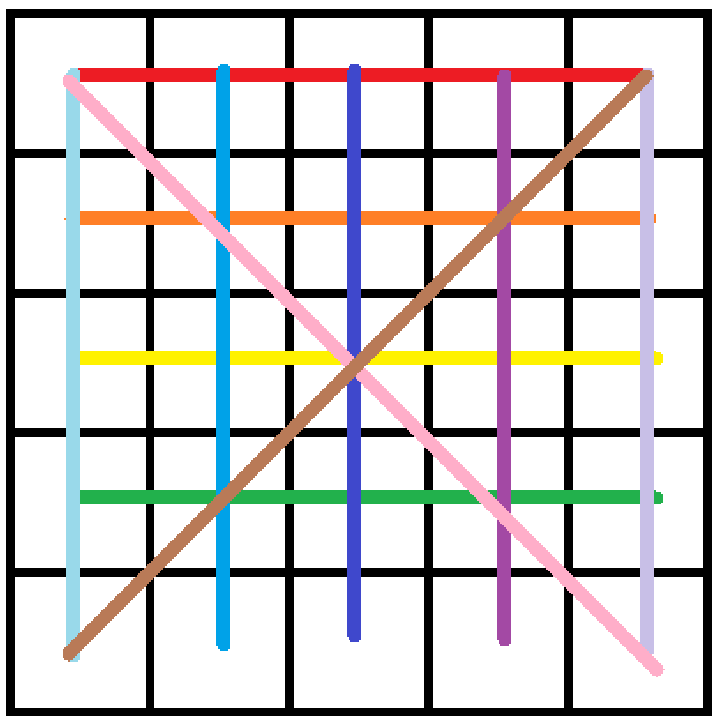
Solution 29

It suffices to find the dimension of V_0, the space of $n \times n$ magic squares with magic number exactly equal to 0, via the natural isomorphism $V_0 \oplus \mathbb{R} \cong V$ where V is the space of magic squares.

We now define a linear transformation $T : \mathbb{R}^{n \times n} \to \mathbb{R}^{2n+1}$ as follows: For $A \in \mathbb{R}^{n^2}$, $T(A)$ is the column vector whose components are the sums of the rows, columns, and diagonals of A, **except** the last row. That is, the components are the sums along the following lines:

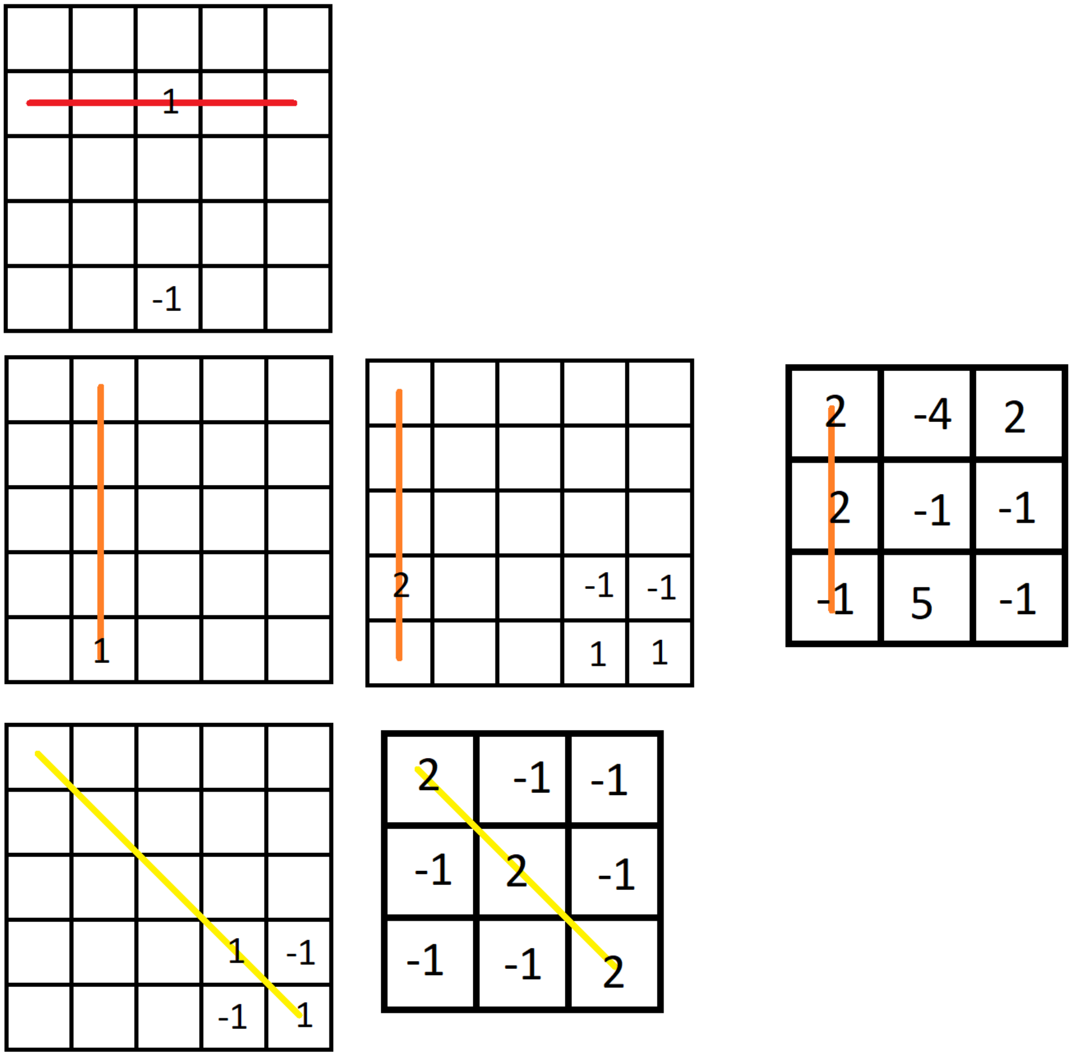


We note that the null space of T is precisely V_0. By the rank-nullity theorem, it follows that:

$$\dim V_0 = n^2 - \operatorname{rk} T$$

So it suffices to find the rank of T. In fact, we claim that T has full rank.

To prove this, we only need to show that for every line above, we can find a matrix such that that line has sum 1 (... or any non-zero real) and all other lines have sum 0. What's nice is that throwing out the last row makes this quite feasible!



As suggested in the above graphic, the scheme is as follows:

- For any row, we can find a cell that lies on neither diagonal and then place a 1 in it. Then, place a -1 in the bottommost cell below 1.
- For a column that is neither the first nor last, we may simply place down a 1 on its bottommost cell. Otherwise, we can do a strange thing, as shown, provided that $n \geq 4$. The case $n = 3$ is scary, and is also shown.
- For a diagonal, we may simply build a 2×2 square of 1's and -1's, provided that $n \geq 4$. The case $n = 3$ is not hard.

This proves that, indeed, T has full rank, so that $\dim V_0 = n^2 - (2n+1) = n^2 - 2n - 1$. Hence, $\dim V = 1 + \dim V_0 = \boxed{n^2 - 2n}$ for $n \geq 3$, whereas for $n = 1, 2$ we see that the dimension is 1.

■

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Solution 30

If the square "rotates" n times, then the angle between two consecutive squares is $\frac{\pi}{2n}$. By geometrical and symmetrical reasoning, we see that if the side length of some square is x then the next side length y must satisfy

$$y \sin\left(\frac{\pi}{2n}\right) + y \cos\left(\frac{\pi}{2n}\right) = x$$

thus the ratio between the side lengths of two consecutive squares is just $\frac{1}{\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)}$. After n "rotations", we see that the side length of the last square is given by $\frac{1}{\left(\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)\right)^n}$. It remains to compute $\lim_{n\to\infty}\left(\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)\right)^{-2n}$.

Letting $x = \frac{\pi}{2n}$, this limit will be equal to $\lim_{x\to 0}\left(\sin x + \cos x\right)^{-\frac{\pi}{x}}$, if it exists. We may write this as:

$$\exp\left[\lim_{x\to 0} -\frac{\pi}{x}\log\left(\sin x + \cos x\right)\right]$$

Applying Taylor's Theorem, we may now write $\sin x = x + o(x)$ and $\cos x = 1 + o(x)$ to write the desired as:

$$= \exp\left[\lim_{x\to 0} -\frac{\pi}{x}\log\left(1 + x + o(x)\right)\right]$$

We may moreover write $\log(1+y) = y + o(y)$, and we may take here $y = x + o(x)$ to obtain:

$$= \exp\left[\lim_{x\to 0} -\frac{\pi}{x}\left(x + o(x) + o(x + o(x))\right)\right]$$

$$= \exp\left[\lim_{x\to 0} -\frac{\pi}{x}\left(x + o(x)\right)\right]$$

$$= \exp\left[\lim_{x\to 0} -\pi - \frac{\pi o(x)}{x}\right]$$

Which is $\exp(-\pi) = \boxed{e^{-\pi}}$ by definition of little-o.

■

Remarks: The knowledgeable reader may have found this problem familiar! Indeed, it bears a strong resemblance to the following problem:

> On each corner of a unit square lies an ant. Starting at the same time, each ant moves directly towards their counter-clockwise neighbor at a speed of 1. When they meet at the center, how far has each ant travelled?

The limiting curve that you have witnessed in this problem is precisely the path taken by these ants, albeit cut short after 90° have been traversed about the center.

We can derive the equation of the curve in a cute way. First, observe that it should not be important that the ants move at a constant speed — as long as they all share the same speed, it should be the case that they trace out the same curve. Now, overlay the complex plane unto the square, with the square's center being 0. If an ant is at $z \in \mathbb{C}$, then it is moving towards iz. The observation lets us take the ant's speed at such a location to be $|iz - z|$, so that its path $z(t)$ will modeled by the differential equation

$$z'(t) = iz(t) - z(t) = (i-1)z(t).$$

This naturally solves as $z(t) = z(0)e^{(i-1)t}$.

With this parametrization of the curve, we have that t lives in $[0, \infty)$. Writing it as $z(t) = z(0)e^{-t}e^{it}$, we see that the distance to the center exponentially decays whereas the angular velocity is constant. This formula directly shows that the distance to the center upon a quarter rotation is given by $|z(0)|e^{-\pi/2}$, which is consistent with our deductions in the original problem's solution.

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Solution 31

Let t be the total amount of tea (in cups), and m be the total amount of milk (in cups). Then $t + m = n$, where n is the number of cups. Moreover, we are given that $\frac{t}{4} + \frac{m}{6} = 1$.

We know that there's some positive amount of tea. After all, it's not a department milk! So, $t > 0$, and in particular, $t/4 > t/6$. Thus

$$1 = \frac{m}{6} + \frac{t}{4} > \frac{m}{6} + \frac{t}{6} = \frac{n}{6}$$

so that $6 > n$.

Similarly, we know there's some positive amount of milk, because there is some "contamination" as stated in the problem. So $m > 0$ and in particular $m/4 > m/6$. This gives us

$$1 = \frac{m}{6} + \frac{t}{4} < \frac{m}{4} + \frac{t}{4} = \frac{n}{4}$$

so that $4 < n$.

Since $4 < n < 6$, we must have $\boxed{n = 5}$.

I leave it as an exercise to demonstrate that there indeed exist 5 cups of milk-contaminated tea that satisfies the problem's conditions.

■

Source: Folklore

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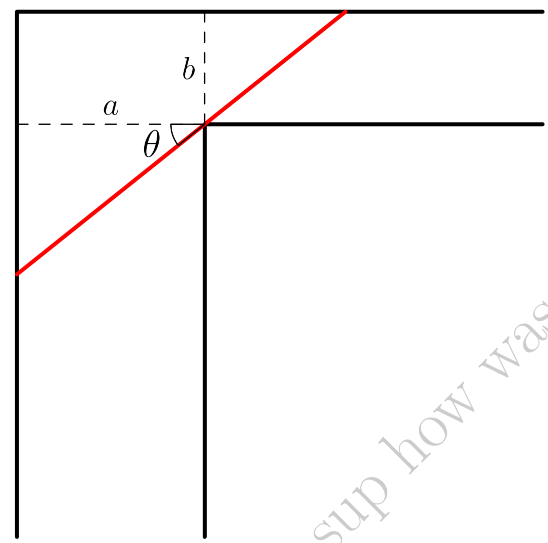
Solution 32

You can argue that the longest such stick length is equal to the minimum possible length of the red segment shown below.



With the angle θ marked as shown, it is clear that this length is given by $\frac{a}{\cos\theta}+\frac{b}{\sin\theta}$. One can use calculus to minimize this quantity. Alternatively, by Hölder's inequality with exponents $3/2$ and 3, we can more directly write

$$\left(\left(\frac{a^{2/3}}{\cos^{2/3}\theta}\right)^{3/2}+\left(\frac{b^{2/3}}{\sin^{2/3}\theta}\right)^{3/2}\right)^{2/3}\left(\left(\cos^{2/3}\theta\right)^{3}+\left(\sin^{2/3}\theta\right)^{3}\right)^{1/3}\geq a^{2/3}+b^{2/3}.$$

This rearranges to $\frac{a}{\cos\theta}+\frac{b}{\sin\theta}\geq\boxed{(a^{2/3}+b^{2/3})^{3/2}}$, and by examining the equality case one can see that this lower bound is indeed obtainable.

■

Remarks: If you have not heard of it, this is an easy variant of the *Moving Sofa Problem*, which asks for the area of the largest *region* that can be passed through this bend of the hallway. Mathematicians have had a lot of fun drawing interesting telephone-like shapes that can get through, but the problem is still officially open at the time of writing. Recently, Jineon Baek has claimed a very promising resolution to the problem (https://arxiv.org/abs/2411.19826).

Here's a fun variant: What is the maximum area of a *rectangle* that can be carried through such a bend?

Source: Folklore

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Solution 33

Claim: The set of all $P \in \mathbb{R}[x]$ that are the sum of two squares is closed under multiplication.

Proof. Take $P = A^2 + B^2$ and $Q = C^2 + D^2$. Then by magic:

$$PQ = |A + iB|^2|C + iD|^2 = |(AC - BD) + i(BC + AD)|^2 = (AC - BD)^2 + (BC + AD)^2$$

□

Without loss of generality suppose P is monic. Since $P \geq 0$, its roots come in conjugate pairs, and by multiplying such corresponding terms in the factorization of P, we see that P is a product of quadratics $x^2 + bx + c$, each of which is non-negative. From discriminant analysis it follows that $b^2 - 4c \leq 0$, so particularly we may write

$$x^2 + bx + c = (x + b/2)^2 + \sqrt{c - (b/2)^2}^2$$

which is a sum of two squares. Now inductively apply the claim!

■

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Solution 34

We compute Clara's probability of winning. If she goes second or third with probability $\frac{2}{3}$, then the first shooter (either Alex or Blaire) must shoot the other expert markswoman or else she guarantees her own death. Then it will be Clara's turn, and her probability of winning will be exactly the probability that she lands the shot on the other living duelist, i.e. $\frac{1}{2}$.

If Clara goes first with probability $\frac{1}{3}$, it actually doesn't matter who she decides to fire at. If she shoots at Alex, Blaire, or even herself (!), she will lose if she lands the hit. Conditioned on the event that she misses (with probability $\frac{1}{2}$), we reduce to the case in which Clara goes third, and we know her survival probability here is $\frac{1}{2}$.

Altogether, Clara's odds of winning are

$$\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}.$$

By symmetry it follows that Alex and Blaire each have survival odds of $\frac{7}{24}$. Despite having the worst aim, Clara is the most likely one to get out alive.

■

Source: I found this in a puzzle book.

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Solution 35

Solution 1

Let X be the set of duelists and define the map $f : X \to X$ via:

$$f : x \mapsto \text{duelist shot by } x$$

If we assume that nobody is alive, then f is a surjection. By finiteness it follows that f is actually a bijection.

Thus we may view f as a permutation that can be decomposed into cycles. Since 31415 is odd, there is at least one odd cycle $(x_1\ x_2\ \ldots\ x_k)$. $k \neq 1$ because nobody shoots themselves, so $k \geq 3$.

Now, by virtue of the shooting cycle, we see that

$$d(x_1, x_2) > d(x_2, x_3) > \ldots > d(x_{k-1}, x_k) > d(x_k, x_1) > d(x_1, x_2),$$

contradiction.

■

Solution 2

Assume for contradiction that everyone dies. First, observe that there are 31415 bullets and 31415 people. So it cannot be the case that someone is shot more than once, since then someone must be shot less than once, i.e. not at all.

By finiteness, we may find the two duelists that are of the shortest distance apart. By minimality, these two duelists shoot each other. By the observation, nobody else shoots these two duelists. Thus we may essentially ignore these two duelists. Inductively repeating this argument, we eventually end up with one person. They must die, but they can't shoot themselves, contradiction.

■

Source: Folklore

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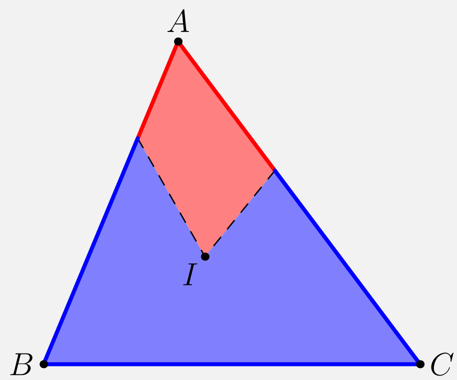
Solution 36

The miracle point in question is the incenter.

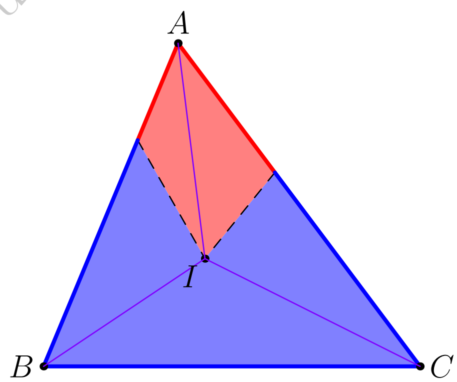
The credits for this approach go to "tenth". We will start by showing a strong variant of the converse.

Lemma 1

Let $\triangle ABC$ be a triangle with incenter I. Let X, Y be points on the boundary of $\triangle ABC$. Then cutting $\triangle ABC$ along segment XI and then along segment IY will divide the area and perimeter of $\triangle ABC$ into the same ratio.



Proof. The key idea is to subdivide the regions into triangles along the angle bisectors, as shown.



If we view each triangle as having height equal to the inradius r (and thus having a base along the perimeter of the triangle), then we can see that its area is $\frac{r}{2}$ multiplied by the length of the perimeter it occupies. Thus, for any triangle, the ratio $\frac{\text{Area}}{\text{Occupied Perimeter}}$ is constant, being equal to $r/2$. It follows that the ratio $\frac{\text{Red Area}}{\text{Red Occupied Perimeter}}$ is also $r/2$, and the blue region is no different. $\square$

With this proven, we are ready to solve the problem swiftly and with style. Suppose that we cut the pizza along segment XY, and that this cut divides the area and perimeter of the pizza into the same ratio.

Consider also cutting the pizza along segments XI and IY. As we have just shown, this also divides the pizza's area and perimeter into the same ratio.

But both the XY cut and the XI and IY cuts divide the pizza's perimeter into the same ratio! Thus, by the previous two paragraphs, they must divide the pizza's area into the same ratio.

This can only be possible if $\triangle XYI$ has zero area. That is, X, I, Y are collinear, so $\overline{XY}$ passes through the incenter.

$\blacksquare$

Source: This is called Haider's Theorem.

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Solution 37

The missing digit is 4.

The key property that we will use is that an integer n and the sum of the digits of n will always have the same remainder upon division by 9. Let us calculate the remainder of 2^{29} upon division by 9.

One simple way to determine this is to seek a pattern in the remainders among the powers of 2. If you know modular arithmetic, this gives a swift way to evaluate the remainder:

$$2^{29} = 4 \cdot 8^9 \equiv 4 \cdot (-1)^9 = -4 \equiv 5 \pmod 9$$

So the remainder is 5.

We know that if all ten possible digits were present, then the digit sum would be 45, which is divisible by 9. So, the missing digit must be a digit d such that $5 + d$ is divisible by 9. There is only one digit for which this holds: $\boxed{d = 4}$.

■

Source: Classic

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Solution 38

The situation encountered by the Hearthians is possible, and it turns out that Desmine must be missing.

The key ideas are:

- If you do the exploration procedure with a distance that is not an integer multiple of the distance along a quarter turn on the planet, then your path is likely too chaotic to expect being able to reunite with many other Hearthians.
- Doing the exploration procedure with a number of quarter turns around the planet that is congruent to 0, 1, or 3 mod 4 will return you to where you started. If instead it is congruent to 2, you actually end up on the antipode of where you started.

  This may be hard to believe, so I will sketch out why this is true. Imagine that a Hearthian starts at the north pole.

  - If this Hearthian returns to the north pole after travelling x km forwards, then evidently they will still arrive at the north pole again after turning, travelling x km forwards, turning, and travelling x km forwards again.
  - If this Hearthian does a single quarter-turn around the planet by walking forwards x km, then they will arrive at the equator after the first leg of their exploration procedure. After turning 90° counter-clockwise, the direction they face will be aligned with the equator, and so they will remain on the equator after waking x km forwards. Finally, after turning 90° counter-clockwise again, they will be facing north, and so walking x km forwards will get them back to the north pole.
  - If this Hearthian does a half-turn around the planet by walking forwards x km, then they will find themselves at the south pole after the first leg of their exploration procedure. The direction they turn does not matter: After travelling x km again in any direction, they arrive at the north pole, thus after yet another x km in any direction, they end their journey at the south pole.
  - If this Hearthian does a three quarters-turn around the planet by walking forwards x km, then as in the one quarter-turn case, they will arrive at the equator. Following the same logic reveals that they indeed will end up at the north pole at the end of their journey.

So doing the procedure with 1, 3, 4, 5, 7, 8, 9, 11, 12, ... quarter turns will take you to where you started. Inspired by the subsequence of numbers 3, 4, 5, 7, 8, 9, we can take the quarter-turn length to be 10 so that Desmine (the one with the 60) ends up at the antipode but everyone else ends up reunited back at the ship.

∎

Remarks: The radius of the planet is not unique. For example, the radius could also be such that the length of a quarter-turn is $\frac{10}{3}$.

It is also true that Desmine is the *only* Hearthian that could be missing as a result of this procedure. I will spare the details, but it turns out that if we assume that the ship is at the north pole, then the z-coordinate (which I take to be up and down) of one's location at the end of the exploration procedure can be modeled by the function $f(\theta) := \cos^3\theta + \sin^2\theta$, where θ is the "angle" traversed in in a single leg of the procedure. It follows that showing uniqueness reduces to proving that if θ is such that exactly 6 of the 7 values in the list $[f(3\theta), f(4\theta), \cdots, f(9\theta)]$ are equal to each other, then the odd element out must be $f(6\theta)$. Unfortunately, this seems difficult to prove without computational aids.

Source: Konstantin Knop

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Solution 39

The statement is false! We present two different methodologies for constructing an uncountable totally-ordered family of subsets of $\mathbb{N}$.

Solution 1

Using a bijection, it is sufficient to tackle the problem when "natural" is replaced with "rational". For each $x \in \mathbb{R}$ we take the *Dedekind cut* $D_x := \{q < x : q \in \mathbb{Q}\}$. We can then take our family to be

$$\mathcal{F} := \{D_x : x \in \mathbb{R}\}.$$

Clearly this is uncountable and is a totally-ordered family of subsets of the countable set $\mathbb{Q}$.

■

Solution 2

Using a bijection, it is sufficient to answer the problem replacing the naturals with the nodes of an infinite binary tree. The family of infinite paths starting from the root is uncountable, but not totally-ordered. To fix this, we simply add to each path all elements to the "left" of the path.

■

Source: I saw this on Math Stack Exchange

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Solution 40

We may save 9. Clearly this is the maximum we can guarantee saving because there is no guarantee that the first prisoner can get their hat color right, by virtue of having no information about their hat. To save everyone else, the first prisoner says "black" if they see an odd number of black hats, and "white" otherwise.

Then the second prisoner can deduce their hat color — if they see an odd number of black hats, then they must be wearing white. Otherwise, they must be wearing black. Now consider any prisoner P thereafter. By listening to the correct guesses of the prisoners behind them and counting the hats in from of them, P will be able to compute the number B of black hats excluding P's and the first prisoner's hats. So the number of black hats excluding just the first prisoner is either B or $B+1$, depending on whether P has a white or black hat, and this can be disambiguated by the first prisoner's information on whether this number is even or odd.

■

Source: Classic

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Solution 41

Recall that a permutation can be either odd or even. A permutation is even if it is created via an even number of swaps ("transpositions"), and odd otherwise. It is a theorem that no permutation can be both odd and even, so this is a well-defined characterization.

A key consequence is that if you take an even permutation and perform one additional swap, then the result is an odd permutation, and vice versa.

Starting from the back of the line, number the prisoners from 1 to 100. In some order, number the hats from 1 to 101. Mark the warden as "prisoner 101" and give him the missing hat.

Using the idea of even and odd permutations, the plan is as follows: Viewing the 101 hats as a permutation of the integers from 1 to 101, prisoner 1 will guess that the permutation is even and guess their hat according to this assumption. That's it.

Note that prisoner 1 sees all hats except their own and the warden's, so from their perspective, there are exactly two possible sequences for the hats, and they differ by exactly one swap. Thus, of the two possibilities that prisoner 1 sees, one represents an even permutation and the other is odd. Hence their assumption that the permutation is even corresponds to a well-defined guess for their hat.

For example, if there are 4 prisoners and prisoner 1 sees

$$? \qquad 2 \qquad 3 \qquad 4,$$

then prisoner 1 guesses 1, because this corresponds to the identity permutation (in which prisoner 1 wears hat 1 and the warden wears hat 5), which is even.

Prisoner 1 may or may not get shot. If they are not shot, then everyone knows that the permutation is indeed even. Otherwise, it must be odd.

Now we get to prisoner 2. We claim that prisoner 2 knows either prisoner 1's hat or the warden's hat. To see this:

- If prisoner 1 was not shot, then obviously by virtue of hearing their correct guess, prisoner 2 knows prisoner 1's hat.
- If otherwise prisoner 1 was shot, then let the hat they guessed be A. Let the hat they're actually wearing be B. The two possibilities were that either prisoner 1 was wearing A and the warden was wearing B, or the other way around. By virtue of hearing the shot, we know that the former was not the case, so it must be the latter, which implies that the warden is wearing A. That is, the warden is wearing the hat guessed by prisoner 1.

Thus, there are only two hat colors that prisoner 2 does not know: That of their own, and that of one other person. So, prisoner 2 is also guessing between two possible sequences, and they differ by a single swap. Since they know the parity of the permutation, they may disambiguate between the two possibilities and guess correctly.

Inductively, prisoner k can guess correctly. This is because prisoners $2, 3, \cdots, k-1$ all guess correctly, and by the same logic, we can argue that prisoner k can either deduce prisoner 1's hat or the warden's. So again, prisoner k needs to decide between two possible sequences of hats that differ by a single swap, which can be disambiguated because prisoner k knows the parity of the permutation.

In all, we see that all prisoners after prisoner 1 will be guaranteed survival. This is clearly the best possible outcome because prisoner 1 has no such guarantee of survival no matter the strategy. Hence 99 prisoners may be saved.

■

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Solution 42

Let X be the set of all possible hat sequences. Define an equivalence relation $\sim$ on X as follows: $A \sim B$ if and only if A and B are eventually the same. That is, they only differ in finitely many places.

The relation $\sim$ partitions X into equivalence classes. The prisoners, in the planning phase, will apply the axiom of choice to agree on a representative of each class.

When the game starts, the hats form some sequence $S \in X$. Every prisoner, by virtue of being able to see the tail of the sequence S, knows the equivalence class of S under $\sim$, and can therefore obtain the agreed-upon representative T of the class $[S]_\sim$. Every prisoner will then guess their hat in accordance to the sequence T.

Since $S \sim T$, we have that S and T will eventually be the same. That is, S and T will differ only in finitely many places, so only finitely many prisoners can die under this scheme. ■

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Solution 43

Solution 1

An incredible approach uses L'Hôpital's rule. Suppose such a function f existed. Then

$$\begin{aligned} 2 &= \lim_{x\to\infty} f(x) \\ &= \lim_{x\to\infty} \frac{f(x)e^x}{e^x} \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}f(x)e^x}{\frac{d}{dx}e^x} && \text{(L'Hôpital's rule)} \\ &= \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} && \text{(Product rule)} \\ &= \lim_{x\to\infty} f'(x) + f(x) \\ &= 1 + 2 \\ &= 3, \end{aligned}$$

contradiction.

To rigorously justify the application of L'Hôpital's rule, we need to check that

- $f(x)e^x$ is differentiable and approaches ∞ as $x \to \infty$,
- e^x is differentiable and approaches ∞ as $x \to \infty$, and
- $\lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x}$ exists and is finite.

Of course, all these are true.

■

Solution 2

The main purpose of this problem was to show off Solution 1, but it is more of an amusing parlor trick then an instructive methodology. A more typical approach is as follows. Suppose for contradiction that such a function f exists. Then there exists $N > 0$ so large that

- $|f(x) - 2| < 1$ for all $x \geq N$, and
- $|f'(x) - 1| < \frac{1}{2}$ for all $x \geq N$.

Now take $a = N$ and $b = N + 100$. By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(b) - f(a)}{100}.$$

Since $c \geq N$, we have that $|f'(c) - 1| < \frac{1}{2}$. Thus $\left|\frac{f(b)-f(a)}{100} - 1\right| < \frac{1}{2}$. In particular, $\frac{f(b)-f(a)}{100} > \frac{1}{2}$, and so $f(b) - f(a) > 50$. But now

$$50 < f(b) - f(a) \leq |f(b) - f(a)| \leq |f(b) - 2| + |2 - f(a)| < 2$$

because $a, b \geq N$, contradiction.

■

Remarks: Let's consider an alternate variant that can't be treated using the approach in Solution 2: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable such that there exists the limit

$$\lim_{x\to\infty} f(x) + f'(x) = L,$$

with L finite. Prove that

$$\lim_{x\to\infty} f(x) = L \text{ and } \lim_{x\to\infty} f'(x) = 0.$$

The intent behind this formulation is to force the methodology used in Solution 1. At first glance, this may seem successful. However, it is not so straightforward. To wit, here is an **incorrect** solution.

"If we apply L'Hôpital's rule as in Solution 1, we can write

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{f(x)e^x}{e^x} \stackrel{?}{=} \lim_{x\to\infty} \frac{f(x)e^x + f'(x)e^x}{e^x} = \lim_{x\to\infty} f(x) + f'(x) = L,$$

as needed."

The error is that the application of L'Hôpital's rule was not justified. The issue is that we do not know that $f(x)e^x \to \infty$ as $x \to \infty$.

To apply L'Hôpital's rule, we would be just as happy if it were the case that $f(x)e^x \to -\infty$ as $x \to \infty$. So it suffices to prove that $f(x)$ is either bounded from below or bounded from above over all $x \geq 0$. This is because if, for example, $f(x) \geq -M$ for all $x \geq 0$, then we may take $g(x) := f(x) + M + 1$ so that $g(x) \geq 1$. We have that $g(x) + g'(x) \to L + M + 1$ and $g(x)e^x \to \infty$ as $x \to \infty$, so we may apply the trick with L'Hôpital's rule to g!

Suppose for contradiction that $f(x)$ is unbounded from above and unbounded from below over $x \geq 0$. Then it is not hard to show that there exists an increasing sequence of local maxima x_n of f, with $x_n \to \infty$ and $f(x_n) \to \infty$. Intuitively, this is because for f to be unbounded from above and unbounded from below, it must obtain higher and higher peaks (as well as lower and lower valleys), and we select x_n to be the x-coordinates of these peaks. It then follows that

$$\infty > L = \lim_{x\to\infty} f(x) + f'(x) = \lim_{n\to\infty} f(x_n) + f'(x_n) = \lim_{n\to\infty} f(x_n) = \infty,$$

contradiction.

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Solution 44

We claim that there is no winning strategy because the game must end in a draw assuming perfect play from both players. This is because the game is isomorphic to Tic-Tac-Toe.

The first observation is that there are exactly 8 ways to make 15. That is, there are exactly 8 three-element subsets of $\{1, 2, \cdots, 9\}$ that sum to 15. Here they are:

- 1, 5, 9
- 2, 5, 8
- 3, 5, 7
- 4, 5, 6
- 3, 4, 8
- 2, 4, 9
- 2, 6, 7
- 1, 6, 8

It turns out that these eight triplets of digits are exactly the triplets that show up among the rows, columns, and two diagonals of a 3×3 magic square!

2	7	6
9	5	1
4	3	8

Thus, if we view the game between Ana and Beth as taking turns claiming digits from this magic square, then this reduces to Tic-Tac-Toe because the eight possible lines of victory in the board are exactly the eight possible obtainable sums by the claim.

■

Remarks: This is a remarkable connection. Though, some may argue that this may not *entirely* be a coincidence! See this blog post for an explanation.

Source: Probably John Conway

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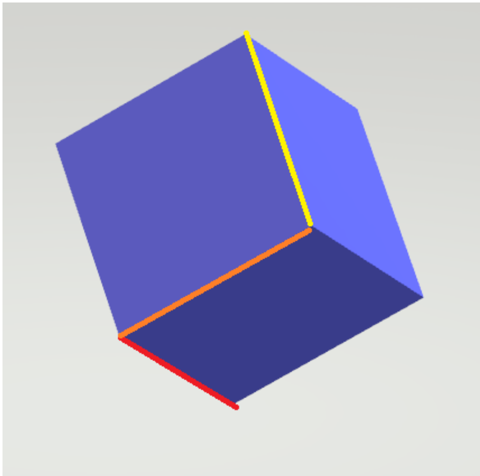
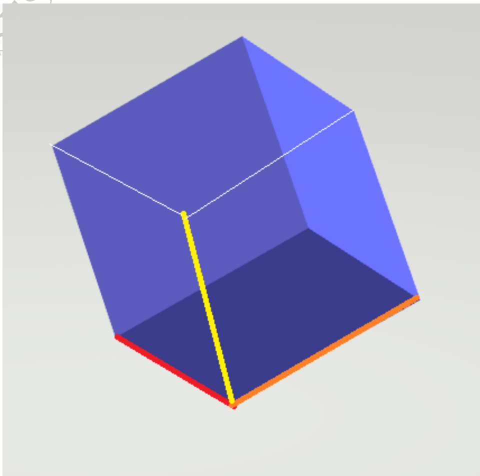
Solution 45

Let us first tackle the cube's "height", i.e. the length of its projection unto the z-axis. Note that the entirety of this height is traversed via the following path along the cube's edges, from the bottom to the top.

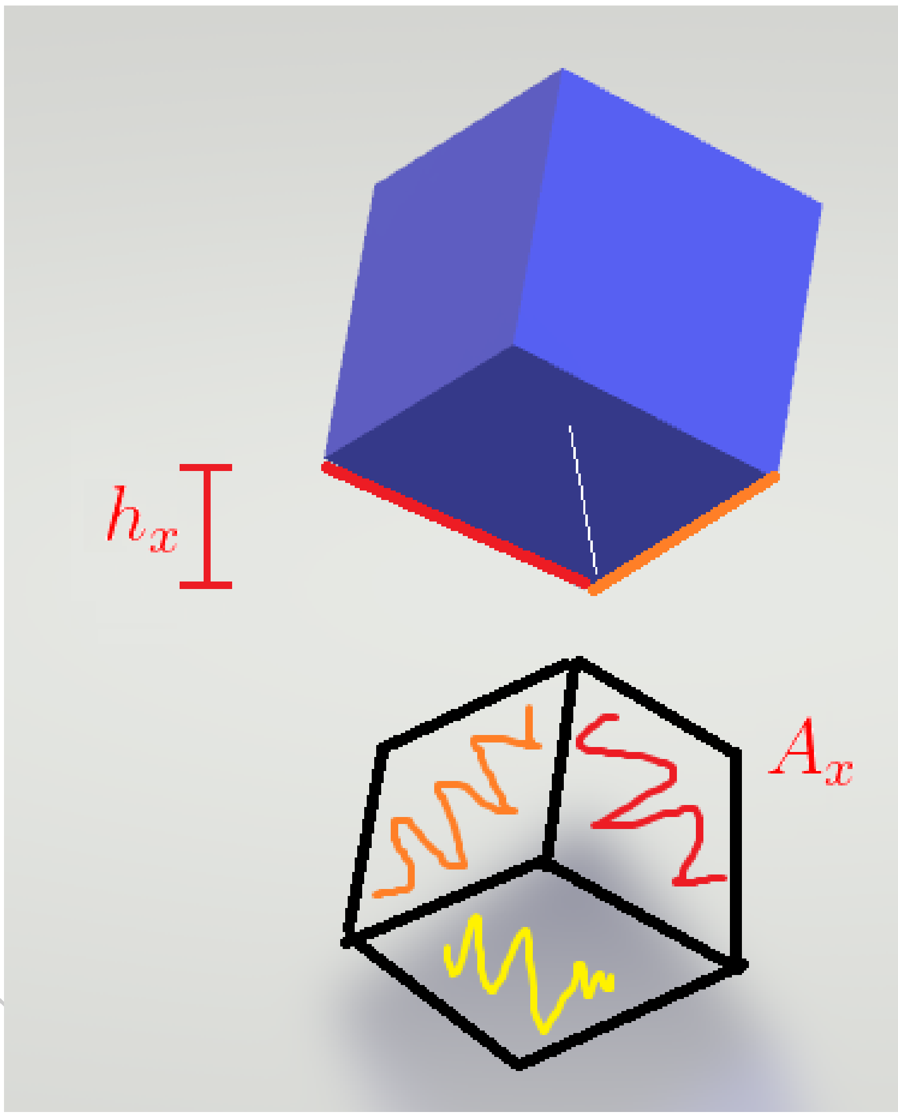


The desired "height" of the cube is given by the sum of the "heights" of these three edges. For easier analysis, we may shift these edges downwards so that they protrude from the bottom-most vertex.



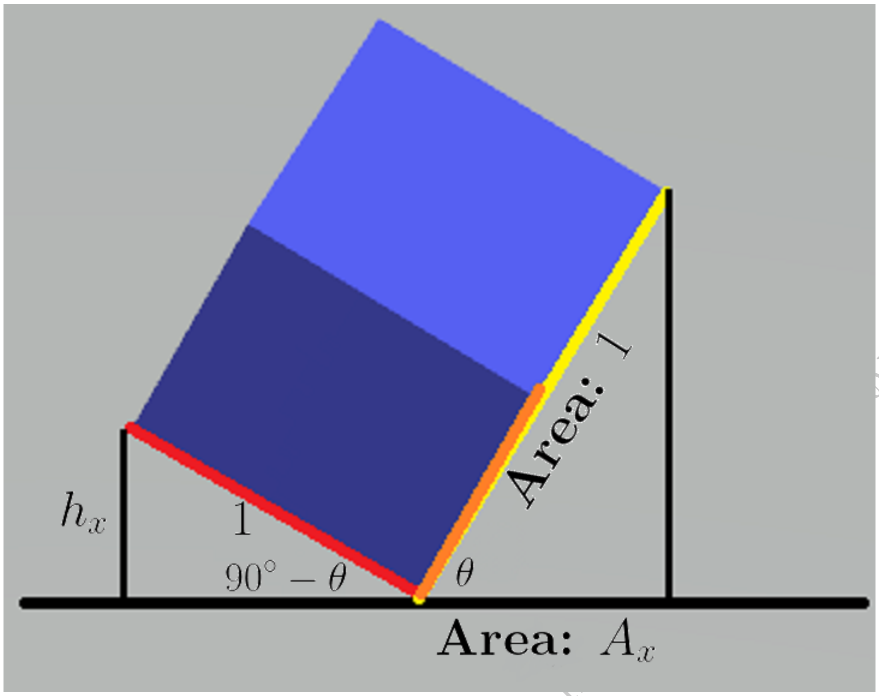
As in the diagram, we call the red, orange, and yellow edges x, y, and z, respectively. Let the "height" of the x, y, and z edges be h_x, h_y, and h_z, respectively. Then the "height" of the cube is $h_x + h_y + h_z$.

Putting this aside, let us now tackle the cube's "shadow". If you view a cube from any perspective, you always see three faces (or less), each being a rhombus. The cube's shadow is no different — it is a hexagon which can be partitioned into three rhombi, each being the projection of a different lower face.



As in the diagram, we label the area of the "shadow" under the face spanned by the y and z edges as A_x. Defining A_y and A_z similarly, we see that the total area of the cube's "shadow" is $A_x + A_y + A_z$.

We claim that $h_x = A_x$. To prove this, we orient our perspective to view the cube from the "side", so that the y and z edges coincide to form a line segment orthogonal to x.



Mark the angle θ as in the diagram. Then, from the lengths triangle formed on the left, we have that

$$\sin(90^\circ - \theta) = \frac{h_x}{1}.$$

The triangle formed on the right, on the other hand, displays a proportional relationship between *areas* instead of lengths. To be precise, it relates the area of the face spanned by y and z (which is 1) to the area of the shadow of this face (which is A_x) via the cosine of the angle between this face and the ground. We get that

$$\cos\theta = \frac{A_x}{1}.$$

But $\sin(90^\circ - \theta) = \cos\theta$, so indeed $h_x = A_x$.

By symmetrical reasoning, we have $h_y = A_y$ and $h_z = A_z$, and so we may conclude that $h_x + h_y + h_z = A_x + A_y + A_z$. That is, the "height" is numerically equal to the area of the shadow.

■

Remarks: The following generalization is true.

Theorem 1

Let $m, n \in \mathbb{N}$. Suppose that a unit cube C lies in $\mathbb{R}^m \times \mathbb{R}^n$ with some orientation. Then the m-dimensional measure of the projection of C unto $\mathbb{R}^m$ (the subspace formed by the first m components) is equal to the n-dimensional measure of the projection of C unto $\mathbb{R}^n$ (the subspace formed by the last n components).

To build up to the proof of this, let us first present two lemmas, both of which are fascinating results in and of themselves.

Lemma 1 (Sylvester's Determinant Identity)

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Then

$$\det(I_m + AB) = \det(I_n + BA),$$

where I_k denotes the $k \times k$ identity matrix.

Proof. Consider the block-form $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix} I_m & A \\ -B & I_n \end{pmatrix}.$$

We evaluate the determinant of this matrix in two different ways. Using "row reduction", we have on one hand that

$$\det \begin{pmatrix} I_m & A \\ -B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ -B + BI_m & I_n + BA \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ 0 & I_n + BA \end{pmatrix} = \det(I_n + BA).$$

On the other hand, we can use "column reduction" to get that

$$\det \begin{pmatrix} I_m & A \\ -B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m + AB & A \\ -B + I_nB & I_n \end{pmatrix} = \det \begin{pmatrix} I_m + AB & A \\ 0 & I_n \end{pmatrix} = \det(I_m + AB).$$

□

Lemma 2 (Complementary submatrices of a unitary matrix have same determinant)

Let

$$U := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an $(m+n) \times (m+n)$ unitary matrix, where A is an $m \times m$ square matrix and D is an $n \times n$ square matrix. Then $|\det A| = |\det D|$.

Proof. We have

$$U^T U = \begin{pmatrix} A^T A + C^T C & A^T B + C^T D \\ B^T A + D^T C & B^T B + D^T D \end{pmatrix}$$

and

$$U U^T = \begin{pmatrix} AA^T + BB^T & AC^T + BD^T \\ CA^T + DB^T & CC^T + DD^T \end{pmatrix}.$$

Since U is unitary, both of the above products must be the $(m+n) \times (m+n)$ identity matrix. In particular, it follows that $B^T B + D^T D = I_n$ and $AA^T + BB^T = I_m$. Now, by Sylvester's Determinant Identity,

$$|\det A|^2 = \det(AA^T) = \det(I_m - BB^T) = \det(I_n - B^T B) = \det(D^T D) = |\det D|^2.$$

□

It turns out that this lemma implies the theorem.

Proof. Without loss of generality, we may let 0 be a vertex of C. Then there are $m + n$ edges protruding from 0, which we may view as a set of $m + n$ orthonormal vectors $v_1, v_2, \cdots, v_{m+n} \in \mathbb{R}^{m+n}$.

Let $U = [v_1\ v_2\ \cdots\ v_{m+n}]$. That is, U is the matrix whose ith column is v_i. Then U is a unitary matrix.

There are $\binom{m+n}{m}$ m-dimensional faces of C that include 0, since each such face corresponds to the "span" of a selection of m of the vectors $v_1, \cdots, v_{m+n}$.

It happens to be the case that the projections unto $\mathbb{R}^m$ of all $\binom{m+n}{m}$ such m-dimensional faces will partition the projection of C unto $\mathbb{R}^m$. We defer the work of reasoning this out to the reader (who hopefully can think in $n+m$ dimensions...). What's important is that due to this, the m-dimensional measure of the projection of C unto $\mathbb{R}^m$ is the sum of the measures of the projections of the $\binom{m+n}{m}$ m-dimensional faces. Analogously, the n-dimensional measure of the projection of C unto $\mathbb{R}^n$ is the sum of the measures of the projections of the $\binom{m+n}{n}$ n-dimensional faces.

To prove this, we use an approach that mirrors the solution to the original problem! Instead of pairing a 2D face with a 1D edge of equal measure after projection, we can pair an m-dimensional face with an n-dimensional face of equal measure after projection.

We pair these faces in the obvious way: Let $\{v_i : i \in S\}$ be a selection of m vectors that determine an m-dimensional face F of C, for a subset $S \subseteq \{1, 2, \cdots, m+n\}$ of m indices. Then we may pair this with the n-dimensional face F' spanned by the unused n vectors, i.e. $\{v_i : i \notin S\}$. We claim that the projection of F unto $\mathbb{R}^m$ has the same measure as the projection of F' unto $\mathbb{R}^n$.

The projection of F unto $\mathbb{R}^m$ is the m-dimensional parallelepiped formed by the m vectors in $\{v_i : i \in S\}$, but with their last n components removed. That is, the m vectors that determine this parallelepiped are given by $\{(v_{i,1}, v_{i,2}, \cdots, v_{i,m}) : i \in S\}$. Naturally, these vectors form an $m \times m$ matrix A whose determinant $|\det A|$ is the m-dimensional measure of the parallelepiped. Moreover, A is the submatrix of U whose columns' indices are given by S and whose rows are the first m rows.

Similarly, the projection of F' unto $\mathbb{R}^n$ is the n-dimensional parallelepiped formed by the n vectors $\{v_i : i \notin S\}$, but with their *first* m components removed. With these components erased, these n vectors form an $n \times n$ matrix D whose determinant $|\det D|$ is the n-dimensional measuure of the parallelpiped.

Crucially, A and D are *complementary* square submatrices of U in the sense that the rows and columns that they extract are all distinct. It follows by Lemma 2 that $|\det A| = |\det D|$ (rearranging rows and columns as necessary so that it may be applied), and this concludes the proof. $\square$

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Solution 46

Suppose the center of the circle is $(0,0)$. Let the radius of the circle be r. Then the equation of the parabola is given by $y = x^2 + r$. Moreover, the line $y = \sqrt{3}x$ must be tangent to the parabola. It follows that the quadratic

$$\sqrt{3}x = x^2 + r$$

must have exactly one root, and hence must be a perfect square trinomial. This occurs exactly when $r = \left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{\frac{3}{4}}$.

■

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Solution 47

On each square of an $m \times n$ board, place a biased coin that comes up heads with probability x. Flip all the coins.

Let A be the event that in each of the n columns, there exists a tails. Let B be the event that in each of the m rows, there exists a heads. Note that $\mathbb{P}(A) = (1 - x^m)^n$, and $\mathbb{P}(B) = (1 - (1 - x)^n)^m$. Moreover, A and B encompass the entire probability space! That is, no matter how the coins flip, either A happens or B happens or both. Thus

$$\mathbb{P}(A) + \mathbb{P}(B) \geq \mathbb{P}(A \cup B) = \mathbb{P}(\Omega) = 1,$$

which is the desired inequality.

■

Source: I saw this problem in the Czech Republic Math Olympiad, but it may be older.

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Solution 48

For simplicity, let us replace "100" with "1", so that the red block's temperature begins at 1 degree and the blue block's temperature begins at 0 degrees. We'll show that the blue block's temperature can get arbitrarily close to 1.

Represent the blocks' temperatures as an ordered pair (Red Temperature, Blue Temperature), so that at the start their temperatures are $(1, 0)$. We first claim that we can go from $(1, 0)$ to $(3/8, 5/8)$. To see this, split each block into halves, so that the temperatures are

$$\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix},$$

with the red blocks on the left. Hit the top blocks together to get

$$\begin{matrix} 1/2 & 1/2 \\ 1 & 0 \end{matrix},$$

and now hit each block with the block diagonally opposite them to get

$$\begin{matrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{matrix}.$$

Lastly, hit the bottom two blocks together to get

$$\begin{matrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{matrix}.$$

Merging back, we indeed end up with $(3/8, 5/8)$.

In general, suppose we have found a way to go from $(1, 0)$ to $(1 - x, x)$. Then by scaling the temperatures up, we can go from

$$(a - b, 0) \to \big((1 - x)(a - b), x(a - b)\big),$$

and by adding a temperature of b to all blocks, we see that we can go from

$$(a, b) \to \big((1 - x)a + xb, xa + (1 - x)b\big).$$

Call this the X-procedure. Let's retry the previous procedure we came up with by using the X-procedure in place of hitting two blocks together normally.

$$\begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}$$

Apply the X-procedure on the top blocks.

$$\begin{array}{cc} 1-x & x \\ 1 & 0 \end{array}$$

Apply the X-procedure between the top-left and bottom-right.

$$\begin{array}{cc} (1-x)^2 & x \\ 1 & x(1-x) \end{array}$$

Apply the X-procedure between the bottom-left and top-right.

$$\begin{array}{cc} (1-x)^2 & 2x-x^2 \\ 1-x+x^2 & x(1-x) \end{array}$$

Apply the X-procedure between the bottom blocks.

$$\begin{array}{cc} (1-x)^2 & 2x-x^2 \\ \text{something awful} & 2x-3x^2+2x^3 \end{array}$$

Bringing the blocks together again, the temperature of the blue block is then $2x-2x^2+x^3$. Let this expression be $f(x)$.

We have shown that, if we can execute the heat transfer $(1,0) \to (1-x, x)$, then we can execute the heat transfer $(1,0) \to (1-f(x), f(x))$. Inductively, it follows that we can execute the heat transfer $(1,0) \to (1-f^{(n)}(x), f^{(n)}(x))$ for all n, where $f^{(n)}$ is f composed with itself n times. It remains to prove that

$$\lim_{n\to\infty} f^{(n)}(1/2) \stackrel{?}{=} 1.$$

First we show that the limit exists. Indeed, by writing $f(x) = x + x(1-x)^2$, we see that $f(x) \geq x$ for all $x \in [0,1]$, which entails that the sequence $\{f^{(n)}(1/2)\}_n$ must be increasing. So the desired limit exists and is some real number L. Now, write

$$f^{(n+1)}(1/2) = f(f^{(n)}(1/2)) = 2f^{(n)}(1/2) - 2f^{(n)}(1/2)^2 + f^{(n)}(1/2)^3.$$

Sending $n \to \infty$ gives

$$L = 2L - 2L^2 + L^3,$$

so either $L = 0$ or $L = 1$. The case $L = 0$ is quite obviously deranged, so the limit is $L = 1$, concluding the proof.

■

Source: Puzzling Stack Exchange

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Solution 49

We are given that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}. \qquad (*)$$

Multiply each side by 1/4, we can get

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{24}. \qquad (**)$$

By subtracting (∗∗) from (∗), we conclude that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\boxed{\frac{\pi^2}{8}}.$$

■

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Solution 50

As suggested by the hint, Ari first flips a fair coin to decide which card she chooses. The remainder of her strategy is as follows: If the real number on that card is t, then Ari will guess "Higher!" with a probability $P(t)$ that we will define later.

Let us show that this works. Suppose that the two cards chosen by Beth are x and y, with $x > y$. Then the probability that Ari wins under this strategy is

$$\frac{1}{2} \cdot P(x) + \frac{1}{2} \cdot (1 - P(y)).$$

If you work out the algebra, this probability is strictly greater than $\frac{1}{2}$ exactly when $P(x) > P(y)$. So Ari's scheme reduces to the following problem: Find a function $P : \mathbb{R} \to [0,1]$ such that whenever $x < y$, we have $P(x) < P(y)$, i.e. P is strictly increasing.

This is clearly possible! For example, Ari can choose

$$P(x) := \frac{1}{2} + \frac{\arctan(x)}{\pi}.$$

■

Source: Classic?

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Solution 51

Claim: The hydra can be moved only to rooms that are multiples of 6.

Proof. For each integer n, we assign a weight w_n to room n as follows:

$$w_n := \begin{cases} 1, & n \equiv 0 \pmod 6 \\ \sqrt{2}, & n \equiv 1 \pmod 6 \\ -1+\sqrt{2}, & n \equiv 2 \pmod 6 \\ -1, & n \equiv 3 \pmod 6 \\ -\sqrt{2}, & n \equiv 4 \pmod 6 \\ 1-\sqrt{2}, & n \equiv 5 \pmod 6 \end{cases}$$

Observe that $w_{n-1} + w_{n+1} = w_n$ for all integers n. Thus the sum of the weights of the rooms occupied by the hydra heads (counting multiplicity) does not change with every move you make. Since we start with one hydra head in room 0, the starting weight sum is 1, so at the end, if there are only $h > 0$ hydra heads left in some room n (and nowhere else), then $h \cdot w_n = 1$. By our choice of weights, this is only possible if n is a multiple of 6 and $h = 1$.

Claim: All multiples of 6 may be achieved.

Proof. The proof is by painful example. Here is one (half of a) procedure that works for moving the hydra from 0 to 6.

-1	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0
1	1	1	0	1	0	0	0	0
0	2	0	0	1	0	0	0	0
0	2	0	1	0	1	0	0	0
0	2	0	1	1	0	1	0	0
0	2	1	0	2	0	1	0	0
0	2	1	1	1	1	1	0	0
0	2	0	2	0	1	1	0	0
0	1	1	1	0	1	1	0	0
0	0	2	0	0	1	1	0	0
0	0	2	0	1	0	2	0	0

The last row is symmetrical with respect to the midpoint between 0 and 6, so by symmetrically reversing the steps we have done, we will end up with a single hydra head in room

6. Inductively, this implies that all multiples of 6 can be reached.

■

Remarks: Here are other interesting approaches that work:

- One could have used the weights $w_n := \omega^n$, where ω is a primitive 6th root of unity. Then the sum of all weights is always equal to 1, due to the miraculous identity

  $$\omega^{n-1} + \omega^{n+1} = \omega^n,$$

  which can be easily verified "visually". Of course, this entails that the hydra can only remain in room numbers that are multiples of 6.

- For any particular moment in time, let a_n be the number of hydras in room n. Then the distribution of hydras may be represented as a "polynomial"

  $$P(x) = \sum_{n\in\mathbb{Z}} a_n x^n.$$

  At the start, we have $P(x) = 1$. Then, every move adds or subtracts a multiple of $x^2 - x + 1$. Thus $x^2 - x + 1$ divides $P(x) - 1$ (or rather, $x^2 - x + 1$ divides the *numerator* of $P(x) - 1$ when expressed as an irreducible rational function). It follows that $P(e^{i\pi/3}) = 1$ at all times.

  If all hydra heads end up in some room k, then P will take the form $P(x) = a_k x^k$. So $a_k e^{ki\pi/3} = 1$. This is possible only when $a_k = 1$ and k is a multiple of 6.

Source: Andreas Blass, Seven Trees in One

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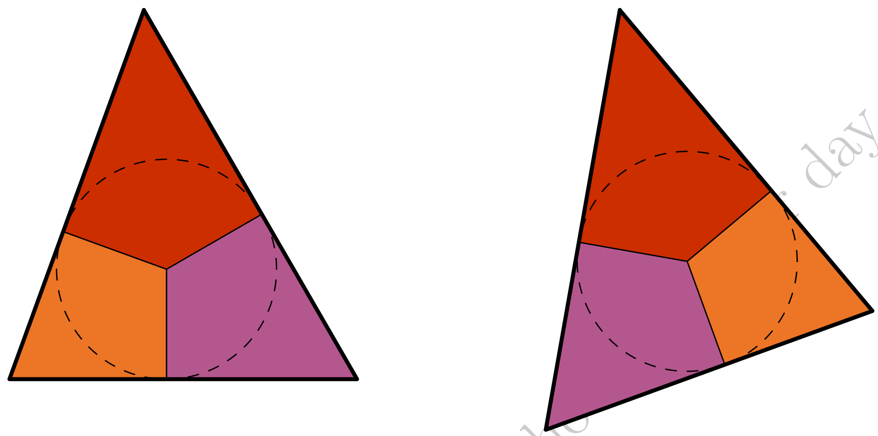
Solution 52

Cut T along the inradii. “Swapping” the two orange and purple pieces below will execute the desired reflection.



■

Remarks: Cutting T along the circumradii does not work if T is obtuse! Unfortunately, this approach cannot be salvaged as far as I'm aware.

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Solution 53

It turns out that such a pair of dice exists. In fact, there is only one such pair of dice!

A 6-sided die whose sides are a, b, c, d, e, f may be represented as the polynomial $x^a + x^b + x^c + x^d + x^e + x^f$, in the sense that the coefficient of x^n is the number of ways that n may be achieved. This sort of property is preserved if we consider multiple dice by multiplying the corresponding polynomials!

For instance, the standard 6-sided die is represented as $x + x^2 + x^3 + x^4 + x^5 + x^6$. Thus, the possible outcomes for rolling two standard 6-sided dice is represented by the polynomial $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$. Our goal is to write this polynomial as the product of two other polynomials $P(x)$ and $Q(x)$, each with positive coefficients summing to 6 (and with no constant term since we want positive integer sides for the dice). Manipulating:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = \frac{x^2(x^6 - 1)^2}{(x-1)^2}$$

$$= \frac{x^2(x-1)^2(x^2+x+1)^2(x+1)^2(x^2-x-1)^2}{(x-1)^2}$$

$$= x^2(x^2+x+1)^2(x+1)^2(x^2-x+1)^2$$

To ensure that P and Q have no constant term, they should each get a factor of x. Next, if we plug in 1 into the above expression, we get something that looks like $(1^2)(3^2)(2^2)(1^2)$. This tells us that each of P and Q need to take a factor of $x^2 + x + 1$ and a factor of $x + 1$ in order to have a coefficient sum of 6. It remains to donate the two $(x^2 - x + 1)$ factors. But if we divide those evenly then we just end up with the original dice. So the *only* reasonable distribution of the factors is as follows:

$$P(x) = x(x+1)(x^2+x+1)$$

$$Q(x) = x(x+1)(x^2+x+1)(x^2-x+1)^2$$

Expanding using Mathematica or something, this gives the generating functions:

$$x^4 + x^3 + x^3 + x^2 + x^2 + x^1$$

$$x^8 + x^6 + x^5 + x^4 + x^3 + x^1$$

Magically, these satisfy the conditions we need! Hence there indeed (uniquely) exists another such pair of dice, and their sides are $1, 2, 2, 3, 3, 4$ and $1, 3, 4, 5, 6, 8$.

■

Source: This pair of dice is known as the Sichermann Dice.

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Solution 54

If we imagine walking along the sides of the tridecagon, then our movement along each side may be represented as a vector, and these 13 vectors sum to 0. We may enforce that the angles of the tridecagon are multiples of 20 degrees by taking the vectors to be 18th roots of unity.

To ensure convexity, the order in which we arrange the roots of unity to form the tridecagon must be by increasing or decreasing argument. Moreover, we cannot use a root of unity twice, otherwise this scheme would force the two repetitions of the root of unity to be adjacent, and hence forming a single side rather than two.

So, the problem reduces to finding a 13 distinct 18th roots of unity that sum to 0. This further reduces to just finding 5 such roots of unity that sum to 0.

The claim is that if we can find 5 such roots of unity, then two of them sum to 0 and the other three also sum to 0 (and thus form an equilateral triangle in the complex plane). By inspection, this would indeed imply that our desired tridecagon is unique up to similarity.

Let ζ be a primitive 18th root of unity, and suppose that $\zeta^{e_1} + \zeta^{e_2} + \zeta^{e_3} + \zeta^{e_4} + \zeta^{e_5} = 0$ for distinct integer powers $0 \le e_i < 18$. If we assume that $e_5 = 0$, then it suffices to prove that either $9 \in \{e_1, e_2, e_3, e_4\}$ or $6, 12 \in \{e_1, e_2, e_3, e_4\}$ (why?).

Let $P(x) = x^{e_1} + x^{e_2} + x^{e_3} + x^{e_4} + 1$. Then $P \in \mathbb{Q}[x]$ and ζ is a root of P, so the cyclotomic polynomial $\Phi_{18} = x^6 - x^3 + 1$ divides P. Let $P/\Phi_{18} = Q$.

Let $Q = Q_0 + Q_1 + Q_2$, where Q_i is the polynomial formed by the terms of Q whose exponents are congruent to i mod 3. Similarly let $P = P_0 + P_1 + P_2$. Then

$$\Phi_{18}Q_0 + \Phi_{18}Q_1 + \Phi_{18}Q_2 = P_0 + P_1 + P_2.$$

The terms on the LHS whose exponents are multiples of 3 are precisely those terms in the polynomial $\Phi_{18}Q_0$ because Φ_{18}'s terms all have degrees that are multiples of 3. Thus $\Phi_{18}Q_0 = P_0$. Similarly, $\Phi_{18}Q_1 = P_1$ and $\Phi_{18}Q_2 = P_2$.

This tells us that $P_0(\zeta) = P_1(\zeta) = P_2(\zeta) = 0$. Translating back to English, what this means is that among the 5 roots of unity we've chosen to sum to 0,

- the ones that are of the form ζ^{3k} sum to 0,
- the ones that are of the form ζ^{3k+1} sum to 0, and
- the ones that are of the form ζ^{3k+2} sum to 0.

The five roots of unity are divided among these three "classes" of roots.

To finish, note that none of these classes may have all five of the chosen roots, because then these roots are five vertices of a regular hexagon, and hence have no hope of summing to 0. And, trivially, none of these classes may have exactly one of the roots. It follows that the only possible distribution for the roots is "0, 2, 3", in some order. If you think hard, this is exactly what we wanted to show.

■

A remark from "tenth": If you want to show that $x^6 - x^3 + 1$ is irreducible for the sake of lowering the amount of "technology" used in this proof, then here you go!

The observation to make is that, over the field $\mathbb{F}_3$, we amusingly have that $x^6 - x^3 + 1 = (x+1)^6$. This motivates looking at the shifted polynomial $(x-1)^6 - (x-1)^3 + 1$. If this is irreducible, then $x^6 - x^3 + 1$ must be too.

By the observation, all coefficients of this shifted polynomial are divisible by 3 (sans the leading coefficient). Also it is not hard to see that the constant term is not divisible by 9. Thus, this polynomial is irreducible by Eisenstein's Criterion. Voilà!

Source: Math Prize for Girls

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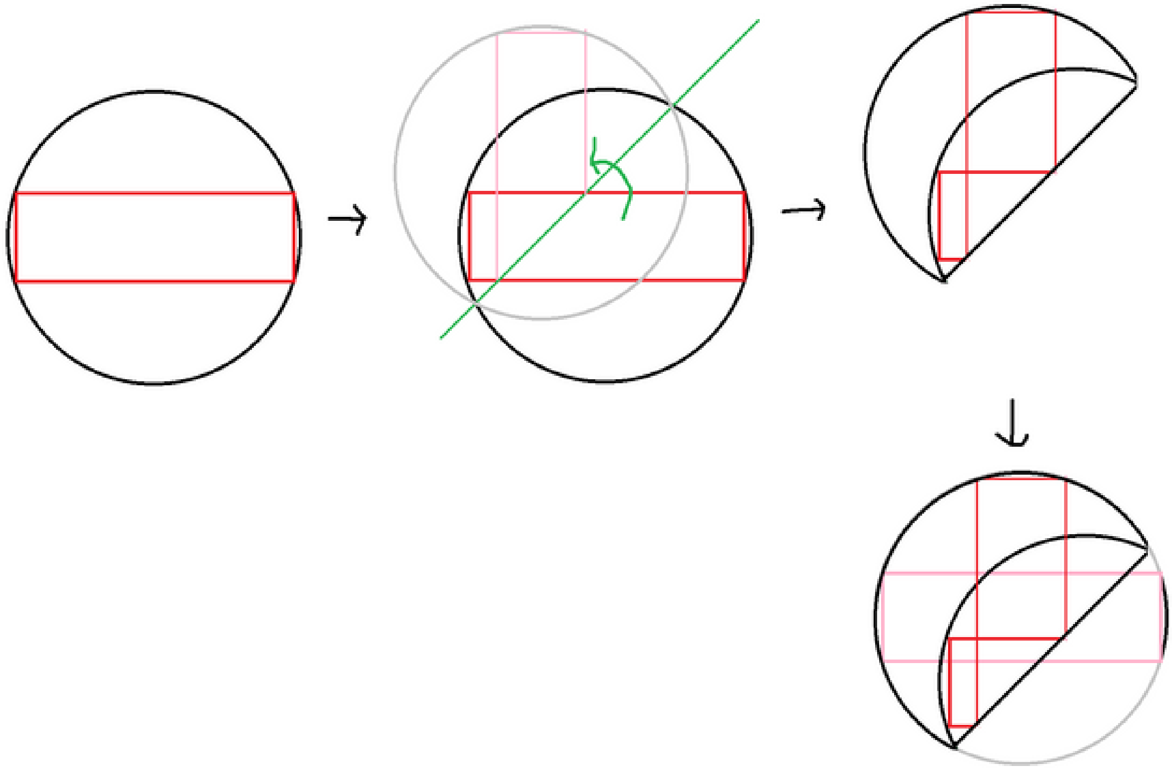
Solution 55

Get a paper plate of the same radius as the original plate. When you fold the rectangular paper, **fold the paper plate along the same crease**!

The folded rectangular paper clearly still fits on the folded paper plate. Moreover, the folded paper plate must fit on the original plate. By "transitivity of fitting", the folded rectangular paper still fits on the original plate.



■

Remarks: By the exact same reasoning, the problem still holds when the paper is non-rectangular!

Source: Art of Problem Solving Forums

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Solution 56

Here is a clever approach that minimizes computation. The expected fraction is given by the integral $\int_0^1 x \left\lfloor \frac{1}{x} \right\rfloor dx$. To evaluate this slickly, we use the crazy identity

$$x \left\lfloor \frac{1}{x} \right\rfloor = \sum_{n=1}^{\infty} x \cdot \mathbf{1}_{\left(0, \frac{1}{n}\right]}(x).$$

This is because of the motto "Baka takes a bite, and if $x \leq 1/2$ then Baka takes another bite, and if $x \leq 1/3$ then Baka takes another bite, ..." and so on.

Now integrate and apply monotone convergence:

$$\begin{aligned}
\int_0^1 x \left\lfloor \frac{1}{x} \right\rfloor dx &= \int_0^1 \sum_{n=1}^{\infty} x \cdot \mathbf{1}_{\left(0, \frac{1}{n}\right]}(x)\, dx \\
&= \sum_{n=1}^{\infty} \int_0^1 x \cdot \mathbf{1}_{\left(0, \frac{1}{n}\right]}(x)\, dx \\
&= \sum_{n=1}^{\infty} \int_0^{1/n} x\, dx \\
&= \sum_{n=1}^{\infty} \frac{1}{2n^2} = \boxed{\frac{\pi^2}{12}}
\end{aligned}$$

■

Remarks: What a crazy answer from a numberless problem!

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Solution 57

The plan is "simple": If n ships have been bombed so far, then the Queen should board the ship with probability p_n that we shall choose later. We will construct the sequence $\{p_n\}_{n=0}^{\infty}$ so as to ensure that the probability of survival is at least $1-\varepsilon$.

Seeing this plan, suppose that the Insurrection has b bombs and chooses to bomb ships $x_1 < x_2 < \ldots < x_b \in \mathbb{N}$. The probability that this kills the Queen is given by:

$$\begin{aligned}&(1-p_0)^{x_1-1}p_0 + (1-p_0)^{x_1}(1-p_1)^{x_2-x_1-1}p_1\\ &+(1-p_0)^{x_1}(1-p_1)^{x_2-x_1}(1-p_2)^{x_3-x_2-1}p_2 + \ldots\\ &\leq p_0 + p_1 + p_2 + \ldots\end{aligned}$$

Taking $p_0 = \varepsilon^2, p_1 = \varepsilon^3, p_2 = \varepsilon^4, \cdots$ we get that

$$\mathbb{P}(\text{Queen Dies}) \leq \frac{\varepsilon^2}{1-\varepsilon} < \varepsilon,$$

so indeed the Queen gets out alive with probability at least $1-\varepsilon$.

■

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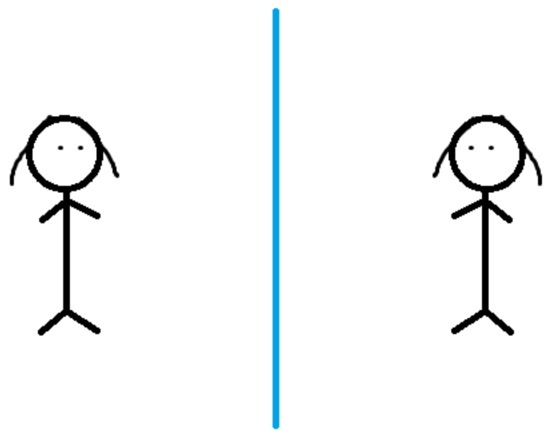
Solution 58

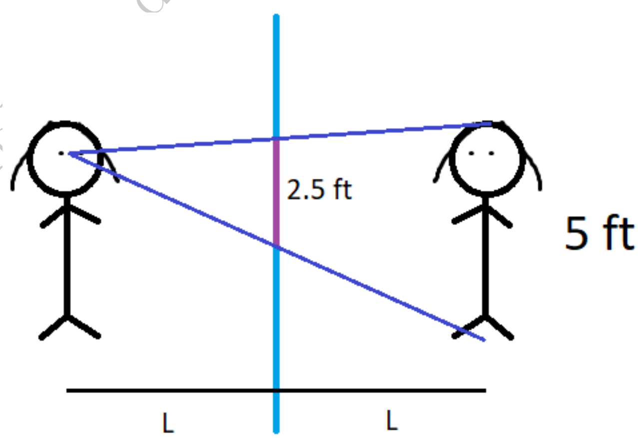
Suppose the mirror were a distance L away from Cherie. (We will see that L does not matter!)

Let the mirror have infinite height. The minimum height that the mirror *could* be is given by the height spanned by Cherie's perceived image in the mirror.

Now replace the mirror with glass and place a doppelganger of Cherie across the glass at distance L.



By similar triangles, we can compute the height spanned by Cherie's image to be exactly half her real height.



■

Source: Me

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Solution 59

Note that a parabola cannot cover all but a finite length of a given ray unless the ray is parallel to the axis of symmetry (This follows from a rate of growth argument; x^2 has higher order of growth than any linear function $ax + b$.).

Find an infinite set of rays, none of which are parallel to each other. A finite covering of the plane must cover these rays.

For each ray there must exist a parabola whose axis of symmetry is parallel to it. Otherwise every parabola will only intersect at most a finite length of the ray, which is bad because rays are infinite and we only have a finite number of parabolas.

But no two rays are parallel, so no parabola can have axis of symmetry parallel to more than one ray. It follows that there are at least as many parabolas as rays. Contradiction, because there are infinitely many rays.

■

Source: VJIMC

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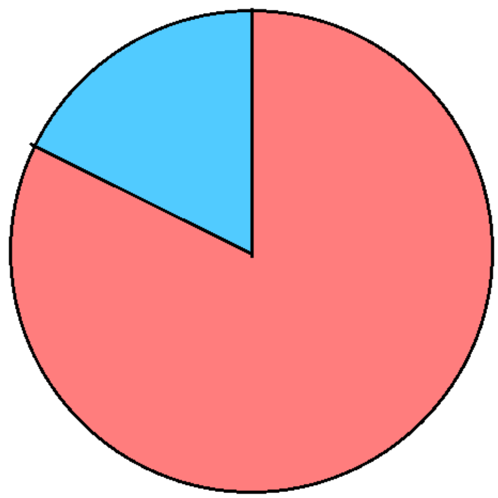
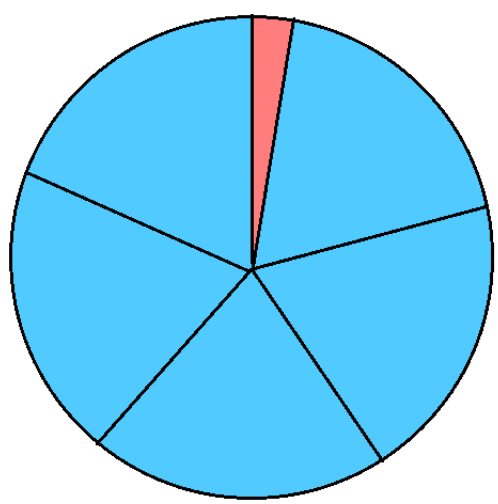
Solution 60

This solution is a highly visual one. Let us start by discovering what precisely is going on behind the scenes.

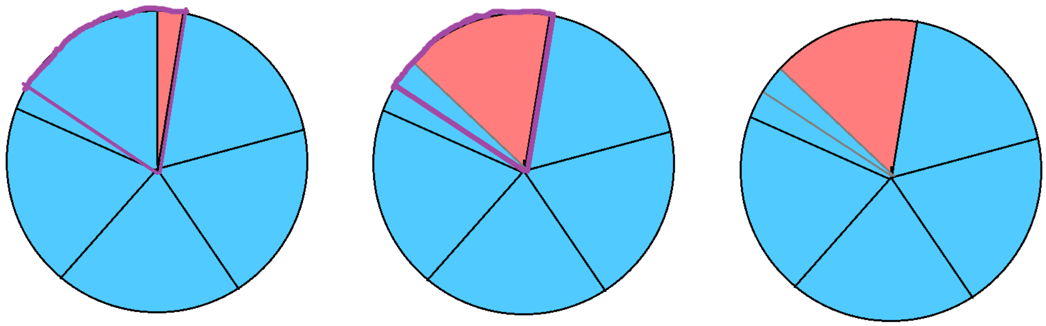
Let's start with one flip.



Now, let us keep flipping until the next piece we flip overlaps with the first piece that we flipped.



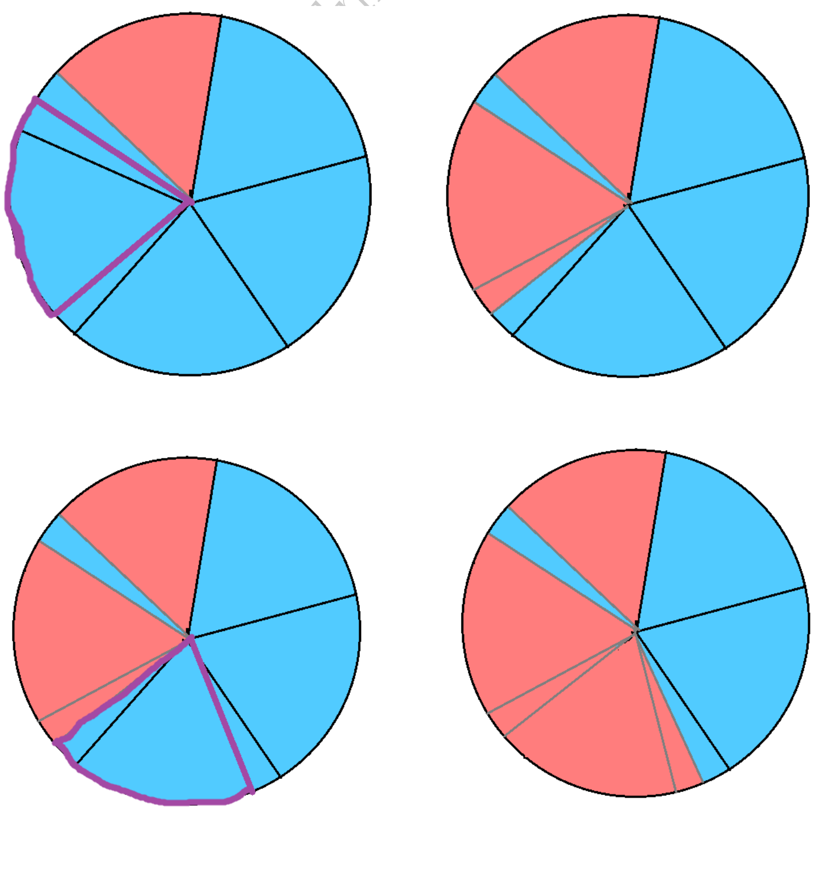
From this point forward, we need to be more careful. As we keep going, any black/gray line represents a cut in the cake as it would in real life. Without further ado, let us do another flip, but slowly!

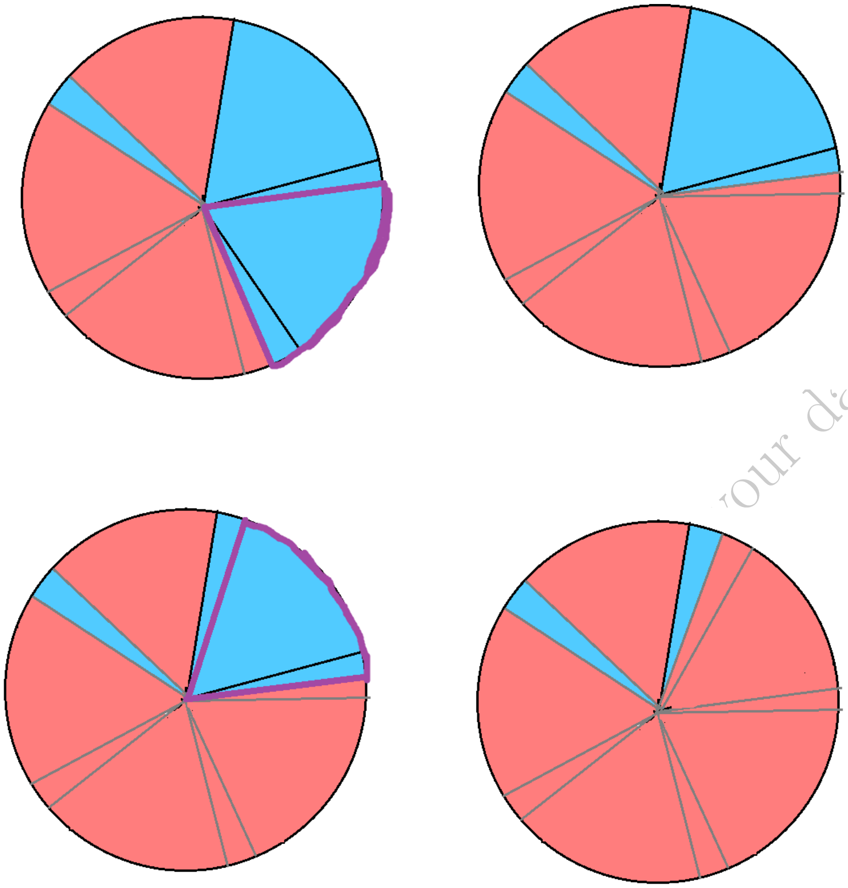


Left: I've outlined in purple the slice that we're about to flip.
Center: I've executed the flip.
Right: The outline is removed.

Notice that we not only inverted the colors. We also had to reflect the piece! That's what it means to flip a slice of cake. Consequently, **the very first cut we've ever made has moved** as a consequence of this flip.

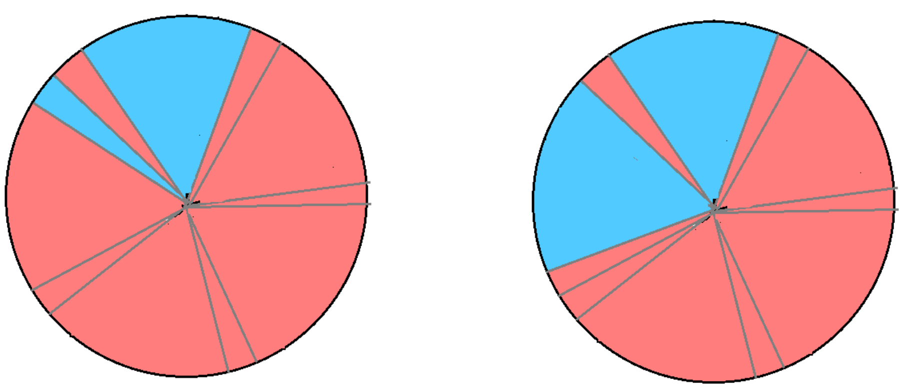
Let's keep doing this for a bit so you can see the pattern.

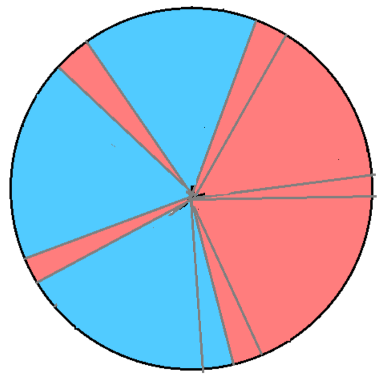




We have arrived at yet another critical point. What happens now?

Here's the big revelation: After this point, **no additional cuts in the cake are ever made**! We're now just taking existing pieces (two at a time) and flipping them over. See for yourself!



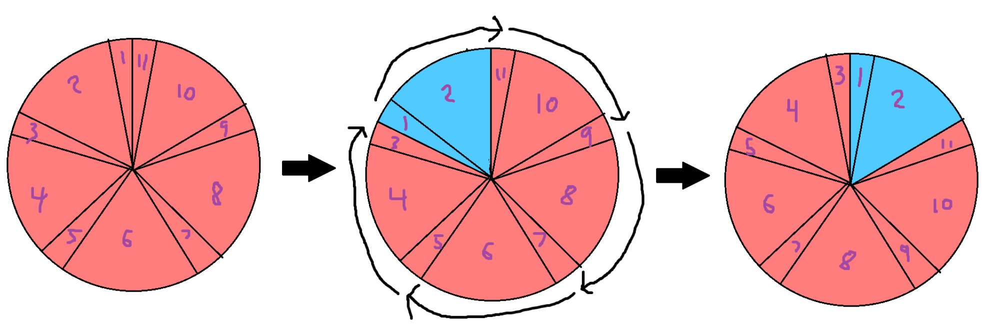


At last, we can piece together the story of what's really going on: There are always two "small pieces" together, and each flip "moves" one of those small pieces to be next to the next small piece, via flipping over that small piece with the adjacent "big piece".

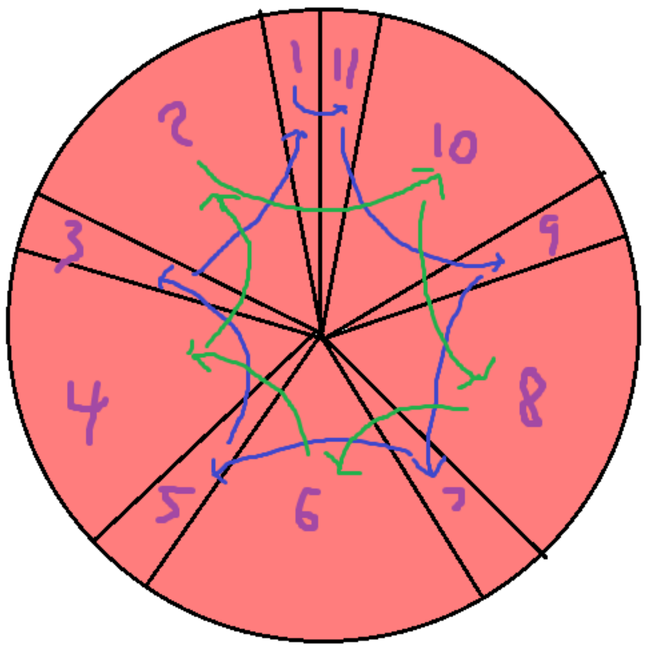
With this insight in hand, we may now proceed to form a proof.

Let's start from the beginning, with all the cuts we're ever going to make already filled in. I'll also number the slices.

Instead of doing a flip, we are going to do a flip and a *rotation*, such that the two small slices are always at the top of the cake.



This sequence of moves executes a permutation on the slices! Since a permutation on a finite set must have a finite order, we see that by repeating this permutation over and over again, we eventually must have all the pieces end up where they started.



But are the orientations (i.e. the colors) of the pieces correct? They might not be. If so, then the long sequence of moves thus far simply executes an algorithm that flips some of the slices. Hence, by executing repeating all those moves just once more, we will flip those same slices, thereby ending with a cake whose top has returned to its original color.

■

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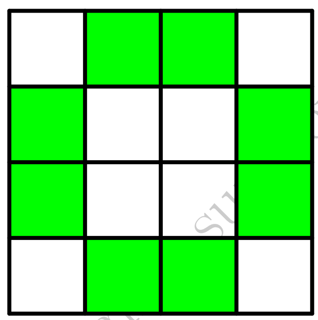
Solution 61

Suppose that every initial configuration of lights may be solved. Then there exists a function f from the set of initial configurations (which has size 2^{16}) to the set of possible solutions (which, as in Hint 1, has size 2^{16}) for which $f(x)$ is a solution to the initial configuration x.

But f must be an injection, because the same solution cannot solve two different boards. We deduce that actually f is a bijection. This implies that no initial board can be solved in two different ways.

We arrive at a contradiction by looking at the all-lights-off board, in which one solution is to do nothing, and another solution is to press the following lights:



■

Remarks: There is a much more remarkable fact about the 4×4 standard Lights Out puzzle.

Theorem 1 (Chasing Lights)

Suppose that a given 4×4 Lights Out puzzle has a solution. Then the puzzle may be solved by the following naive process:

1. Start with the first row.

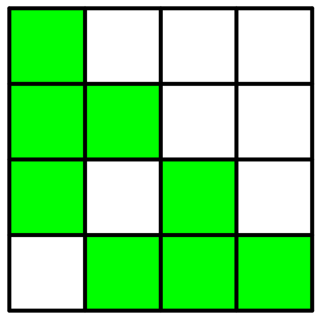
2. For each light that is on in this row, press the light below it.

3. Repeat Step 1 for the next row.

There is a natural proof that involves some linear algebra. Here is a more elementary one.

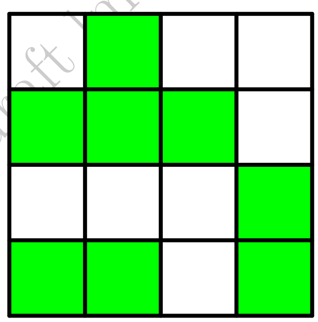
Proof. Take such a solvable puzzle. It suffices to show that it has a solution that does not involve pressing any lights in the top row. This is because if no lights are to be pressed in the first row, then the lights that are pressed in the second row of the solution must precisely be those lights that are under a light that is on in the first row. After pressing those lights, the solution no longer involves pressing any lights in the second row, and we repeat this reasoning for each subsequent row.

Take a solution. If the solution involves pressing the first light in the first row, then add the following light-presses to the solution:



We can do this because pressing these lights does not ultimately change the state of any of the lights.

Likewise, if the solution involves pressing the second light in the first row, then add the following light presses to the solution:



The cases in which the solution involves pressing the third or fourth light in the first row are handled symmetrically. Hence we can obtain a solution that does not press any light in the first row. □

The theory shown here is merely the tip of the iceberg. For more, I highly recommend the article *Two Reflected Analyses of Lights Out* by Óscar Martín and Cristóbal Pareja-Flores.

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Solution 62

There are two cases to consider: When n is odd, and when n is even.

Case 1: n is odd

We proceed via a sort of "induction". Clearly, if all lights are off at the start, then the puzzle is solved in an easy run by virtue of being already solved. This is the "base case". Now for the "inductive step", we show that if some solvable puzzle is solved in an easy run, then this property is preserved after pressing any light. Since every solvable puzzle can be obtained by starting from an empty board and pressing a finite sequence of lights, this would solve the problem.

We say that two distinct lights are *neighboring* or *neighbors* if they are in the same row or the same column.

Consider a puzzle that is solved in one easy run. Then all its lights are turned off, so every light has an even number of neighbors that are on (regardless of whether that light is on or off!). Now press a light L. We need only show that all lights still satisfy this property.

Take a light K distinct from L.

- If K is not a neighbor of L, then exactly two neighbors of K are toggled, so its number of neighbors that are on has changed by either -2, 0, or 2. Hence the number of such neighbors remains even.
- If K is a neighbor of L, then $n-1$ of its neighbors are toggled. Since $n-1$ is even, the parity of the number of neighbors that are on must remain even.

As for L itself, it has $2n-2$ neighbors, all of which are toggled after pressing L, and $2n-2$ is evidently even.

Case 2: n is even

There are multiple approaches to this case. Here, I present the one that I believe to be the most interesting.

The first observation is that every initial configuration of lights constitutes a solvable puzzle. This is because we can toggle any individual light L by pressing L and all neighbors of L. By reasoning as in the previous problem, it follows that every puzzle has a unique solution.

Incidentally, the pattern of lights that need to be pressed to toggle a light L is the same pattern of lights that are toggled when L is pressed. This motivates an interesting notion of *duality*: For a board of lights X, let X' be the board of lights where a light is on iff it is one of the lights pressed in the solution to X. Then X and X' have the same effect on each other: Toggling a light on X' corresponds to pressing the corresponding light on X, and toggling a light on X corresponds to pressing the corresponding light on X'!

This symmetry between the board X and its *dual* board, X', presents several miracles:

- By definition, the on-lights in X' constitute the solution to X. But likewise, the on-lights in X constitute the solution to X'.
- *Duality is an involution:* We have that $(X')' = X$. This follows by the previous bullet and the uniqueness of solutions.
- *Duality is linear:* For boards X and Y, we have that $X' + Y' = (X + Y)'$. Here, addition is done in the sense that "On" is 1 and "Off" is 0, over the field $\mathbb{F}_2$ (so that $1 + 1 = 0$).

Now let us exploit this symmetry for an elegant proof. For a board X, let $E(X)$ be the board obtained after an easy run. We need to show that $E(E(X)) = 0$, where 0 is the all-off board. But a board is solved in one easy run exactly when it is equal to its dual (i.e. $E(Y) = 0 \iff Y = Y'$), so it suffices to show that $E(X) = E(X)'$.

Intuitively speaking, the argument is as follows: An easy run on X simply changes the solution X' by adding X to it, and by duality we can argue that it also changes the original board by adding X' to it. More precisely, we first observe that

$$E(X)' = X' + X. \qquad (*)$$

That is, the solution to the board *after* an easy run is changed by an addition of X, which is evident because an easy run consist of pressing those lights that are on in X, and pressing each such light corresponds to a simple toggle in the dual.

Now, we take the dual of each side of $(*)$ to obtain

$$E(X) = X + X',$$

where we have applied the fact that duality is an involution and is linear. But now we are done since

$$E(X)' = X' + X = X + X' = E(X).$$

■

Remarks: This was the main result in a research project I did in 10th grade. It turns out that the result itself is more well-known than I thought it was back then, but I still quite like the ideas behind the proof.

There's more, though! Going back to the standard Lights Out rules, where pressing a light toggles only that light and its orthogonally-adjacent neighbors, we can still define the notion of an easy run. It turns out that in this case, easy runs still have the potential for solving puzzles. To be specific: In 2011, Bruce Torrence proved that if n is such that an $n \times n$ standard Lights Out puzzle can be solved for every possible initial configuration of lights, then repeatedly applying easy runs will eventually solve any puzzle.

Source: Me

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Solution 63

We proceed by induction. The case $n = 1$ is kinda easy. Now suppose that we may solve the all-on Lights Out for $n - 1$ lights.

Fix a light L. Then by the hypothesis, we may press some lights so that all lights other than L are toggled off. If this process toggles L off, then we already win. Thus we may assume that the lights we pressed ended up toggling only those $n - 1$ lights besides L.

The same logic may be applied to all other lights, so we may assume that we have the power to toggle any $n - 1$ lights of our choice.

Claim 1: We may assume that n is odd.

This is because if n is even, then we may toggle all n subsets of $n - 1$ lights. This toggles every light $n - 1$ times, which is odd, so this constitutes a solution.

Claim 2: We have the power to toggle any two lights of our choice at once.

Suppose the two lights we'd like to toggle are L and K. Toggle all lights except L, and then toggle all lights except K. Tada!

Claim 3: There is a vertex with even degree.

If all vertices have odd degree, then the sum of the degrees is odd (because n is odd by the assumption permitted by Claim 1). But the sum of the degrees is twice the number of edges by the handshake lemma, and hence must be even, contradiction.

Now we may use Claims 2 and 3 to finish. Start with all lights on.

By Claim 3, find a vertex/light L with even degree. Press it. This turns L and all neighbors of L off. L has an even number of neighboring lights, so by Claim 2, we are able to toggle all of L's neighbors back to on. This means that L is off whereas all the other $n-1$ lights are on. As we established at the beginning, we may toggle all those $n - 1$ lights off. ■

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Solution 64

Instead of ants turning around upon collision, it is equivalent to let them pass through each other.

Part (a)

Under this new framing, it is clear that 22 ants fall off the left end, whereas 20 ants all off the right end.

Part (b)

Under this new framing, every ant on the left passes through (i.e. "collides") with every ant on the right (though, this isn't true under the old framing). Thus there are $20 \times 22 = 440$ "collisions" in total.

■

Remarks: Can you figure out which ant(s) endure the most collisions?

Source: Classic

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Solution 65

Draw a line L that intersects that graph at three points A, B, C. The x-coordinates a, b, c of these points must be the roots of the cubic polynomial $x^3 - (mx + n)$, where $mx + n$ is the equation of the line L. By Vieta's Formulae, it follows that $a + b + c = 0$. Thus the center of mass of the three points A, B, and C lies on the y-axis! Construct* said center of mass.

Repeating this procedure, we now have two points on the y-axis, and so by connecting them we will have constructed the y-axis. Construction of the x-axis quickly follows.

*To construct the center of mass, one way is as follows: Since you can construct parallelograms, you can definitely construct $A + B + C$, where the "0 vector" can be anywhere you want. By scaling, you can then construct $\frac{A+B+C}{3}$, which must be the center of mass no matter where you place the "0 vector".

■

Source: Heard this from an internet acquaintance, who in turn saw this on Reddit.

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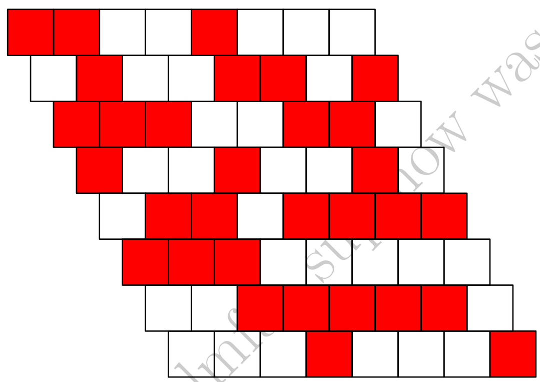
Solution 66

Invent a second king that's trying to get from the left side to the right side. The second king moves like the first king, but steps only on burning squares.

We claim that the first king can reach the top side if and only if the second king cannot reach the right side. If this is true, then we can conclude by symmetry that the probability is exactly 50%.

To prove this, we first modify the chessboard by shifting each row so that both king's movements consist of simply moving to an adjacent square.



Red: Square that is on fire

Viewing the board as such, the proof becomes quite simple.

Suppose that the first king can reach the top edge. Then his path clearly blocks the second king from reaching the right edge.

Conversely, suppose that the second kind cannot reach the right edge. Then the first king can find a path to the top edge by following the boundary of the set of all squares that can be reached by the second king.

■

Remarks: What we have done was reduce the problem to the game of *Hex*, which is played on a board of hexagons.

The *Hex Theorem* states that a draw is impossible in the game of Hex. A fascinating fact is that this theorem is equivalent to Brouwer's Fixed Point theorem!

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Solution 67

If none of the numbers are lying, then any three numbers that lie in three different rows and columns must sum to the perimeter of the outer rectangle. Note that in the following two cases, the sum of the indicated numbers is the same, and equal to 42:

14	16	12
18	14	10
16	18	14

14	16	12
18	14	10
15	18	14

If any of those green numbers were lying, then these sums wouldn't agree. So they're all telling the truth, and moreover the outer perimeter is 42.

The liar is either the 12 or one of the 18s. To narrow it down, sum these three numbers:

14	16	12
18	14	10
15	18	14

These three sum to 48. Treasonous! We know that the green 16 and green 14 are truthful, thus the 18 marked in red is the liar, and really should be 12.

■

Source: Georgia Southern Math Tournament

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Solution 68

Booster can find Mario, guaranteed.

The core mechanism behind Booster's strategy is the following claim:

Claim: If Mario is behind curtain n, and Booster opens curtains $m, m-1, m-2, ..., 1$ for some $m > n$ with the same parity as n, then Booster will find Mario.

Proof. Act out the procedure with your fingers until you are convinced. □

From this, we get the next claim.

Claim: For any positive integer N, if it is assumed that Mario is behind one of the first N curtains, then Booster has an algorithm to catch Mario.

Proof. First, Booster guesses that Mario is on an even parity. Then Booster opens every curtain from $2N$ to 1 (the choice of $2N$ is extremely sub-optimal). This would find Mario if he's indeed on an even parity. If Booster does not find Mario, then we now know that Mario was on an odd parity. In fact, he still is. More specifically, Mario must be behind an odd curtain between 1 and $N + 2N = 3N$. So if Booster opens every curtain from $4N + 1$ to 1, then he will find Mario.

If Booster still doesn't find Mario, then Booster has verified that Mario was not in the first N curtains. □

Now we construct Booster's algorithm.

0. Let $N = 1$. Let C be the number of curtains opened thus far.

1. Booster checks if Mario *started* within the first N curtains by assuming that Mario is currently within the first $N + C$ curtains, and using the claim to test this hypothesis.

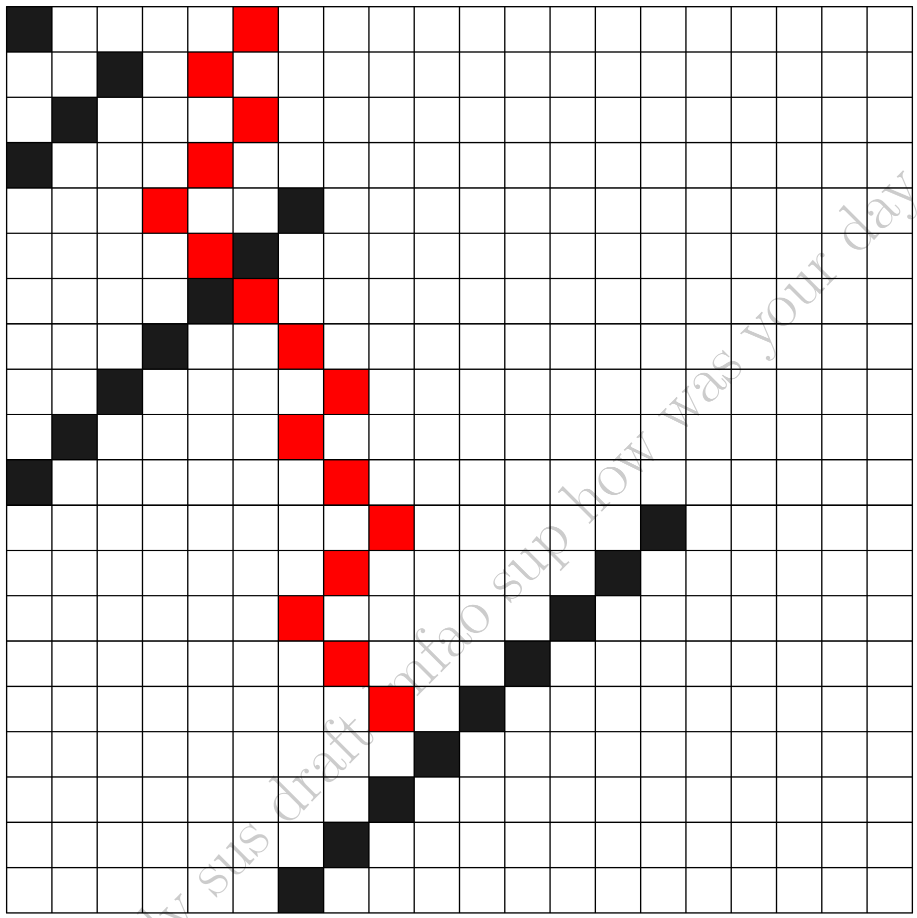
2. If Mario is not found, we increment N, update the value of C, and loop back to Step 1.

Note that Step 1 works because if Booster has opened C curtains so far, then Mario must be within the first $N + C$ curtains, assuming that Mario started within the first N curtains.

Eventually, N will equal Mario's starting curtain, and then on that loop of the algorithm, Booster will catch Mario. It just might take a very long time.

■

Remarks: As per the hint, Booster's strategy can be viewed as constructing increasingly-long diagonal "barriers" on an infinite grid. One of these diagonals must "catch" Mario, who is descending downwards diagonally.



Red: Mario's position *Black: Curtain checked by Booster*

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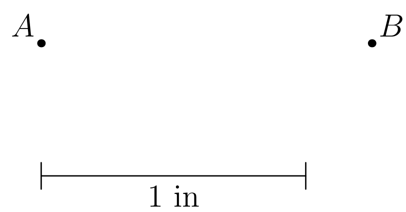
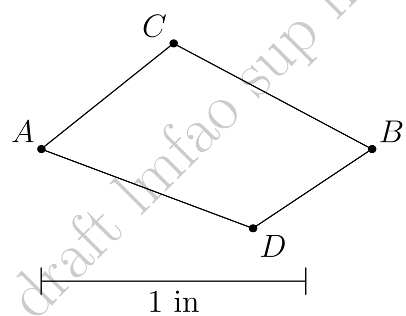
Solution 69

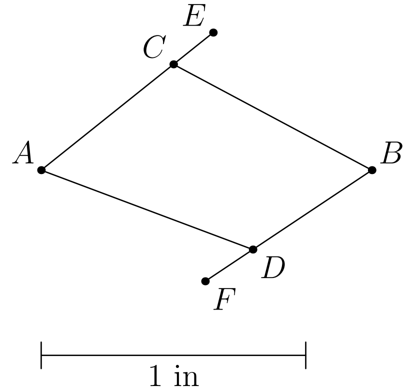
Let us instead suppose that A and B are only just out of reach — say, at most 1.5 inches apart.



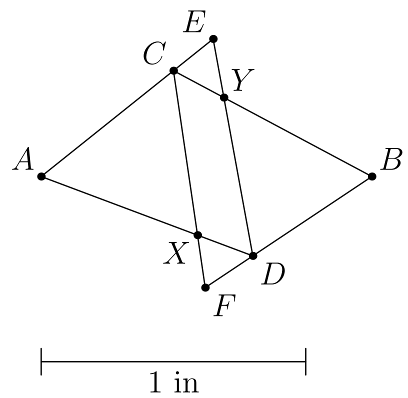
Now, since A and B are not too far apart, we may identify two points C and D that are both within one inch of both A and B. (Also, we must ensure that C and D are within 1 inch.)



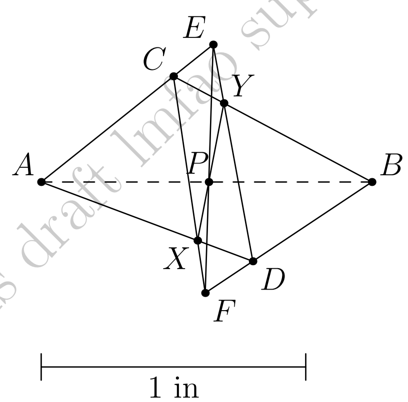
Next, we extend ray AC slightly to a point E. Likewise we extend ray BD slightly to a point F. (We must ensure that E and F are within 1 inch.)



If the steps thus far were followed sufficiently delicately, then our straightedge will be long enough to connect C with F and E with D. We label their intersections with AD and CB respectively as X and Y, respectively.



Lastly, we mark the intersection of XY and EF as P.



As suggested above, we claim that P lies on $\overline{AB}$, so that we may connect A with P and P with B to construct the line segment between A and B.

This follows from *Pappus's Theorem* from projective geometry. To be specific, let Z be the intersection of $\overline{AB}$ and $\overline{EF}$. Then Pappus tells us that the points $X = \overline{AD} \cap \overline{CF}$, $Y = \overline{CB} \cap \overline{ED}$, and $Z = \overline{AB} \cap \overline{EF}$ are collinear. This implies that $\overline{AB}$, $\overline{EF}$, and $\overline{XY}$ are concurrent at the point Z. Hence, in fact, $Z = P$, so P is collinear with A and B.

Our work thus far gives a scheme for connecting two points that are just out of reach — say, at most 1.5 inches apart. Now we may finish with absurdity: If we can connect any two points that are at most 1.5 inches apart, then this means that we can simulate a 1.5-inch straightedge by using a 1-inch straightedge. Scaling up the argument, it follows that we can use a 1.5-inch straightedge to simulate a $(1.5)^2$-inch straightedge. So our 1-inch straightedge can simulate a $(1.5)^2$-inch straightedge. Inductively, we deduce that we can simulate a $(1.5)^n$-inch straightedge for all positive integers n. That is, arbitrarily large straightedges can be simulated. In particular, we must be able to connect any two points in the plane, no matter their distance.

■

Remarks: If you enjoyed that, here is a variant. You are on a plane and have been tasked with drawing the ray $\overrightarrow{AB}$ until it hits a point C far into the distance using your straightedge (which, in this problem, is as long as you would like). Unfortunately, about halfway through, you have come to a standstill: A sleeping cat!

This is a serious problem. If you were to continue extending this ray, your pen would touch the cat and wake it up, which is unacceptable. Can you find a way to continue the ray *past* the cat? That is, using only your straightedge, can you construct the rest of the ray's extension (sans the area around the cat)?

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Solution 70

Emily wins (provided that the table can fit at least one quarter).

She starts by placing a quarter right in the center of the table. Then, if Sydney places a quarter centered at a point P, then Emily will place a quarter at the reflection of P about the center. It is clear that whenever Sydney can move, Emily must be able to execute her move as well, so Emily cannot lose. So she has to win because the game eventually ends. ∎

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Solution 71

Let the integral be I, so that

$$I = \int_0^\infty e^{-x^2-1/x^2}\,dx.$$

Then, by substituting x with $1/x$, we get

$$I = \int_0^\infty \frac{1}{x^2} e^{-x^2-1/x^2}\,dx.$$

Now add the above two equalities to obtain

$$2I = \int_0^\infty \left(1+\frac{1}{x^2}\right) e^{-x^2-1/x^2}\,dx.$$

At first this looks dumb. But if we rewrite the integrand as

$$2I = \int_0^\infty \left(1+\frac{1}{x^2}\right) e^{-(x-1/x)^2-2}\,dx,$$

then miraculously we see that the u-substitution $u = x - 1/x$ is applicable because $\frac{du}{dx} = 1 + \frac{1}{x^2}$! Now we have that

$$2I = \int_{-\infty}^\infty e^{-u^2-2}\,du = \frac{\sqrt{\pi}}{e^2}$$

by the Gaussian integral (note the new limits on the integral). Thus $\boxed{I = \dfrac{\sqrt{\pi}}{2e^2}}$.

■

Remarks: This is also a textbook application of the miraculous *Glasser's Master Theorem*.

Theorem 1 (Glasser's Master Theorem)

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable. (That is, $\int_{-\infty}^\infty |f(x)|\,dx < \infty$.) Then

$$\int_{-\infty}^\infty f\left(x - \frac{1}{x}\right)\,dx = \int_{-\infty}^\infty f(x)\,dx.$$

Essentially, if the integrand is a function of $x - \frac{1}{x}$, then $x - \frac{1}{x}$ can be replaced with x without any fuss.

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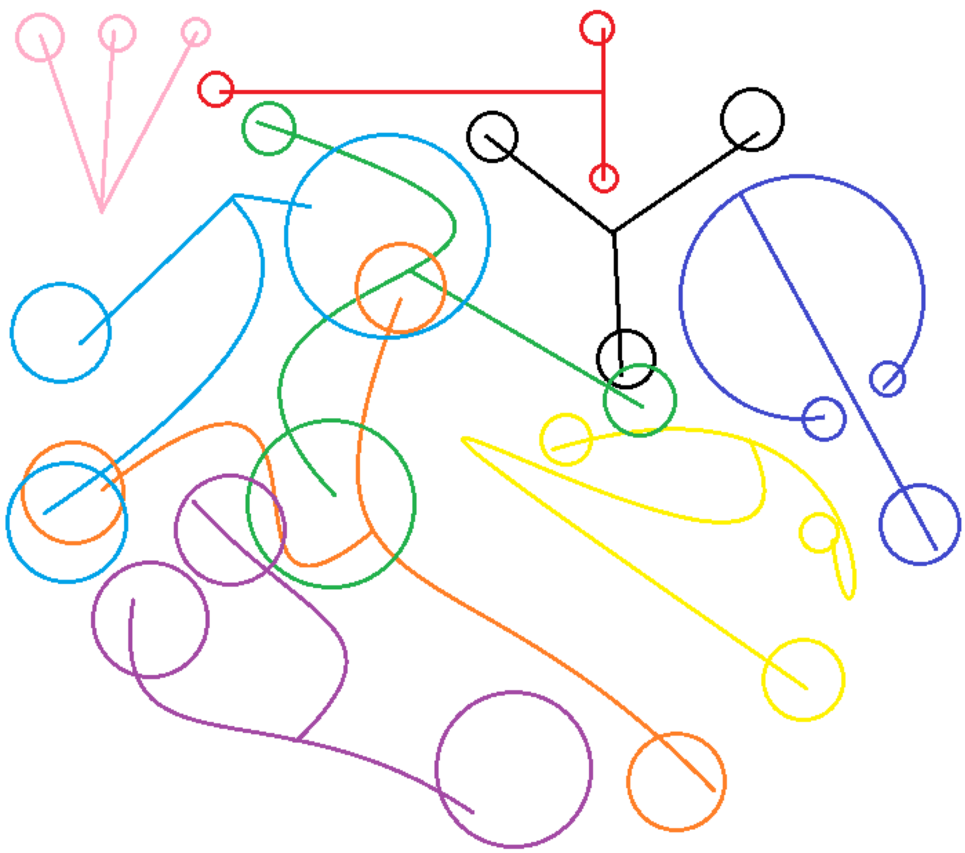
Solution 72

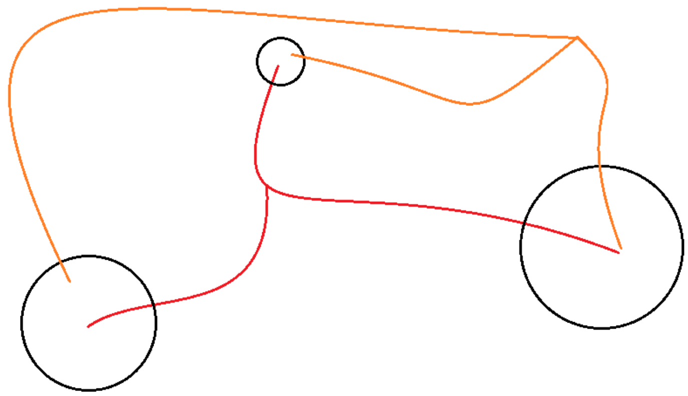
Define a *rational ball* to be a ball (i.e. a circle) whose radius is rational and whose center has rational coordinates. Evidently, there exist countably many rational balls.

For each "Y-set", draw three pairwise-disjoint rational balls containing its endpoints, such that each of the balls does not intersect either of the other two "branches" of the Y-set.



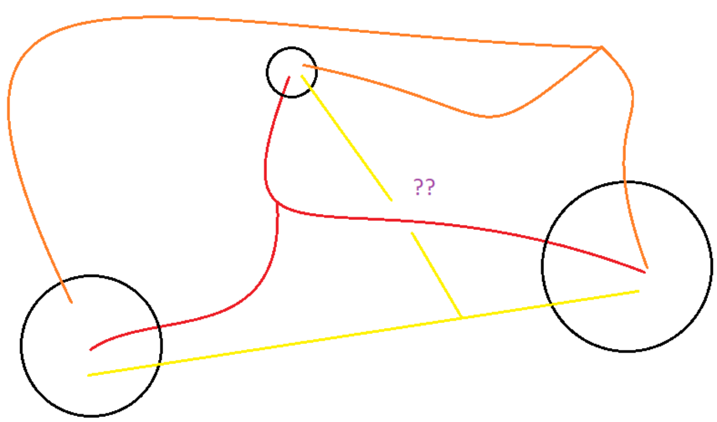
Define the map f which sends each "Y-set" to the 3-tuple of rational balls around its endpoints (in any order). Evidently, the number of such tuples is countable.

As suggested in the hints, we were motivated to try and construct an injection from the collection of Y-sets to a countable set, and the initial hope is that f is the desired injection. Unfortunately, f is not an injection. There could exist distinct Y-sets x, y for which $f(x) = f(y)$, as shown below.

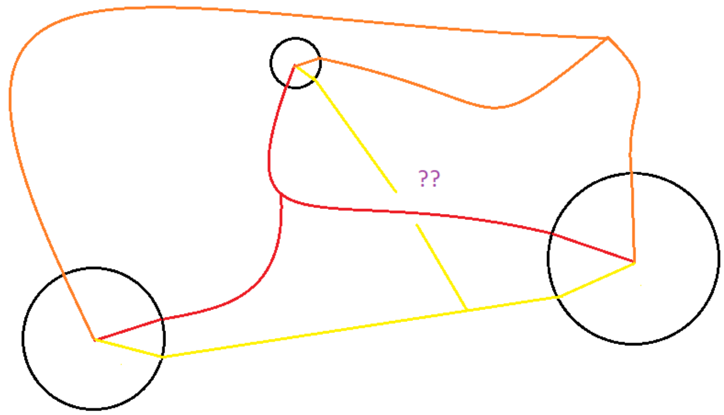


However, I claim that we can't do better! That is, there do not exist distinct Y-sets x, y, z for which $f(x) = f(y) = f(z)$. That is, there do not exist three distinct Y-sets whose endpoints circles are identical. Proving this is sufficient for showing that the domain of f is at most countable, which is what we want to show.

To see this, suppose that we have indeed found three such distinct Y-sets.

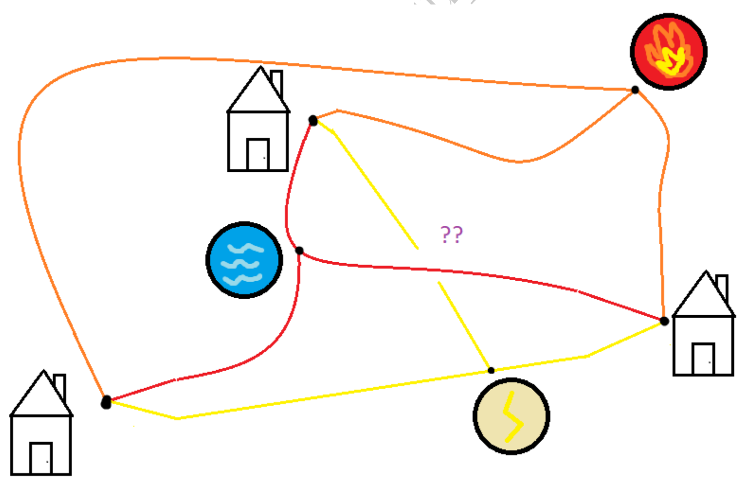


Inside each ball, erase all parts of the Y-sets, and replace those parts with radii to the center.



The condition that each ball does not intersect any of the other two branches ensures that after this operation, the three sets are still homeomorphic to the letter Y.

Now, place a house on the center of each circle and a utility at the "3-way crossing point" of each Y-set.



By the assumption that none of the Y-sets cross each other (i.e. are disjoint), we see that we have constructed a solution to the **three utilities problem (!!!)**, contradiction.

■

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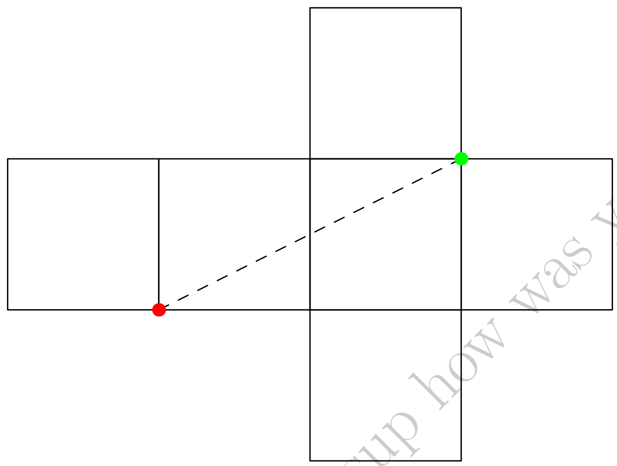
Solution 73

The shortest distance between two points is a straight line, but there is no straight line between a cube's opposite vertices that does not exit the cube's surface. Fortunately, this issue can be solved by unfolding the cube.



No matter how you unfold the cube, the ant will be at least a "knight's move" away from the opposite vertex, and this distance is $\sqrt{5}$.

■

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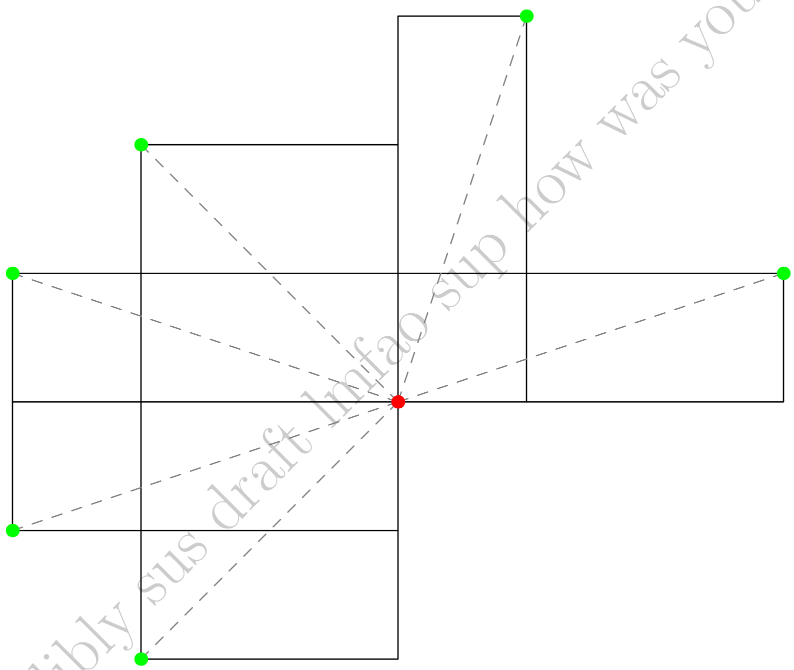
Solution 74

We begin by computing the shortest distance to the opposite vertex.

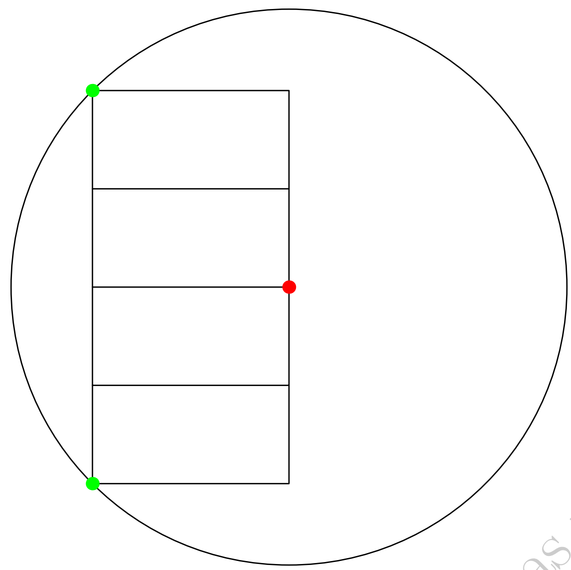
Let us unfold the box as below, where some faces of the box are repeated.



The ant begins at the red point and seeks to reach the vertex opposite this point, which can be represented by any of six green points in the above unfolding, as depicted.

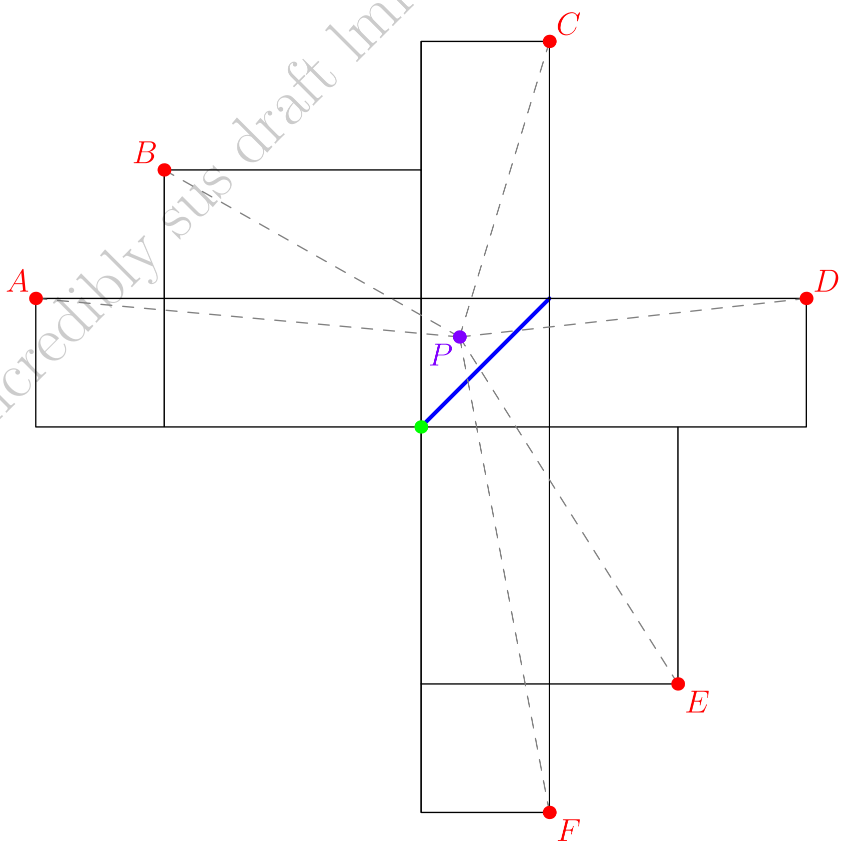
With some thought, it is not hard to see that the six paths above (drawn with dashed lines) are the only six sensible paths to the opposite vertex, and the shortest paths among these have length $2\sqrt{2}$. This is the shortest distance to the opposite vertex.

This is crucial for our arguments, as we can notice that all points on the four 1×2 faces are with $2\sqrt{2}$ of the ant's starting position.



We deduce that the point on the surface that is farthest from the ant must lie somewhere on the 1×1 face opposite the ant.

Let us now unfold the box in a different way, as below.



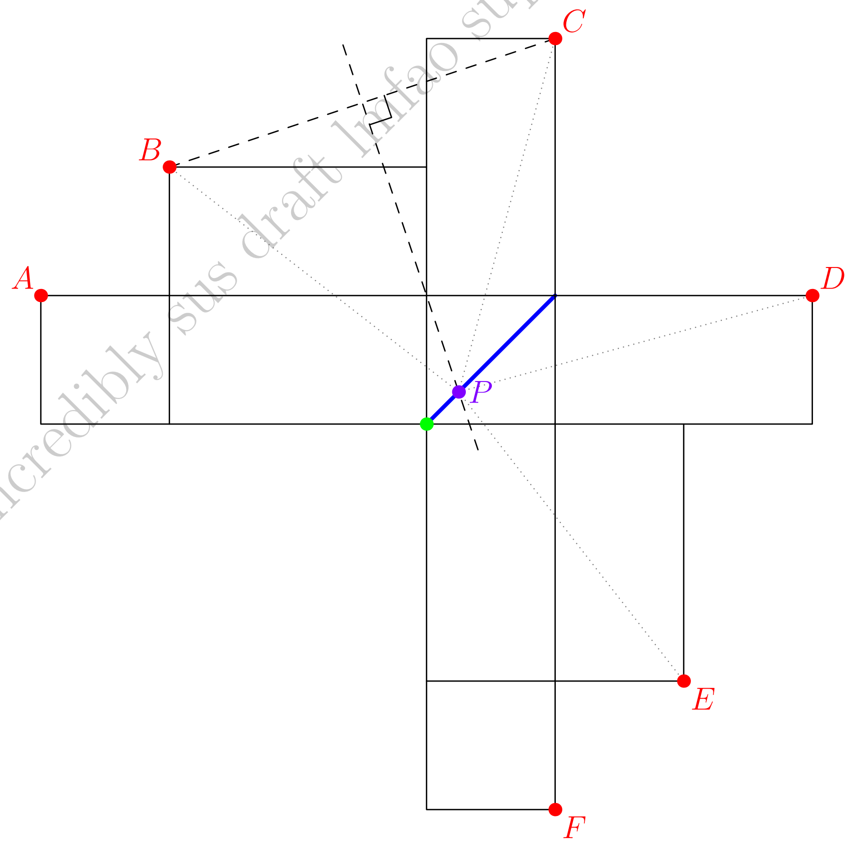
Here, it is the ant's starting vertex which is being duplicated (with six different labels from A to F), and the purple point P is an arbitrary point on the opposite 1×1 face. As before, there are six sane paths that the ant could choose to travel to P, as shown. We seek to select P so that the shortest of these paths is maximized in length.

To that end, we make two observations:

1. The distances from A and F to P in the above net will always be strictly greater than the distances from B and E to P, respectively. So we may disregard the paths from A and F.

2. If the point P maximizes the shortest path, then it must lie on the blue segment. Otherwise, the shortest path may be increased by moving P slightly towards the blue segment.

From this, we find that the point P that maximizes the distance is given by the intersection of the blue segment and the perpendicular bisector of segment $\overline{BC}$.



Working out the coordinate geometry or otherwise, we find that P must be a quarter of the way up the blue segment, and its distance from any of B, C, D, or E will be $\boxed{\dfrac{\sqrt{130}}{4}}$. ■

Remarks:

- $\frac{\sqrt{130}}{4}$ is indeed larger than the distance to the opposite vertex, $2\sqrt{2}$, but just barely — the difference is about 0.022.
- For a $1 \times 1 \times 1$ box, the farthest point is the opposite vertex. For a $1 \times 1 \times 2$ box, this is not so. Is there a critical value $1 \leq A \leq 2$ for which the farthest point for a $1 \times 1 \times A$ box switches from being the opposite vertex to a point strictly on the opposite face? Yes, there is, and this value is $A = \frac{3+\sqrt{17}}{4}$.

Source: This is called Kotani's Ant.

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Solution 75

You cannot stop me from assuming that P is monic, so let us do so. Suppose that the roots of P are $r_1, \cdots, r_n$, so that we may write

$$P(x) = \prod_{i=1}^{n} (x - r_i)$$

by the Fundamental Theorem of Algebra. Then by the product rule, we have that

$$P'(x) = \sum_{i=1}^{n} \frac{P(x)}{x - r_i}.$$

Let z be a root of $P'(x)$, so that

$$\sum_{i=1}^{n} \frac{P(z)}{z - r_i} = 0.$$

Either z is a root of P (in which there is nothing to prove), or we may divide each side by $P(z)$ to obtain

$$\sum_{i=1}^{n} \frac{1}{z - r_i} = 0.$$

Now take the conjugate of each side,

$$\sum_{i=1}^{n} \frac{1}{\overline{z} - \overline{r_i}} = 0$$

and rationalize (...*real*ize?) the fractions by multiplying the top and bottom of each by $(z - r_i)$ to obtain

$$\sum_{i=1}^{n} \frac{z - r_i}{|z - r_i|^2} = 0.$$

This now rearranges to

$$z = \sum_{i=1}^{n} \left(\frac{1/|z - r_i|^2}{\sum_{j=1}^{n} \frac{1}{|z - r_j|^2}} \right) r_i,$$

which implies that z is a convex combination of the $\{r_i\}$ because

$$\sum_{i=1}^{n} \frac{1/|z - r_i|^2}{\sum_{j=1}^{n} \frac{1}{|z - r_j|^2}} = 1.$$

■

Source: This is the Gauss-Lucas Theorem.

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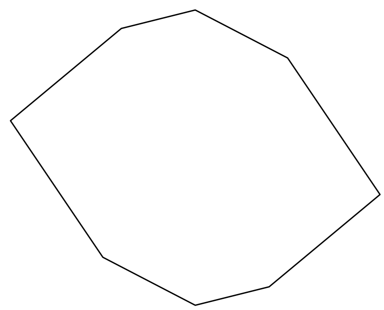
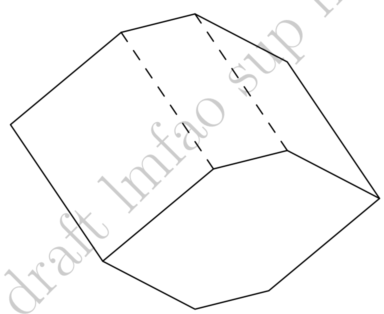
Solution 76

This will be a "proof by demonstration" because I cannot be bothered to formalize this.

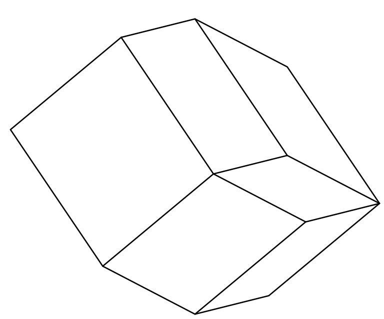
Let us suppose that this is our polygon.



The key idea is to draw the segments obtained by "sliding down three sides".



The dashed segments partition the top region into parallelograms. As for the bottom region, it is a smaller polygon with 180-degree rotational symmetry, so we may induct down on the number of sides to partition it into parallelograms.



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Solution 77

Part (a)

Instead of Heads/Tails, use 0/1 for the sides of the fair coin.

Let a binary expansion of p be $p = (0.d_1 d_2 d_3 \cdots)_2$. We keep flipping our fair coin until the binary sequence generated by our results is different from $d_1, d_2, d_3, \cdots$.

At any step, we will stop flipping with 50% odds, so we must eventually stop flipping the coin almost surely (i.e. with probability 1). We claim that our final flip is 0 (i.e. Heads) with probability p.

Indeed, our final flip is 0 iff we stopped at the kth flip with $d_k = 1$. So the probability is given by

$$\sum_{k:d_k=1} \frac{1}{2^k}$$

since there is a probability of $1/2^k$ that we stopped at the kth flip. By definition of binary expansion, this sum is exactly p.

Part (b)

Flip the coin twice. If the results are the same, then flip the coin twice again. Keep doing this until the results are different, which will eventually happen with probability 1. We claim that when this happens, these two flips are equally likely to be HT or TH. Indeed, this is true because HT and TH are equally likely to occur among any two flips, so conditioned on the event that we got either HT or TH, the odds of getting either are $50 : 50$.

■

Remarks: By combining both parts, we see that one can simulate any biased coin using any biased coin, as long as all probabilities are strictly between 0 and 1. Moreover, the simulation is possible even if the provided biased coin's probability of flipping heads is unknown.

The problem also shows that you can simulate a fair coin by using a fair coin, but this is not as groundbreaking.

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Solution 78

Part (a)

We claim that $A > 4$ is large enough.

First, some housekeeping: Suppose the side lengths of our collection of squares are given by $a_1, a_2, \cdots$. We may assume countability, because the sum of uncountably many positive reals cannot be finite. We can also assume that $a_i < 1$ for all i, otherwise this is dumb. Since $\sum_{i=1}^{\infty} a_i^2 = A > 4$, we may find n so large that $\sum_{i=1}^{n} a_i^2 > 4$.

From the above, we see that we may replace "collection" with "finite collection" in the problem (at least, for proving that $A > 4$ works). We want this so that we may apply an inductive argument.

This also lets us strengthen the claim we are proving as follows: If the sum of the areas of a finite collection of squares is $4Ns^2$, and none of the squares has area greater than s^2, then we may cover N squares of side length s with them. This would solve the problem.

It's sufficient to prove the case for $s = 1$ by scaling. Again, let the side lengths be $a_1, \cdots, a_n$, so that $\sum_{i=1}^{n} a_i^2 > 4N$. Suppose a_1 is the largest sidelength. Find $k \in \mathbb{N}$ for which $\frac{1}{2^k} \le a_1 < \frac{1}{2^{k-1}}$. Subdivide the N unit squares into $N(2^k)^2$ squares of side length $1/2^k$. We'll use the square with side length a_1 to cover exactly one of these.

Now there are $4^k N - 1$ more squares of side length $1/2^k$ to cover. Moreover, the sum of the areas of the rest of the squares we may use for covering is

$$\sum_{i=2}^{n} a_i^2 > 4N - a_i^2 > 4N - \frac{1}{4^{k-1}} = 4(4^k N - 1)\left(\frac{1}{2^k}\right)^2.$$

Thus, by **taking** $s = 1/2^k$ **in the claim we're proving** and using induction (the number of covering squares we may use has decreased), we can cover the rest.

Part (b)

We can show that $0 < a < 1/4$ is small enough by using the same argument as above, replacing $>$ with $<$ and replacing "cover" with "fit".

■

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Solution 79

Magellan captains a ship A and orders some of his crew on ship B to follow him as he sails forward. Once they're a quarter of the way along the equator, both ships A and B are halfway through supplies, so Magellan plunders the supplies of ship B so that ship A has full supplies.

Once Magellan is halfway across the equator, where he again is halfway through his supplies, he magically orders the third ship C to start sailing backwards, so that once Magellan is 3/4 done with his circumnavigation, he will rendezvous with ship C, which is halfway through their supplies. Since Magellan has no more supplies, he steals ship C's supplies so that he finishes the circumnavigation.

■

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Solution 80

A nice candidate for such a function would be

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!},$$

but this is not in closed form.

The trick is that we may write $f(x)$ as

$$f(x) = \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3}$$

where ω is a primitive third root of unity! To see why this works, expand e^x, $e^{\omega x}$, and $e^{\omega^2 x}$ into power series, add them up, and watch the magic.

To simplify, we may write

$$\begin{aligned} f(x) &= \frac{1}{3}\left(e^x + e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)x} + e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)x}\right) \\ &= \frac{1}{3}\left(e^x + e^{-x/2}\operatorname{cis}\left(\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\operatorname{cis}\left(\frac{-\sqrt{3}}{2}x\right)\right) \\ &= \frac{1}{3}\left(e^x + 2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)\right). \end{aligned}$$

This is in a nice closed form! Also, one might notice that some of these terms here are useless, so for maximal simplicity we may take the function $\boxed{e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)}$, and this works!

■

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Solution 81

Let p_k be the kth prime. For all $n \in \mathbb{N}$, define

$$S_n := \{1/n\} \cup \left\{ \frac{1}{n+1} + \frac{1}{kp_n} : k \in \mathbb{N}, \frac{1}{n+1} + \frac{1}{kp_n} < \frac{1}{n} \right\}.$$

Let $S = \bigcup_{n=1}^{\infty} S_n$. (Note that S is closed because it contains all its accumulation points, which all have the form $1/n$.) Then, identifying the points on the boundary of a circle with the interval $(0, 1]$, we define our convex set K to be the convex hull of S.

To construct a Venn-diagram with N copies of K, we take (for $i = 1, 2, \cdots, N$) $K^{(i)}$ to be K "rotated back by $1/i$", so that the "$1/i$" point of $K^{(i)}$ lies on top of the "1" point of K. Analogously, let us define $S_n^{(i)} := -1/i + S_n$ and $S^{(i)} := -1/i + S = \bigcup_{n=1}^{\infty} S_n^{(i)}$. We claim that $K^{(2)}, K^{(3)}, \cdots, K^{(N+1)}$ forms the desired Venn diagram.

To see that our construction works, consider any subset A of $\{2, 3, \cdots, N+1\}$. We claim that there exists $x \in (0, 1]$ such that:

- $x \neq 1/i$ for $i = 2, 3, \cdots, N+1$ (this ensures that x is not an accumulation point!)
- $x \in S^{(i)}$ for all $i \in A$
- $x \notin S^{(i)}$ for all $i \notin A$

Let $m = \prod_{i \in A} p_i$. Then we simply take $x = \frac{1}{m^k}$ where k is large enough so that $x < \frac{1}{N(N+1)}$. The bound $x < \frac{1}{N(N+1)}$ ensures that $x \in \left(\frac{1}{i}, \frac{1}{i-1}\right) - \frac{1}{i}$ for $i = 2, \cdots, N+1$.

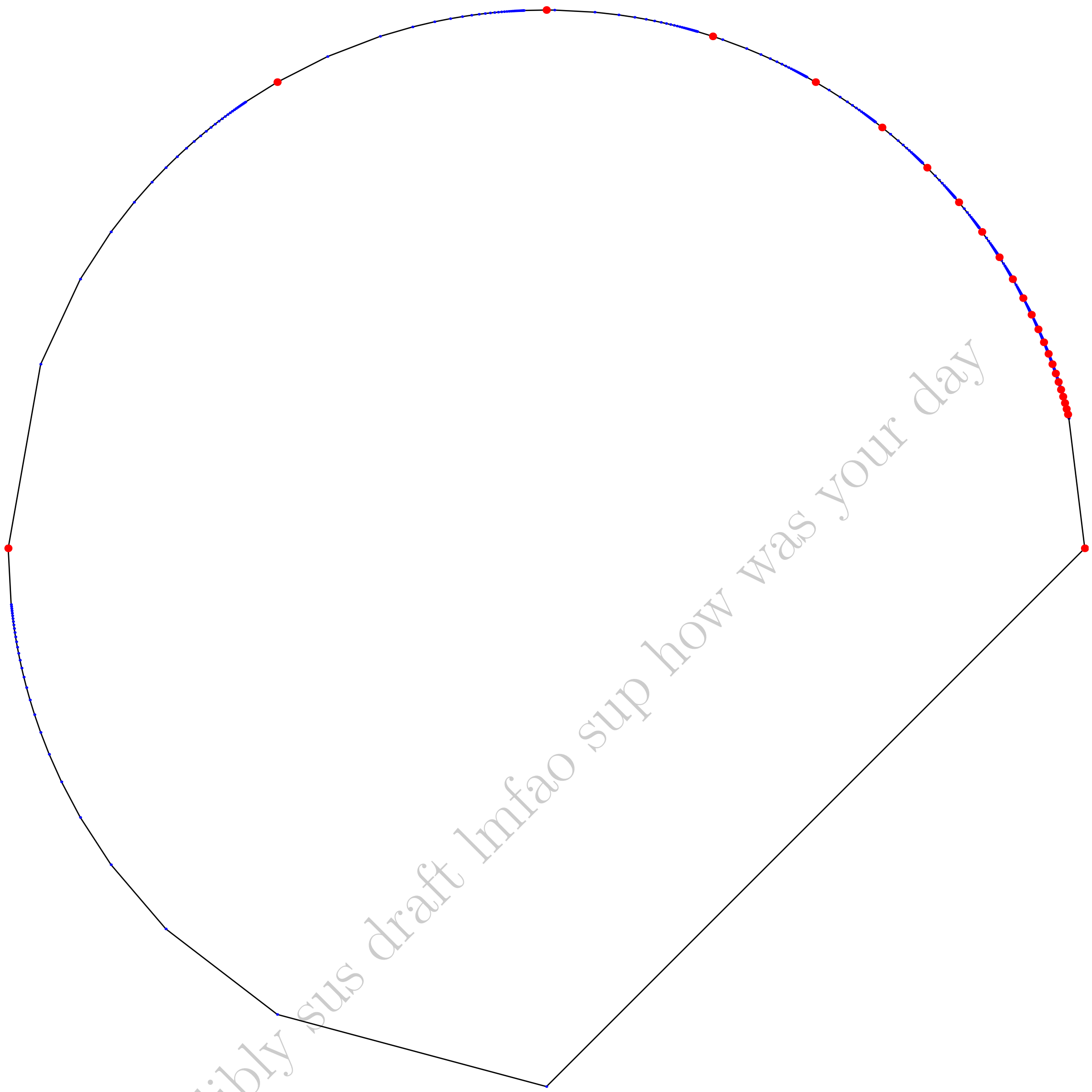
Consider $i \in A$. To see that $x \in S^{(i)}$, we claim that $x \in S_{i-1}^{(i)}$. Indeed, $\frac{1}{i} + \frac{1}{m^k} \in S_{i-1}$ because m^k is large enough to ensure $\frac{1}{i} + \frac{1}{m^k} < \frac{1}{i-1}$, and p_i divides m^k. Thus after rotating back we have $\frac{1}{m^k} \in S_{i-1}^{(i)}$.

Consider $i \notin A$. Then $x \notin S^{(i)}$ because we may follow the reasoning above, and then note that p_i does *not* divide m^k.

We conclude that $\bigcap_{i \in A} K^{(i)} \cap \bigcap_{i \notin A} (K^{(i)})^c$ is non-empty for all A. Hence we have formed a Venn diagram using N copies of K.

■

Remarks: The convex set we have constructed looks something like the figure on the next page.



The red points are the points represented by $\{1/n : n \in \mathbb{N}\}$. As you may have noticed, not all of them are drawn due to a shortcoming of the algorithm used to generate the figure.

CMU Alumni Isaac Browne and Edward Hou have shown a stronger construction that solves an infinite version of the problem, that is, they found a sequence $\{K_i\}_{i\in\mathbb{N}}$ of congruent open convex sets such that for any *finite* $A \subseteq \mathbb{N}$, the intersection $\bigcap_{i\in A} K_i \cap \bigcap_{i\notin A} K_i^c$ is non-empty. This was proposed as Problem 12424 for the American Mathematical Monthly. You can find a solution here ("A Universal Venn Diagram").

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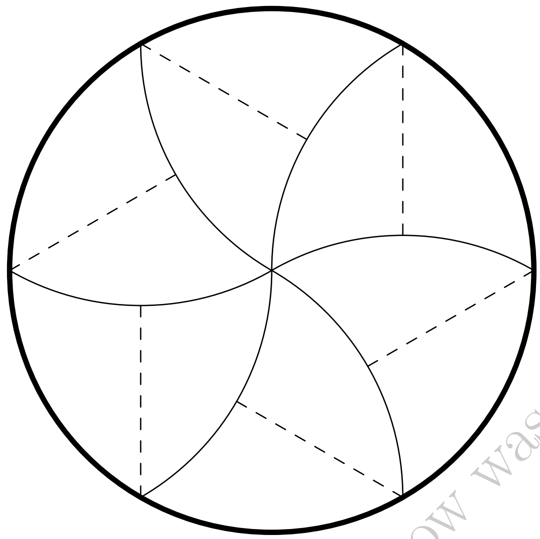
Solution 82



■

Remarks: What if Timmie is even pickier than usual, and doesn't want *any* part of the crust, not even a point? It turns out that it is still possible to satisfy Timmie! It's quite a bit harder though. See the paper https://arxiv.org/pdf/1512.03794.pdf for some ways to do it, and more!

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Solution 83

The problem in question is one variant of the "Pizza Theorem". We begin by proving a key lemma.

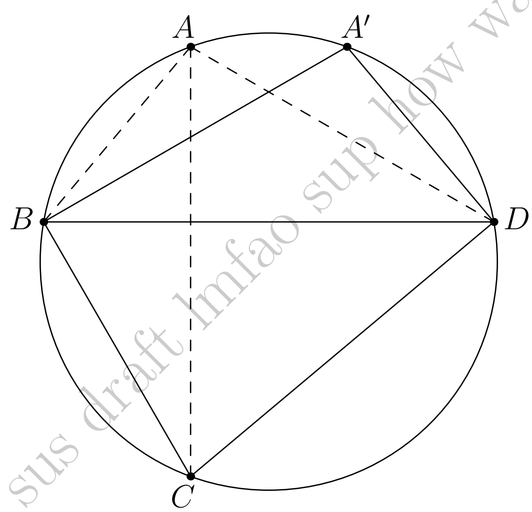
Lemma 1

Suppose that $ABCD$ is an orthodiagonal, cyclic quadrilateral. Then

$$AB^2 + CD^2 = BC^2 + AD^2 = 4R^2,$$

where R is the radius of the circumcircle of $ABCD$.

Proof.



Swap chords AB and AD, so that A becomes A'. We claim that $\angle A'BC$ is a right angle, so that $A'C$ is a diameter. Indeed, observe that

$$\angle A'BD = \angle ADB = \angle ACB,$$

So $\angle A'BC = \angle A'BD + \angle DBC = \angle ACB + \angle DBC = 90°$.

From this claim, we now have

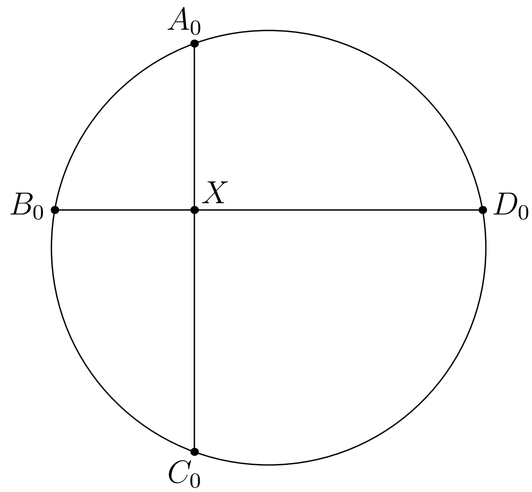
$$A'B^2 + BC^2 = A'C^2 = 4R^2$$

and

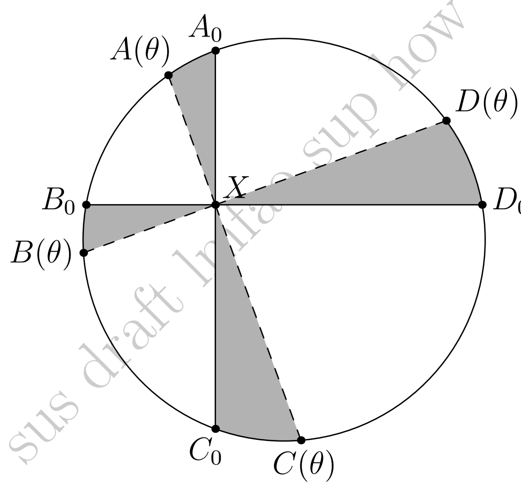
$$A'D^2 + CD^2 = A'C^2 = 4R^2.$$

Substituting back $A'B = AD$ and $A'D = AB$ proves the desired equalities. □

We now may resolve the original problem. Fix some arbitrary "right-angled cross".



If we "rotate" the cross by some angle θ, then a certain amount of area is displaced.



Note that if we can prove that the shaded area depends only on the radius R and the angle θ, then we are done!

We find the area by integration. Let $A(t)$ be the point on the arc between A_0 and $A(\theta)$ such that $\angle A_0XA(t) = t$. So, $A(0) = A_0$ and, well, $A(\theta) = A(\theta)$... Define $B(t)$, $C(t)$, and $D(t)$ similarly.

By polar integration, the area of the "A"-shaded region is given by the integral

$$\int_0^\theta \frac{1}{2} XA(t)^2 \, dt,$$

where $XA(t)$ denotes the length of the segment connecting X and $A(t)$. Hence the total shaded area is

$$\frac{1}{2}\int_0^\theta XA(t)^2 + XB(t)^2 + XC(t)^2 + XD(t)^2 \, dt.$$

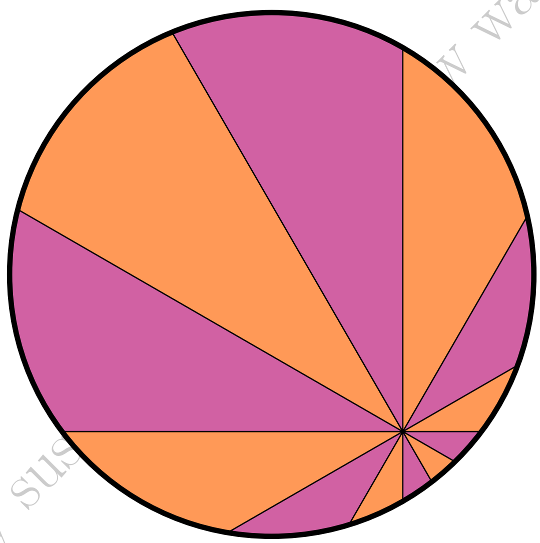
But the segments $XA(t), XB(t), XC(t), XD(t)$ are equally spaced, at angles of 90°. Thus by the lemma, we have that

$$XA(t)^2 + XB(t)^2 + XC(t)^2 + XD(t)^2 = 4R^2.$$

Therefore the area is just $\frac{1}{2}\int_0^\theta 4R^2\,dt = 2R^2\theta$. This indeed depends only on R and θ, proving the Pizza Theorem.

■

Remarks: A more "romantic" version of the Pizza Theorem (and, perhaps, the version that is more popular) goes as follows: Let $N \geq 2$. If $2N$ cuts are made at equal angles through a point, dividing the pizza into $4N$ slices, then taking alternate slices will share the pizza among two people. More succinctly, the number of slices created in this manner must be one of $8, 12, 16, 20, \cdots$.



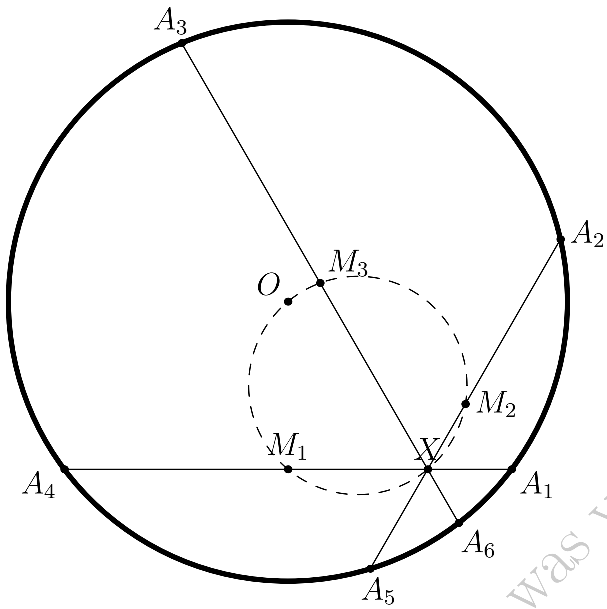
Unfortunately, this version of the Pizza Theorem is not implied by the version we have proven, as shown by the case depicted above with 12 slices ($N = 3$).

On the bright side, we can still apply the same calculus approach: It is sufficient to show that for any point X in a circle, if we draw $2N$ rays at equal angles emanating from X that hit the circumference at points $A_1, A_2, \cdots, A_{4N}$, then

$$\sum_{i=1}^{2N} XA_i^2$$

is a constant in the sense that it does not depend on which $2N$ rays we draw.

The following beautiful approach is due to "tenth". Assume that the labeling of the points $A_1, A_2, \cdots, A_{2N}$ is in counter-clockwise order.



We first pair up the terms in the sum as

$$\sum_{i=1}^{2N} XA_i^2 = \sum_{i=1}^{N} XA_i^2 + XA_{i+N}^2.$$

The key idea is that

$$XA_i^2 + XA_{i+N}^2 = (XA_i - XA_{i+N})^2 + 2(XA_i)(XA_{i+N}).$$

The product $(XA_i)(XA_{i+N})$ is the *power* of X with respect to the circle (to be precise, it is $R^2 - OX^2$), so it is a constant. As for $(XA_i - XA_{i+N})^2$, this is the square of the distance between X and the midpoint of A_iA_{i+N}, which we shall denote as M_i for $i = 1, 2, \cdots, N$. It is now sufficient to prove that the sum $\sum_{i=1}^{N} XM_i^2$ is a constant.

The conclusion follows by two miracles.

Miracle 1: The midpoints $M_1, M_2, \cdots, M_N$ form a regular N-gon! Proof: $\angle OM_iX = 90^\circ$ for all i by virtue of M_i being the midpoint of a chord, so the points $M_1, M_2, \cdots, M_N$ are concyclic with O and X. Then the regularity of polygon $M_1M_2\cdots M_N$ follows from the fact that the cuts were made at equal angles.

Miracle 2: This lemma exists!

Lemma 2

Let $X, M_1, M_2, \cdots, M_N$ be points. Then the sum

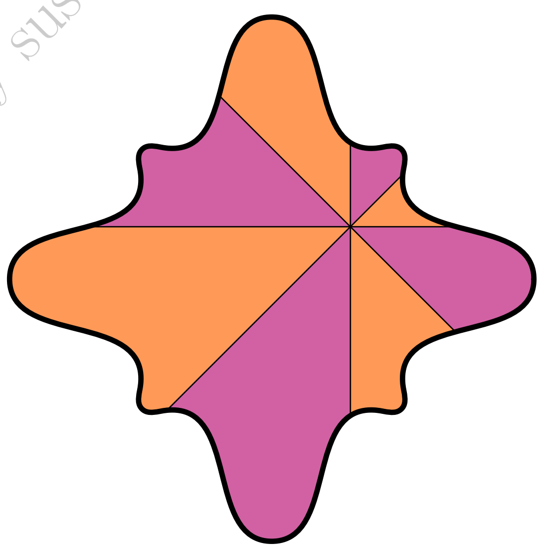
$$\sum_{i=1}^{N} XM_i^2$$

depends only on the distance between X and the centroid of the points $M_1, M_2, \cdots, M_N$.

I leave it to the reader to verify this (...or perhaps this fact will come up in a different problem in this book, where it shall be proven?). Now, since $M_1M_2 \cdots M_N$ is regular, its centroid coincides with its circumcenter, which is the midpoint of $\overline{OX}$. So the sum $\sum_{i=1}^{N} XM_i^2$ depends only on the distance between X and this midpoint, which is a constant $\frac{OX}{2}$.

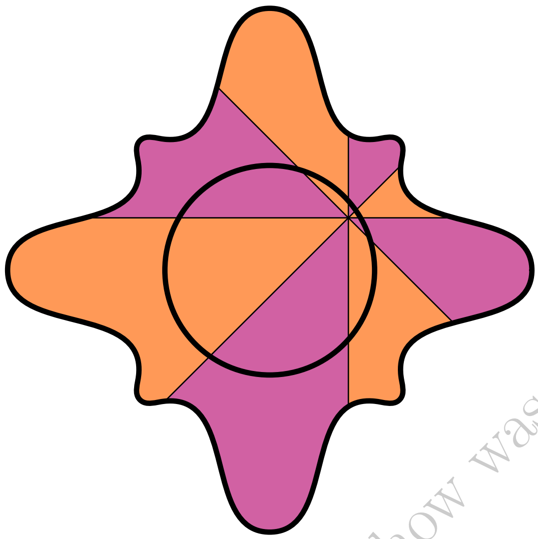
This completes the proof. But what goes wrong when $N = 1$? The issue is that when $N = 1$, the centroid of the "regular 1-gon" M_1 (which is M_1 itself) no longer coincides with the circumcenter of $\triangle OXM_1$, due in part to the fact that a regular 1-gon does not have a well-defined circumcenter. A similar issue occurs for $N = 2$, but this case is handled just fine by the original problem.

Even More Remarks: The Pizza Theorem generalizes slightly to very poorly-made pizzas. Consider, for example, the following pizza with D_8 symmetry.

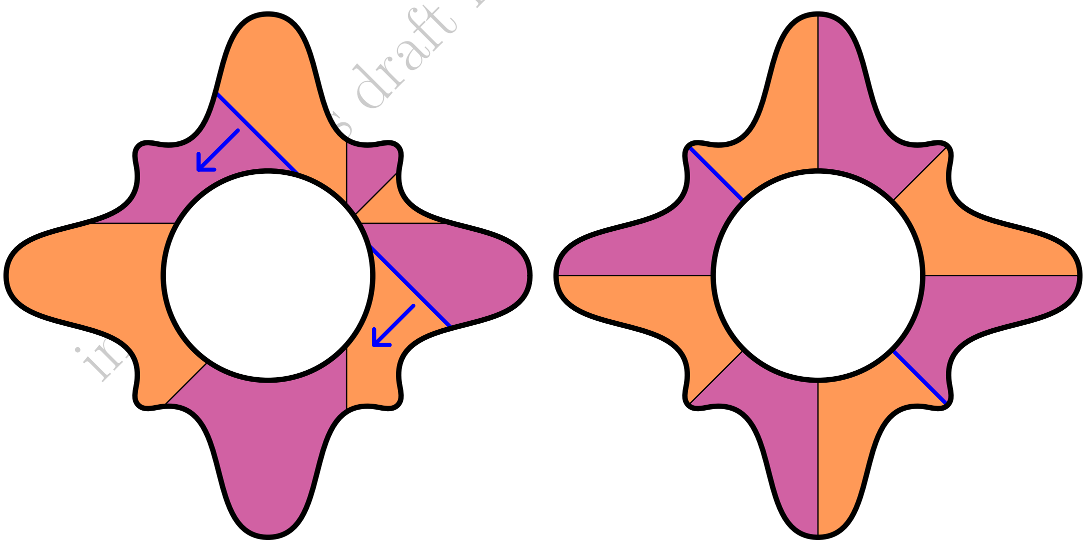


Amazingly, when we divide this pizza into 8 slices as shown, we once again obtain equal shares by alternating slices.

To demonstrate this, we begin by drawing a circle centered at the center of this oddly-shaped pizza, large enough to contain the point through which the cuts intersect.



By the Pizza Theorem, the pizza contained inside the circle is equally shared. So it remains to show that the pizza outside the circle is equally shared. This is far easier to argue once we remove the inside of the circle.



The divisions of the remaining pizza consist of 8 segments from the circle to the crust. As we move a pair of opposite such segments towards the middle (such as the two blue segments above), we can see that neither color gains nor loses pizza. Doing this for every such pair, we may reach a configuration in which the cuts are perfectly symmetrical, so that we may conclude that, in fact, the amount of pizza of each color is equal.

In general, if we include more cuts, then the shape of the pizza must exhibit an appropriate amount of symmetry in accordance to the version of Pizza Theorem that is applied.

Source: Pizza Theorem

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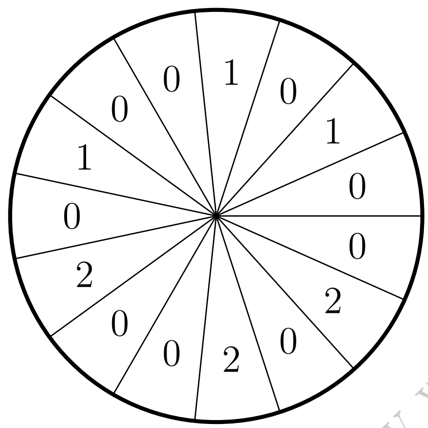
Solution 84

The answer is yes. Here is a construction.

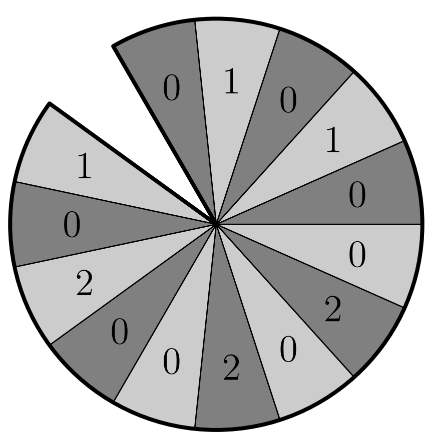


The numbers indicate the sizes of the slices. Realistically, Beth cannot make a slice of size 0, but she could make those slices have some negligible size such as 0.01 instead. This does not change the strategy.

Let us now describe Beth's strategy. There are two cases, based on the first slice Allison chooses.

Case 1: Allison chooses a slice of size 0

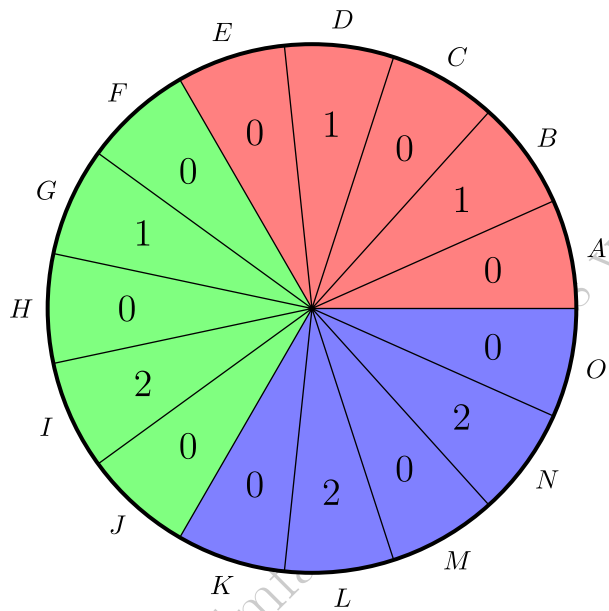
If this is the case, then Beth two-colors the remaining slices in an alternating fashion.



She then determines which color contains more pizza than the other, and proceeds to take slices of only that color. This ensures that she gets all slices of said color, and Allison gets the slices of the other color.

All slices are of integer size and the total size of the pizza is 9, so there necessarily will be one color that contains strictly more than half the pizza (to be precise, one color must have size at least 5).

Case 2: Allison chooses a slice of positive size



Beth's strategy here is more complicated. Refer to the above coloring of the pizza which partitions the pizza into three colored "wedges". We also have labeled all 15 pieces with letters for convenience.

- Beth starts by taking the 0 slice that is adjacent to the edge of the colored wedge that Allison ate from. For example, if Allison ate slice I, then Beth takes slice J.
- From then on, there are three cases:
  - If they have finished eating one colored wedge, and the other two colored wedges are untouched, then Beth takes the slice from the wedge of smaller size. For example, if the green wedge is all finished, and both the red and blue wedges are untouched, then Beth takes slice E.
  - If Beth can avoid eating from an untouched wedge, then she will.
  - If neither of the above points apply, Beth simply "copies" Allison by always taking the slice adjacent to the one that Allison picks.

The key idea is that this strategy ensures that Beth can claim two things:

- The largest wedge untouched by Allison's first bite (either the blue or green wedge)
- Either the other wedge untouched by Allison's first bite, or the rest of the wedge that Allison first ate from.

This will always ensure that Beth gets at least $\frac{5}{9}$ of the pizza!

Let us go through a quick example: If Allison starts with slice N, then Beth takes slice O. Beth can now guarantee that she will get slices G and I, and she will also either get slices B and D or the slice L. Indeed, if Allison takes M, then Beth can claim L, and if otherwise Allison wants to prevent Beth from claiming L, then Allison must take slices A and then C, conceding to Beth the B and D slices. In any case, Allison will eventually be forced to take F or J because it will be Allison's turn once the red and blue wedges are finished, so Beth can guarantee herself all positive slices of the green wedge.

■

Remarks: $\frac{5}{9}$ is the best Beth can ensure! This was conjectured by Peter Winkler and proven by Knauer, Micek, and Ueckerdt in 2011. See the following paper for the proof and more: https://arxiv.org/pdf/0812.2870.pdf

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Solution 85

The map $1/x$ is a bijection on $(0, \infty)$. Thus, the minimum of x^x is the minimum of $(1/x)^{1/x}$, which is equal to $\frac{1}{x^{1/x}}$. This minimum is equal to the reciprocal of the maximum value of $x^{1/x}$. Thus the maximum value of $x^{1/x}$ is the reciprocal of the minimum of x^x, which is $1/M$.

■

Source: Shamelessly stolen from user "juliankuang" of AoPS , who allegedly came up with this in the shower.

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Solution 86

Suppose for contradiction that such a polygon existed. Imagine that I start on one of the vertices, and then traverse the perimeter until I come back to where I started.

Let L, R, U, and D be the number of units I move left, right, up, and down, respectively.

Since I came back to where I started, we have $L = R$ and $U = D$. And, since the sides alternate between horizontal and vertical, we must have $L + R = U + D$. By some algebra, we may deduce from these facts that $L = R = U = D$.

Since $L + R + U + D$ is the perimeter, we conclude that the perimeter is divisible by 4. But 314 is not divisible by 4, contradiction.

■

Source: The Brilliant.org community. Rest in peace.

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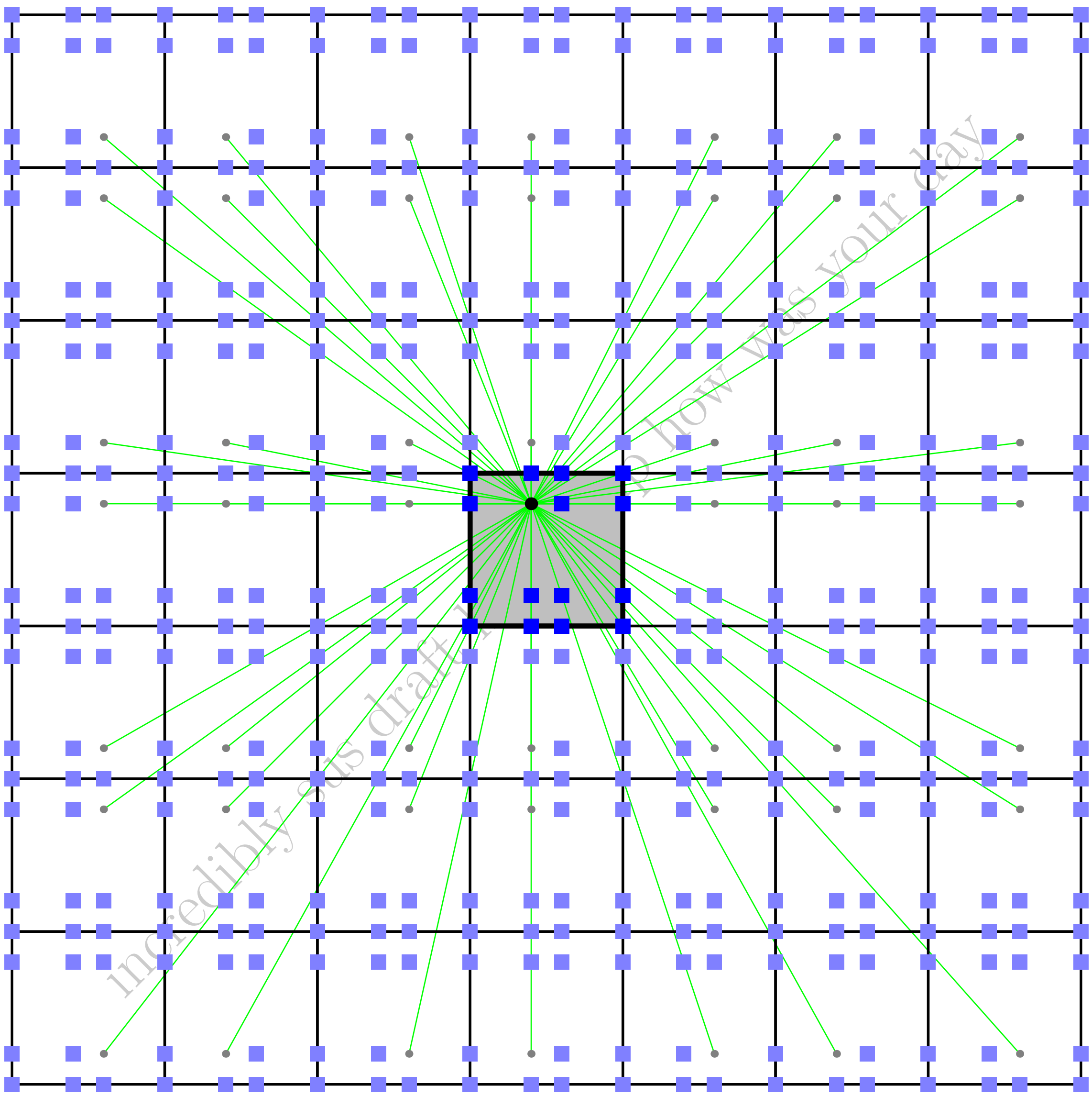
Solution 87

My friends can indeed arrange themselves in such a way, and 15 friends is sufficient to do this.



The gray square is the original mirror room, the black dot is me, and the 15 dark blue squares are my friends. All other squares depict reflections of the mirror room, and the green rays are the possible lines of sight from me to a reflection of myself. Each such ray is blocked by a friend, as needed.

To be more precise: If we view the room as $[0,1]^2$ and I decide to stand at (a,b), then my friends will stand at the following locations:

- The four corners
- $(a, 0)$, $(a, 1)$, $(0, b)$, and $(1, b)$
- $(1-a, 0)$, $(1-a, 1)$, $(0, 1-b)$, and $(1, 1-b)$
- $(1-a, b)$, $(a, 1-b)$, and $(1-a, 1-b)$

Now, why does this work? First, note that the coordinates of any of my reflections will take the form $(2m \pm a, 2n \pm b)$ for integers m and n. The key claim is that there will be a friend blocking my line of sight to $(2m \pm a, 2n \pm b)$ at precisely the midpoint! (Or the midpoint is a reflection of me, in which case we induct downwards.)

Indeed, the midpoint of the segment connecting (a, b) and $(2m \pm a, 2n \pm b)$ is $(m \text{ or } m + a, n \text{ or } n + b)$. This gives 16 cases since we also must consider the parities of m and n. When all cases are reduced to an equivalent point inside the unit square, we obtain the 16 coordinates of me and and my 15 friends.

■

Source: The Leningrad Olympiad, but it is likely more famous.

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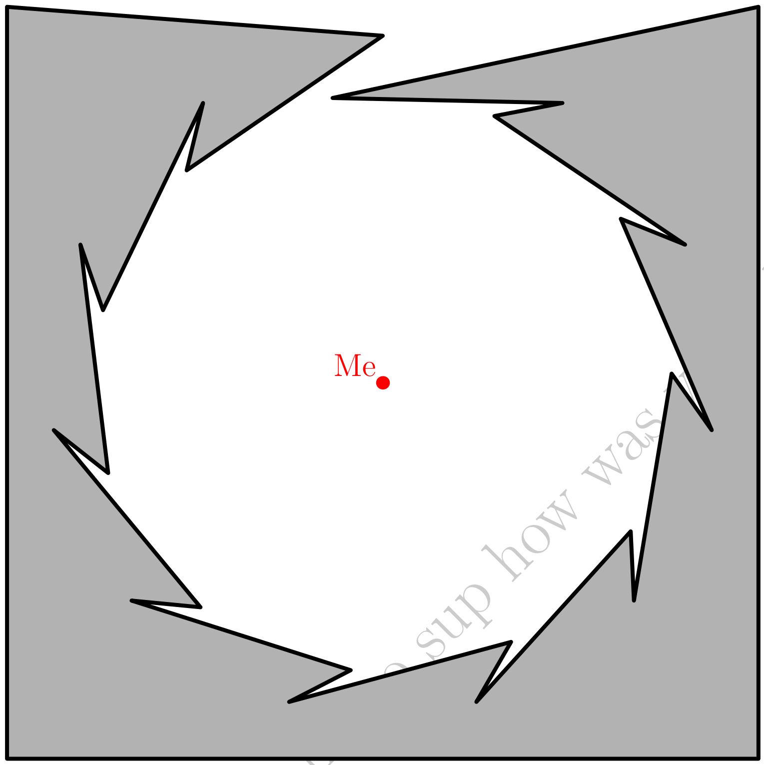
Solution 88



■

Remarks: The least number of sides you can get for a counterexample is 8. Probably. Someone claimed a proof but didn't give one. Maybe *you* can prove it!

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Solution 89

We begin with an intuitive sketch for the process. Let A, B, C, D be the bottoms of the table's legs, in counter-clockwise order.

1. Place the table somewhere so that the two opposite corners A and C touch the floor. It's alright if the B and D legs clip through the floor.

2. Pivot the table about $\overline{AC}$ until the B and D are the same vertical distance above/below the floor. Assume without loss of generality that they're both above the floor after the pivoting, and let the points on the floor under B and D at this time be B' and D'. View these two points as fixed.

3. (The key step) "Rotate" the table continuously so that A and C are always touching the floor, and B and D are always the same distance vertically above the floor, until points A and C arrive at points B' and D' respectively.

4. Now B and D must be under the floor (Why?), so by the Intermediate Value Theorem there must have existed a time during the "rotation" during which both of them are on the floor.

This is not completely rigorous since it is not clear that we can "rotate" the table continuously in the manner described. To finish, we must sketch out this process precisely. That is: Given that A and C are on the floor, and B and D are the same vertical distance above the floor, we can move $A, B, C,$ and D continuously so that $ABCD$ remains a square of the same size, A and C always remain on the floor, B and D always remain the same vertical distance above/below the floor, and A and C switch positions at the end.

In general, if it is only given that the floor is continuous, then this is quite a difficult issue to tackle. We will see that the Lipschitz condition on the floor will make this much more feasible.

First, note that it suffices to find a continuous motion of the two points A and C so that they stay on the graph, switch positions, and their distance is fixed throughout. If so, then at all times there are unique positions for B and D so that $ABCD$ is a square and B, D are the same distance above the graph. The composition of continuous functions entails that these positions move continuously.

The following key result allows us to "rotate" C around A (and vice versa).

Lemma 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a 1-Lipschitz continuous function and fix a point $x_0 \in \mathbb{R}^2$. Fix $R > 0$ and let S be the set of points in $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$ at a distance R from $(x_0, f(x_0))$. For each $\theta \in \mathbb{R}$, let $V(\theta) := \{x_0 + (t\cos\theta, t\sin\theta) : t > 0\}$ be the ray emanating from x_0 at the angle θ. Then, for all $\theta \in \mathbb{R}$, there exists a unique point $g(\theta) \in \mathbb{R}^3$ on the surface S whose projection unto $\mathbb{R}^2$, $(g_1(\theta), g_2(\theta))$, lies in $V(\theta)$. Moreover, $g : \mathbb{R} \to \mathbb{R}^3$ is a continuous function.

The rigorous statement above is quite atrocious. Intuitively, the picture to have in mind is as follows: Draw a sphere around some point on the graph of f. Then the sphere should intersect the graph at some curvy "ring".

Proof. For existence and uniqueness of the point $g(\theta)$, it is sufficient to consider the case $\theta = 0$ by symmetry, in which case we need only consider the cross section of f and S obtaining by intersecting these surfaces with the xz-plane.

To wit, we may rephrase the problem as follows: Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-Lipschitz continuous function, let $x_0 \in \mathbb{R}$, let $R > 0$, and let S be the circle of radius R centered at $(x_0, f(x_0))$. Then S intersects the graph of f at exactly two points: One to the "left" of x_0, and one to the "right" of x_0. Particularly, the case $\theta = 0$ concerns itself with the existence and uniqueness of such an intersection to the "right" of x_0.

Roughly speaking, the existence is quite simple and comes from applying the Intermediate Value Theorem or the Jordan Curve Theorem properly. For uniqueness, we suppose there are two distinct intersections at $(y_1, f(y_1))$ and $(y_2, f(y_2))$ with $y_1, y_2 > x_0$. Observe that $y_1 \geq x_0 + \frac{R}{\sqrt{2}}$, otherwise the two points x_0 and y_1 fail the Lipschitz condition on f. The same is true for y_2. But now we may argue that the circle S is "too steep" between $x_0 + \frac{R}{\sqrt{2}}$ and $x_0 + R$, so that the points y_1, y_2 fail the Lipschitz condition.

$g(\theta)$ must be continuous because f is. □

To be precise, the sort of "rotation" that this Lemma allows us to execute is as follows: We may slide C continuously along the intersection between the graph of f and the surface S of points of distance $\sqrt{2}$ from A. And, viewed from above, this motion for C will appear to be "circling around A".

We may now describe the procedure for swapping A and C.

1. "Rotate" C about A as described above until C is a distance of $\sqrt{2}$ from its starting position. An application of the Intermediate Value Theorem will show that this is possible.
2. "Rotate" A about C until A arrives at the starting position of C.
3. "Rotate" C about A until C arrives at the starting location of A.

This completes the proof.

■

Remarks: The proof sketch works quite well in practice. If you have a well-shaped table that is wobbling on an uneven ground, it can be stabilized by "rotating" the table.

The Lemma is extremely false when f is not assumed to be 1-Lipschitz (Do you see why?). See the paper https://arxiv.org/pdf/math/0511490.pdf for the general proof.

One of the authors of said paper happens to be the Youtuber "Mathologer"! Naturally, he made a video about the problem at https://www.youtube.com/watch?v=aCj3qfQ68m0. A very nice exercise from the video (which is certainly nicer to rigorously reason about than the problem!) is as follows: Given any bounded figure, prove that there exists a square all of whose sides are tangent to the figure.

Lastly, the solution I wrote completely ignored any issues that could occur with the legs or top of the table intersecting the graph of f. I will leave it to you to think about whether or not this could be a problem.

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Solution 90

We claim that 6 soldiers is the minimum number of soldiers required to ensure the capture.

Sufficiency of 6 soldiers

For each positive integer n, let $f(n)$ be the greatest positive integer for which n soldiers can guarantee the capture of any criminal hiding in a tree-shaped palace of $f(n)$ rooms. We seek to show that $f(6) \geq 1000$.

For a room R of a tree-shaped palace, the *branches from* R are the connected components of rooms that arise when R is deleted from the palace. For example, if S is a neighbor of R, then the set of all rooms that can be reached from S without passing through R is a branch of R. The number of branches of R is equal to the degree of R.

The key observation is as follows.

Claim: Suppose that within a tree-shaped palace, there exists a path of distinct rooms $R_1, R_2, \cdots, R_m$ that, when deleted from the palace, will result in the palace being split into smaller connected components of rooms, each with no more than $f(n)$ rooms. Then $n+1$ soldiers can guarantee the capture of a criminal in this palace.

This is because, given the existence of such a path of rooms $R_1, \cdots, R_m$, a strategy for the $n+1$ soldiers is as follows:

1. Soldier 1 waits at room R_1.
2. For each room R besides R_2 that is adjacent to R_1, the other n soldiers check if the criminal lies within the branch of R_1 that lies past R. This branch has no more than $f(n)$ rooms, so n soldiers are sufficient.
3. Soldier 1 advances to room R_2, and the rest of the n soldiers checks each branch of R_2 besides the ones that contain R_1 and R_3.
4. Soldier 1 advances to room R_3, and we continue until all rooms are searched.

This observation lets us prove the essential result we require.

Claim: For each positive integer n, we have $f(n+1) \geq 3f(n)+3$.

To show this, we take a tree-shaped palace of $3f(n)+3$ rooms and show that there exists a path of rooms that splits the palace into components of size no more than $f(n)$.

We generate this path via the following algorithm.

Main Algorithm

1. Start with any room. Call it S_1. Set $m = 1$.
2. Consider the branches of S_m that do not include the tentative path thus far (i.e. $S_1, \cdots, S_{m-1}$).
3. If each such branch has size at most $f(n)$, then we terminate the algorithm, with the desired path being $S_1, \cdots, S_m$.
4. If there is exactly one branch of S_m of size greater than $f(n)$, then we take S_{m+1} to be the neighbor of S_m leading into this branch, increment m, and loop back to Step 2.
5. Otherwise, there are exactly two branches with size greater than $f(n)$, and particularly their sizes are at least $f(n)+1$. It follows that the branch of S_m containing the tentative path thus far has size at most $3f(n)+3-(f(n)+1+f(n)+1+1) = f(n)$. Armed with this deduction, we **cancel the current path** and construct a new one as follows: Run the subroutine (described below) on each of the two branches of size at least $f(n)+1$ to obtain two paths from $R := S_m$, each of which splits their respective branches into further branches of size at most $f(n)$.
6. Concatenate these two paths with R to form the desired path and terminate.

Subroutine

1. Let R_1 be the room leading into the branch of R on which we call this subroutine. The branch has at least $f(n)+1$ rooms and at most $3f(n)+3-(f(n)+1+1) = 2f(n)+1$ rooms. Set $m = 1$.
2. Consider the branches of R_m that do not include the path thus far (i.e. $R_1, \cdots, R_{m-1}$).
3. If each such branch has size at most $f(n)$, then we terminate the algorithm, with the desired path being $R_1, \cdots, R_m$.
4. Otherwise, there is exactly one branch with more than $f(n)$ rooms. Take R_{m+1} to be the neighbor of R_m that leads into this branch, increment m, and loop back to Step 2.

The correctness of this algorithm is mostly self-evident, though I should justify the exhaustion of cases. In the main algorithm, the number of branches of S_m (excluding the one with the tentative path) of size greater than $f(n)$ can only be 0, 1, or 2. If there were 3, then there are at least $3(f(n)+1)+1 = 3f(n)+4$ rooms, where we have also included S_m in the count, which is bogus. Similarly, in the subroutine, the number of branches of R_m (excluding the one with the tentative path) of size greater than $f(n)$ can only be 0 or 1, as if there were 2, then the branch on which we call the subroutine would have at least $2(f(n)+1)+1 = 2f(n)+3$ rooms, which is not the case.

This algorithm proves the essential result, which we shall now use to compute a lower bound on $f(6)$. It is not hard to see that $f(1) = 3$. It follows that

- $f(2) \geq 3(3 + 1) = 12$,
- $f(3) \geq 3(12 + 1) = 39$,
- $f(4) \geq 3(39 + 1) = 120$,
- $f(5) \geq 3(120 + 1) = 363$, and
- $f(6) \geq 3(363 + 1) = 1092$.

In particular, $f(6) \geq 1000$, which is what we wanted to show.

Necessity of 6 soldiers

We will construct a tree-shaped palace of size at most 1000, inside of which a lucky criminal could evade 5 soldiers.

First, if four rooms are arranged in a "Y" shape, then one soldier is clearly insufficient for capturing a criminal hiding in such an arrangement of rooms. Let us call this structure Y_1, and call the room in the middle the "central room" of Y_1.

Recursively, for each positive integer n, we build the structure Y_{n+1} as follows: Place a room R, which we take to be the central room of Y_{n+1}. Then, we place three copies of Y_n and attach their central rooms R_1, R_2, R_3 to R.

Assume that n soldiers are insufficient to guarantee capture of a criminal inside a palace in the shape Y_n. We claim that $n + 1$ soldiers are insufficient to guarantee capture of a criminal inside a palace of shape Y_{n+1}.

Indeed, this is not too difficult to reason out. Let the branches of R that contain R_1, R_2, and R_3 be T_1, T_2, and T_3, respectively. The criminal, who knows how to dodge n soldiers in a Y_n-shaped palace with positive probability, employs the following "strategy":

- Pick a branch of R to hide in (T_1, T_2, or T_3) at random.
- With sufficient luck, dodge any search of the branch that uses at most n soldiers.
- If $n + 1$ soldiers are all present in a common branch, then with sufficient luck, this branch is not the same branch that the criminal is in. So the criminal can decide to switch to the third unoccupied branch with 50% odds.

We claim that with sufficient luck, the criminal can dodge any search of Y_{n+1} that uses $n+1$ soldiers.

Since the criminal could be hiding in T_1, any search that is guaranteed to work must still be guaranteed to work if the criminal promises to stay in T_1. That is, the search of Y_{n+1} must include an exhaustive search of T_n, and for this, n soldiers is insufficient. Hence, the search pattern must at some point involve all $n+1$ soldiers inside T_1 simultaneously. The same is true for T_2 and T_3.

Take the first time that all soldiers are within some T_i for some i, and assume for ease that $i = 1$. Then, the criminal will not be in T_1 with sufficient luck, and by the criminal's strategy, they could now be hiding in either T_2 or T_3. It is impossible to verify which one the criminal is in without taking all $n+1$ soldiers and putting them all in one of these branches at some point — say, T_2. Then, with sufficient luck, the criminal will actually have been in T_3, and at this point in time, they have the opportunity to switch to T_1. The soldiers cannot ascertain whether the criminal is in T_1 or T_3, and this cycle may continue forever.

Inductively, we have shown that n soldiers is insufficient for searching a Y_n-shaped palace. In remains to compute the size of Y_5. Since $|Y_1| = 4$ and $|Y_{n+1}| = 3|Y_n| + 1$, we may compute $|Y_2| = 13$, $|Y_3| = 40$, $|Y_4| = 121$, and $|Y_5| = 364$. Since $364 \leq 1000$, we have proven the necessity of 6 soldiers.

■

Source: Leningrad Math Olympiad

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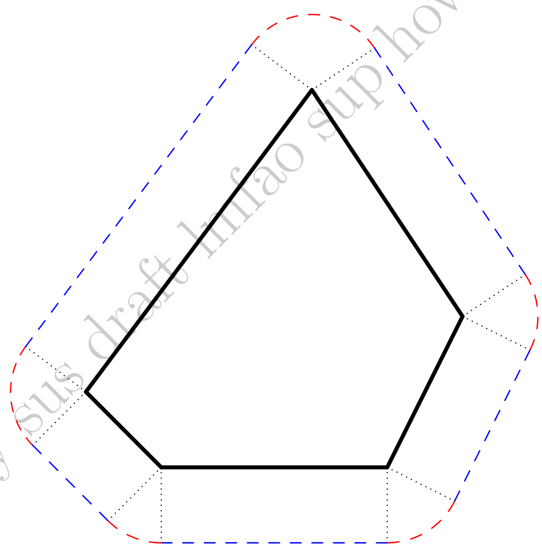
Solution 91

Part (a)

It is unlikely that you had trouble with this part, but I will spell it out anyways. If the radius of Gloria's house is R, then the fence is a circle of radius $R+1$. Their perimeters are $2\pi R$ and $2\pi(R+1)$, respectively, so the difference is 2π, no matter the value of R.

Part (b)

This part is much more interesting. It turns out that the fence is obtained by pushing each side of the polygon one foot "outwards", and then connecting the obtained segments via circular arcs.



The sum of the length of those "outwards" segments (marked in blue) is evidently the perimeter of the house. Hence the length of the fence is longer by exactly the sum of the lengths of the arcs, and it can be seen that they may combine to form a circle of unit radius. Thus the difference must still be 2π.

Part (c)

For appropriate sets $E \subseteq \mathbb{R}^2$, we denote by $P(E)$ its perimeter. For a line L and a set or point A, let us write $\text{proj}_L(A)$ for the orthogonal projection of A unto L. We will make use of the following incredibly useful theorem.

Theorem 1

Let $K \subseteq \mathbb{R}^2$ be a convex set and let (U, V) be a random unit vector whose angle is uniformly distributed. If L is the line through $(0, 0)$ and (U, V), then

$$\mathbb{E}\,\mathrm{length}(\mathrm{proj}_L(K)) = C \cdot P(K)$$

for some universal constant C.

Proof. Let $\varphi = (\varphi_1, \varphi_2) : [0, T] \to \mathbb{R}^2$ be a parametrization of the boundary ∂K. Then the perimeter of K may be expressed as

$$P(K) = \int_0^T \|\varphi'(t)\|\, dt.$$

Our goal is to massage the expected length into the right hand side.

Note that the point $\mathrm{proj}_L \varphi(t)$ stays on line L as t varies from 0 to T, and in doing so traces out an interval on L, visiting each point on that interval *twice!* (This is because it has to go back and forth.) This technically induces a parametrization of the projection of K, and so if we associate L with the real line then we can find the "length" of this curve to be:

$$2\,\mathrm{length}(\mathrm{proj}_L(K)) = \int_0^T \left| \frac{d}{dt}(U, V) \cdot \varphi'(t) \right| dt$$

To reiterate, we need to include the factor of 2 on the left hand side since we're double-counting the interval length when we view it as the length of the "curve traced out by the projection".

Anyways this is pretty nice because we can now mess with the right side to obtain

$$= \int_0^T \left| U\varphi_1'(t) + V\varphi_2'(t) \right| dt.$$

Taking the expectation gives

$$2\mathbb{E}\,\mathrm{length}(\mathrm{proj}_L(K)) = \mathbb{E} \int_0^T \left| U\varphi_1'(t) + V\varphi_2'(t) \right| dt.$$

And hey, expectations are just integrals, and the integrand is non-negative, so we can swap the expectation and integral by Tonelli's Theorem to arrive at

$$= \int_0^T \mathbb{E} \left| U\varphi_1'(t) + V\varphi_2'(t) \right| dt.$$

This expectation is actually quite nice to compute! In order to evaluate it, we switch gears to geometry: The expectation is the expected distance from the origin of the projection of $\varphi'(t)$ unto the line L. We can instead view this as taking a random point on a circle centered at $(0,0)$ and radius $\|\varphi'(t)\|$ and computing the expected absolute value of its x-coordinate. I don't actually need to compute this to prove the theorem, but I'll do it anyway. By four-fold symmetry this is just $\frac{2}{\pi}\int_0^{\pi/2}\|\varphi'(t)\|\cos\theta\,d\theta = \frac{2}{\pi}\|\varphi'(t)\|$. So our nasty integral is really just:

$$= \int_0^T \frac{2}{\pi}\|\varphi'(t)\|\,dt = \frac{2}{\pi}P(K)$$

Thus the theorem has been proven, and we have shown that $C = 1/\pi$. □

This beautiful result has many applications, and demolishing this problem is just one of them. Here is the argument: The projection of the fence unto a line L will always be 2 feet greater in length than the projection of the house unto L. Thus, when L is selected at random, then the expected lengths will differ by exactly 2 as well. By the Theorem, we conclude that the perimeters differ by exactly $2\cdot\frac{1}{C}$, where $\frac{1}{C} = \pi$.

■

Remarks: How can we be assured that the perimeter can be computed as $\int_0^T \|\varphi'(t)\|\,dt$ for an appropriate parametrization ϕ, which may not even be differentiable? In the unlikely event that you've bothered to ponder such a question, I shall answer it because I feel obligated to put my masters degree in mathematics to good use. Let K be a convex bounded set. It is clear that its boundary, ∂K, can be parametrized (i.e. "traced in a continuous way", by e.g. taking a ray emanating from inside K, marking its intersection with the boundary and "spinning" the ray), and roughly speaking, this means that ∂K is a *curve*.

Curves always have a notion of *length*, which is computed by using an increasingly large number of segments that approximate the curve. There is a theorem which states that basically every curve can be traced out with a careful selection of a parametrization $\varphi : [0,T]\to\mathbb{R}^2$ such that the length of the curve can be found by integrating the "speed" of the parametrization over time, i.e. $\int_0^T \|\varphi'(t)\|\,dt$.

It's possible for the "speed" to not exist at some points. For example, a square has sharp corners, and since a parametrization isn't traveling in any certain "direction" at such corners, we cannot define its "speed". But this is alright, since as long as there aren't too many such "corners", the integral can still be computed.

If you're wondering, we can also define perimeters for (most) arbitrary sets! For any (measurable) $E \subseteq \mathbb{R}^2$, we can define its perimeter as

$$\sup\left\{\int_E \operatorname{div}\phi\,dx : \phi\in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2) \text{ and } \|\phi\|_\infty \le 1\right\}.$$

For sufficiently nice sets E, it can be shown that this definition is consistent with simpler definitions of perimeter via the divergence theorem.

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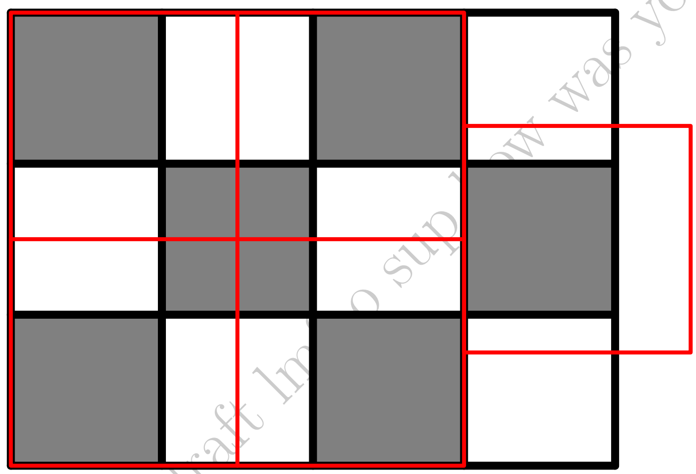
Solution 92

Part (a)

This is a classic Pigeonhole Principle argument. Partition the rectangle into six 1×2 dominoes. Then two of the 7 points must lie in the same domino. Since such dominoes have a diameter of $\sqrt{5}$, these two points must be at most $\sqrt{5}$ apart.

Part (b): Solution 1

The problem is still true if there were only 6 points in the rectangle.

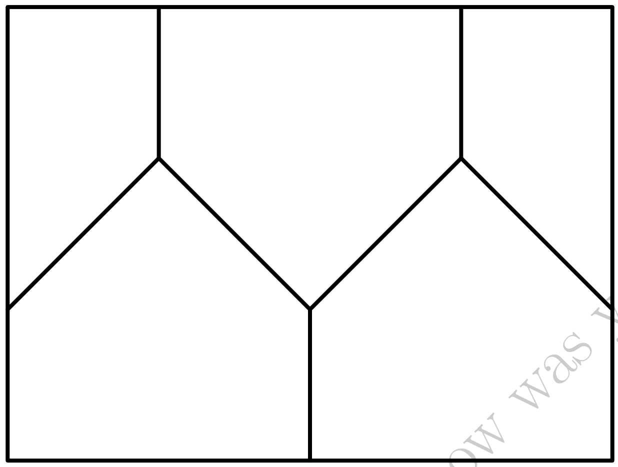


Subdivide the 4×3 rectangle into 1×1 cells, and color the cells like a checkerboard. Clearly the six points must lie in different cells, and the cells in which they lie cannot be orthogonally adjacent to one another. From some inspection, it follows that the six cells in which the six points lie must be all the same color — either all white or all black.

Without loss of generality we may assume that they lie in the black cells. Draw five red squares of side length $\frac{3}{2}$ as above. These squares cover the black cells, thus they contain the six points. By the Pigeonhole Principle, two of the points lie in a common red square. Since each red square has diameter $\frac{3}{2}\sqrt{2} < \sqrt{5}$, we are done.

Part (b): Solution 2 (From "asbodke")

The Pigeonhole Principle can be applied directly by using the clever partition shown below.



■

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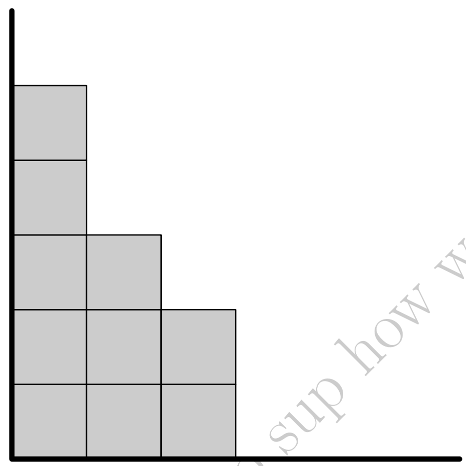
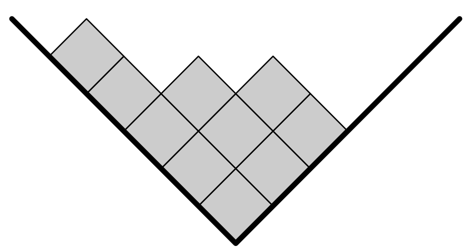
Solution 93

As suggested by the hints, we view the piles of stones as stacks of boxes, sorted from largest to smallest. For example, if $n = 4$ and the pile sizes are 3, 2, and 5, then we represent this configuration with the following diagram.

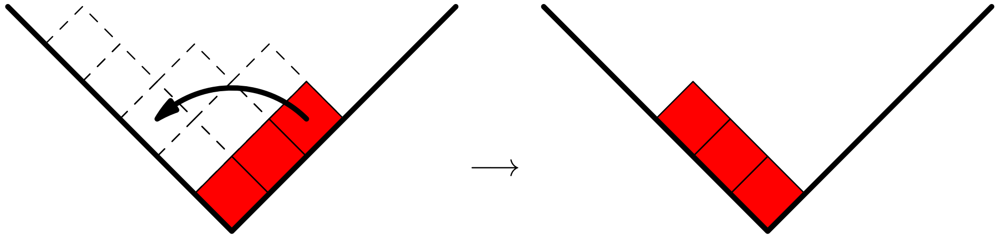


Each column represents a different pile, and the number of boxes in each column is the number of stones in the corresponding pile.

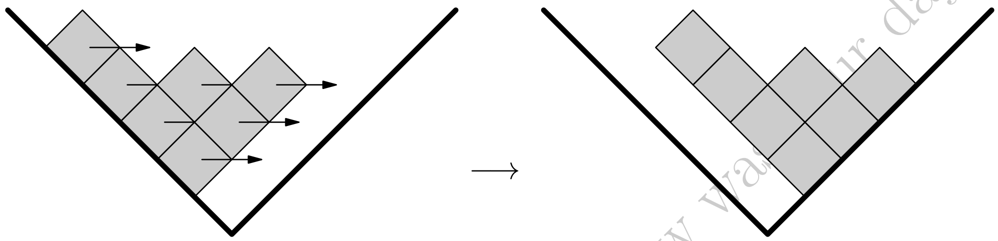
As the process described in the problem statement progresses, we will update the diagram in such a way that the columns will always be in decreasing order. If we do this, then a monovariant will appear, which is more easily seen if we rotate the diagram 45°!



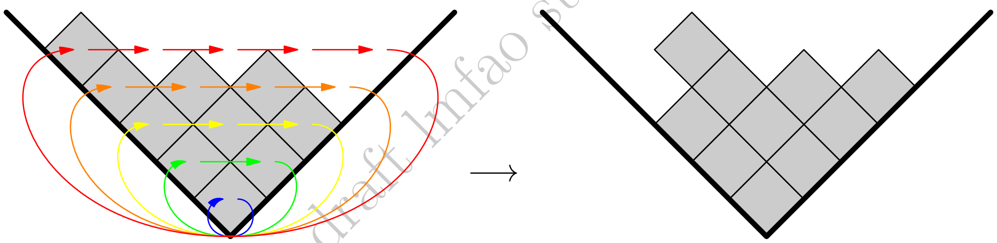
How does this diagram change when we take one stone from each pile to form a new one? One way to represent this process is via two steps. For the first step, we take the red boxes shown below (which represents one stone from each pile) and "rotate them over" to the other side (to form the new pile),



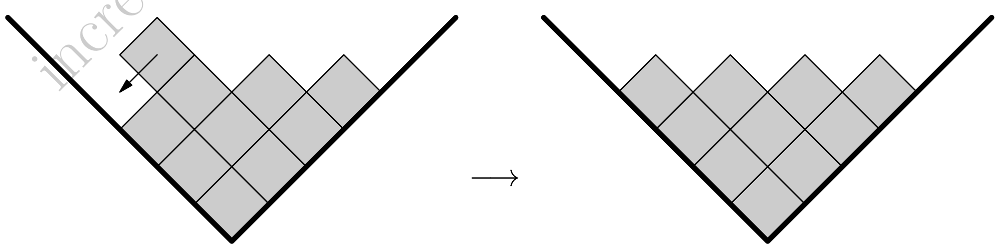
and then we take all the other boxes (what remains of the other piles) and shift them "one slot" to the right.



In sum, the first step of the process can be viewed as a set of simultaneous "cycles".



For the second step, we simply sort the piles' sizes to be in decreasing order. In terms of the diagram, this would entail rearranging the columns to be in order of decreasing height. However, it's more revealing to view the sorting process as *letting all boxes slide down*!



This hence completes the visual representation of the process. Our goal is to show that upon repeating these two steps, the arrangement of boxes eventually forms the pattern shown above: a "perfect staircase".

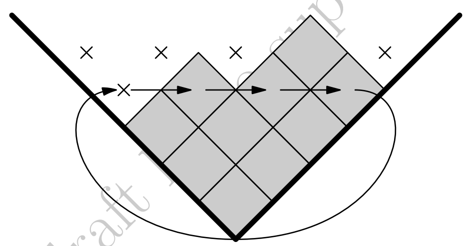
With this visual representation in hand, the proof is incredibly slick. We make two observations:

1. Under the first step of the process, every box's altitude remains the same. For fancy points, we can say that this implies that **the total gravitational potential energy of the boxes does not change.**

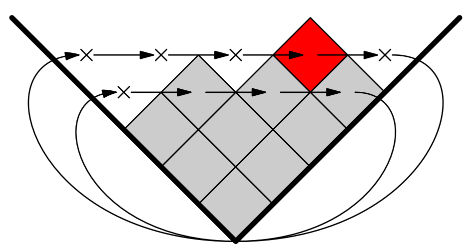
2. Under the second step, the altitudes of boxes can only decrease. In other words: Since we let gravity act on the boxes, **the total gravitational potential energy energy can only decrease** (though it could stay the same).

Hence the total gravitational potential energy is a decreasing monovariant. The desired "perfect staircase" arrangement is clearly the arrangement with the minimum total gravitational potential energy. Hence, it remains to prove that if the arrangement is not a "perfect staircase", then the total gravitational energy must eventually decrease.

If not, then boxes never slide down, so the second step does not move any boxes. Hence only the first step moves boxes around, in the "cycles" shown on the previous page. Find the first such "cycle" of boxes from the bottom that isn't full, i.e. has a space not occupied by a box.



Since the diagram isn't a "perfect staircase" by assumption, the cycle *above* this one must have a box, which we shall color red.



These two cycles have lengths r and $r+1$ for some integer r, so in particular their lengths are relatively prime. Thus, with enough iterations of the cycling, the red box must eventually hover over the empty space which we know to exist in the cycle below it. When this happens, gravity will pull it down to decrease the potential energy of the system, contradicting the assumption that this never occurs.

■

Remarks: I stole the beautiful idea behind this proof from the expository paper https://arxiv.org/pdf/1503.00885.pdf of V. Drensky, which contains more results.

Source: This is called Bulgarian Solitaire.

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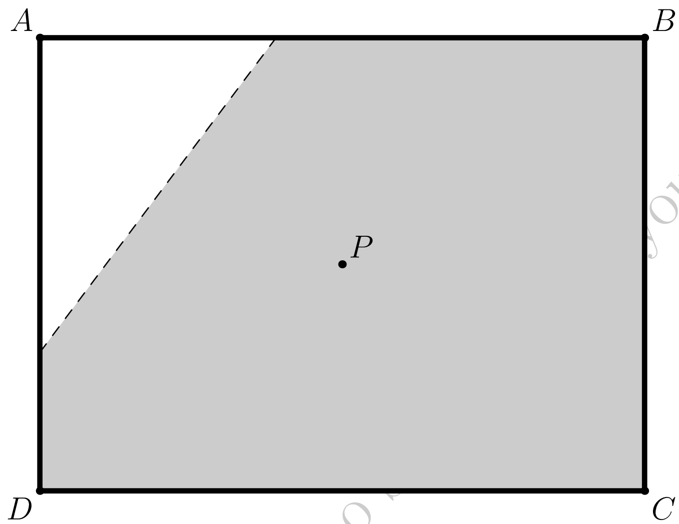
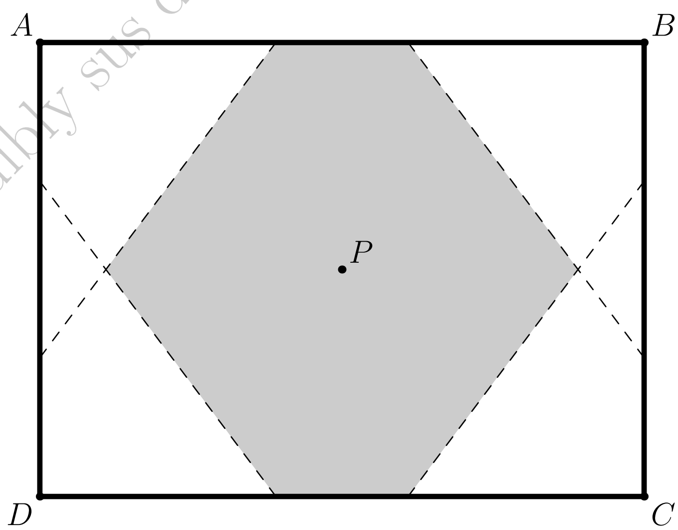
Solution 94

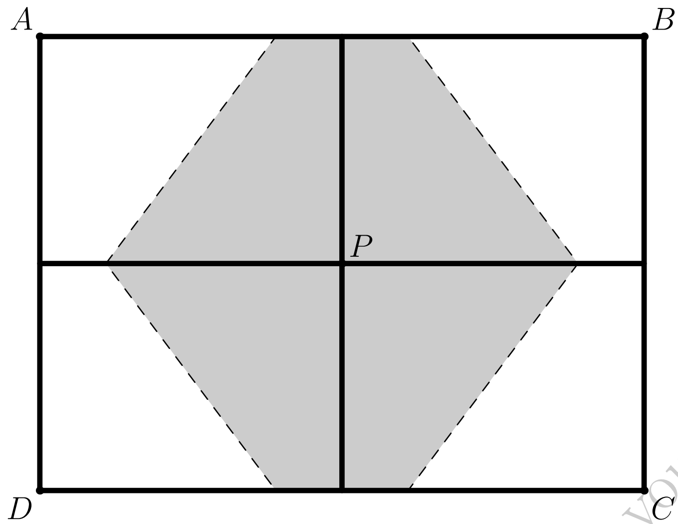
The side lengths of the rectangle are irrelevant. Let the vertices of the rectangle be A, B, C and D. Let the center be P. Then the set of points inside the rectangle that are closer to P than A is given by cutting the rectangle along the perpendicular bisector of $\overline{AP}$.



Arguing in the same way for B, C, and D, we find that the set of points inside the rectangle that are closer to P than any of A, B, C, or D is given by the shaded region below.



The desired probability is given by the fraction of the rectangle's area that is taken up by the shaded region. To determine this fraction, divide the rectangle into quarter rectangles as shown.



Exactly half of each of these quarter rectangles are shaded! This is because each of the four dashed lines are perpendicular bisectors. We conclude that the whole rectangle is exactly half-shaded. Thus the probability is $\boxed{\frac{1}{2}}$.

■

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Solution 95

Part (a)

Let $E = \{x \in [0,1] : x \leq f(x)\}$ and $x_0 = \sup E$.

Claim 1: $x_0 \leq f(x_0)$

This follows from writing $x \leq f(x) \leq f(x_0)$ for $x \in E$, and then taking the sup on the LHS to get $x_0 \leq f(x_0)$.

Claim 2: $x_0 \geq f(x_0)$

From $x_0 \leq f(x_0)$ we have by the increasing condition that $f(x_0) \leq f(f(x_0))$, so $f(x_0) \in E$ by definition of E, hence $f(x_0) \leq x_0$ by definition of x_0.

From the two claims, we have $f(x_0) = x_0$, so x_0 is a fixed point.

■

Part (b)

Let $\mathcal{F} = \{E \in P(X) : E \subseteq f(E)\}$ and $E_0 = \bigcup_{E \in \mathcal{F}} E$.

Claim 1: $E_0 \subseteq f(E_0)$

This follows from writing $E \subseteq f(E) \subseteq f(E_0)$ for $E \in \mathcal{F}$, and then taking the union on the LHS $E_0 \subseteq f(E_0)$.

Claim 2: $E_0 \supseteq f(E_0)$

From $E_0 \subseteq f(E_0)$ we have by the increasing condition that $f(E_0) \subseteq f(f(E_0))$, so $f(E_0) \in \mathcal{F}$ by definition of $\mathcal{F}$, hence $f(E_0) \subseteq E_0$ by definition of E_0.

From the two claims, we have $f(E_0) = E_0$, so E_0 is a fixed point.

■

Remarks: These two solutions are basically the same.

Source: Probably a classic exercise

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Solution 96

Let $f(n)$ be the number of digits of n that are at least 5. By the hint, we have that the desired sum is rational exactly when $\sum_{n=0}^{\infty} \frac{f(2^n)}{2^n}$ is. Now note that we may express $f(m)$ as the sum

$$f(m) = \sum_{k=0}^{\infty} 1_{\text{the } 10^k \text{ place is } \geq 5}(m),$$

so the desired sum is

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 1_{\text{the } 10^k \text{ place is } \geq 5}(2^n).$$

But

$$\begin{aligned}\text{The } 10^k \text{ place of } 2^n \text{ is } \geq 5 &\iff \frac{2^n \bmod 10^k - 2^n \bmod 5 \cdot 10^{k-1}}{5 \cdot 10^{k-1}} = 1 \\ &\iff \frac{2^n \bmod 10^k - \frac{2^{n+1} \bmod 10^k}{2}}{5 \cdot 10^{k-1}} = 1.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 1_{\text{the } 10^k \text{ place is } \geq 5}(2^n) &= \frac{1}{5 \cdot 10^{k-1}} \sum_{n=0}^{\infty} \frac{2^n \bmod 10^k}{2^n} - \frac{2^{n+1} \bmod 10^k}{2^{n+1}} \\ &= \frac{1}{5 \cdot 10^{k-1}},\end{aligned}$$

and this is clearly rational once we sum over k.

■

Remarks: I'm too lazy to perturb these computations to get the actual answer, but supposedly the sum comes out to $\frac{10}{9}$.

One person misread the problem as determining whether $\sum_{n=0}^{\infty} \frac{o(n)}{2^n}$ is rational, and apparently the solution to this is also nice.

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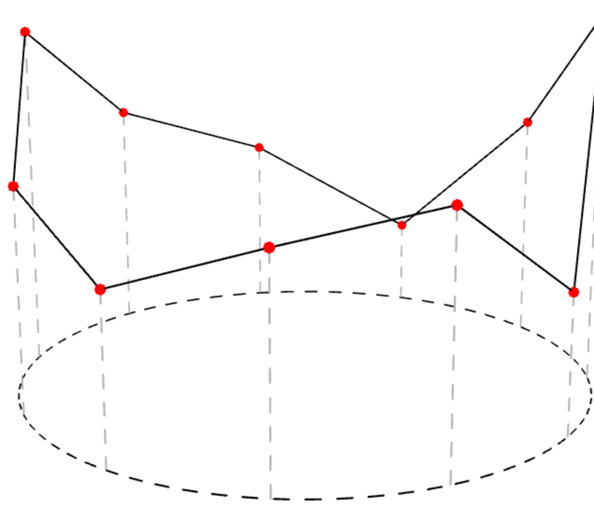
Solution 97

Imagine connecting the tops of everyone's heads with line segments, as shown. This creates a periodic "line graph".



Then each occurrence of "taller" coincides with a "peak" along the line graph, whereas each occurrence of "shorter" coincides with a "valley". The key insight from this visualization is that **between any two consecutive valleys, there exists exactly one peak**! This proves that the number of valleys and peaks are equal.

We deduce that exactly 5 people said "shorter", because 5 people said "taller". Hence, the number of people who say "in-between" is $25 - 5 - 5 = \boxed{15}$.

■

Remarks: Did you spot the joke in the problem statement?

Source: Math Hour Olympiad

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Solution 98

The answer is 10 inches.

The turtle cannot crawl more than 10 inches

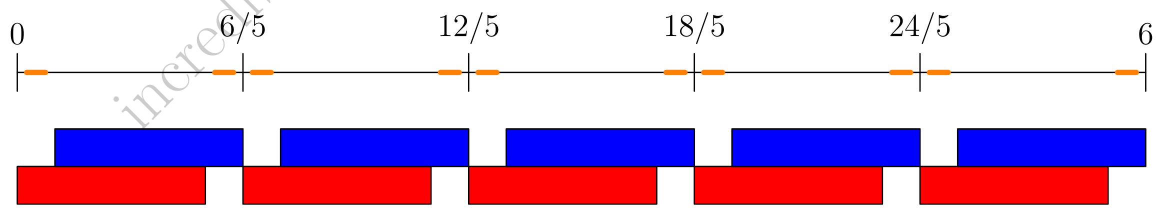
Model the turtle's crawling time via the interval $[0, 6]$. Since the turtle is watched at all times, there exists an enthusiast watching during $[0, 1]$ and another during $[5, 6]$. We claim that we may select 8 other enthusiasts whose watching intervals cover the rest of $[0, 6]$. To see this, select the latest enthusiast that starts watching at a time in $(0, 1]$. If their interval is $[a, a + 1]$, then $0 < a \leq 1$ and a is maximal (in the sense that there is no enthusiast that watches $[a', a' + 1]$ with $0 < a' \leq 1$ and $a' > a$).

Now select another enthusiast that watches the interval $[b, b + 1]$ where $1 < b < a + 1$ (which exists by maximality of a; indeed, if there were no such enthusiast, then a small moment of time after a would be left unwatched). These two watchers, together, cover the interval $[1, 2]$. By repeating this reasoning, we may find $2 \times 3 = 6$ more enthusiasts that cover the three intervals $[2, 3]$, $[3, 4]$, and $[4, 5]$.

We have hence found 10 enthusiasts that watch the whole interval. During each of their watching intervals, the turtle can move at most one inch. Thus a seemingly rough bound on the most the turtle can move is 10 inches.

The turtle could crawl 10 inches

In fact, 10 inches can be obtained. See the diagram below. The red and blue rectangles represent the watching-intervals of ten enthusiasts. During each of the 10 marked orange periods of time, let the turtle move 1 inch forward, and otherwise let the turtle stay still.



■

Remarks: The solution does not change if we interpret time intervals as being open rather than closed. A possible concern is that the solution breaks if there are infinitely many turtle enthusiasts observing the turtle (in which case we are not guaranteed a maximum). In this case, we can modify the solution as follows. Fix $\epsilon > 0$. Then, since the turtle's movement is continuous on a compact interval, it is in particular uniformly continuous, so there exists $\delta > 0$ such that the turtle never moves more than ϵ within a time interval of length δ. Now extend all enthusiasts' watching intervals by δ in each direction to form open intervals of length $1 + 2\delta$. By compactness of $[0, 6]$ we may select a finite number subcollection of enthusiasts whose extended watching intervals cover $[0, 6]$. Rerunning the logic of the solution, we find 10 of these enthusiasts whose watching intervals cover $[0, 6]$. It follows that the turtle cannot crawl more than $10 + 20\epsilon$ inches. Sending $\epsilon \to 0^+$ gives the expected conclusion.

Source: Mathematical Circles (Russian Experience)

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Solution 99

The possible values for t are $t = 0$ and $t = 1/n$ for $n \in \mathbb{N}$.

Let the continuous and increasing function $f : [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$ represent the turtle's movement. Observe that a value of t works if and only if for every such function f, we can find x such that $f(x + t) = f(x) + t$.

The claimed values of t work

Clearly $t = 0$ works. As for $t = 1/n$, let us suppose that for some f, there does not exist $x \in [0, 1 - \frac{1}{n}]$ for which $f(x + 1/n) = f(x) + 1/n$. Then the continuous functions $f(x + 1/n)$ and $f(x) + 1/n$ never intersect in $[0, 1 - \frac{1}{n}]$, hence one of these functions is strictly greater than the other for all x in $[0, 1 - \frac{1}{n}]$.

The first case is that $f(x + 1/n) > f(x) + 1/n$ for all $x \in [0, 1 - \frac{1}{n}]$. If so, then

$$1 = f(1) > f\left(\frac{n-1}{n}\right) + \frac{1}{n} > f\left(\frac{n-2}{n}\right) + \frac{2}{n} > f\left(\frac{n-3}{n}\right) + \frac{3}{n} > \ldots > f(0) + \frac{n}{n} = 1,$$

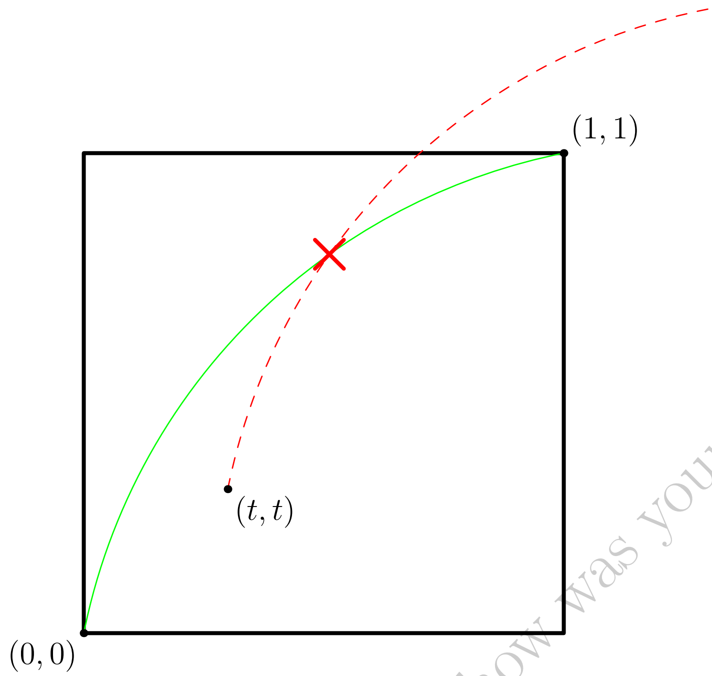
contradiction. If instead $f(x + 1/n) < f(x) + 1/n$ for all $x \in [0, 1 - \frac{1}{n}]$, then we accordingly obtain

$$1 = f(1) < f\left(\frac{n-1}{n}\right) + \frac{1}{n} < f\left(\frac{n-2}{n}\right) + \frac{2}{n} < f\left(\frac{n-3}{n}\right) + \frac{3}{n} < \ldots < f(0) + \frac{n}{n} = 1,$$

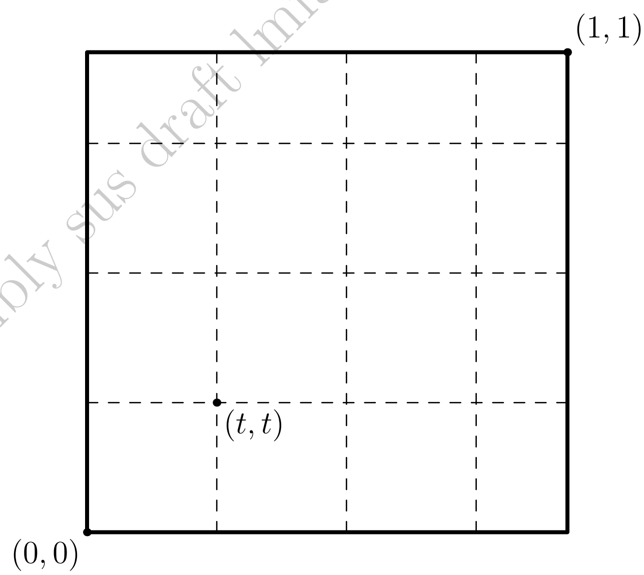
again a contradiction.

All other possible values of t fail

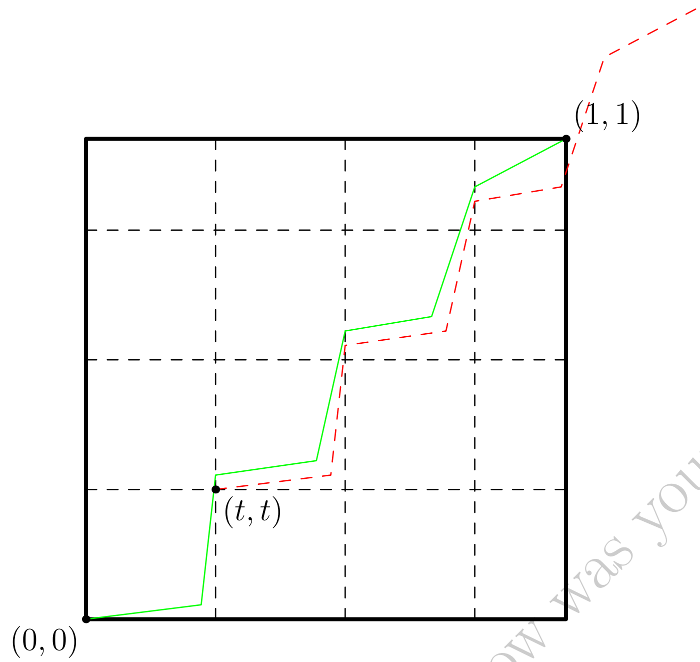
Suppose $t \neq 0$ and $t \neq 1/n$ for any $n \in \mathbb{N}$. We wish to construct an f for which $f(x + t) \neq f(x) + t$ for all $x \in [0, 1 - t]$. Equivalently, we want $f(x) \neq f(x - t) + t$ for all $x \in [t, 1]$, so we want the graphs of $f(x)$ and $f(x - t) + t$ to never intersect. The graph of $f(x - t) + t$ is that of $f(x)$, except it is shifted horizontally to the right by t and vertically upwards by t. We can hence think of this problem as the following "game": We start at $(0, 0)$ and a doppelganger at (t, t) mimics our movements. The goal is the reach $(1, 1)$ without ever crossing the doppelganger's path.



To achieve this, begin by drawing gridlines at all multiples of t, subdividing the square into $t \times t$ cells and some residual rectangles along the upper and right edges.



If we wish to dodge the doppelganger, we must always stay above it or always stay below it. The choice does not matter by symmetry, so let us choose to stay above it. To "make room" above the doppelganger, the idea is to stay *low* for as long as possible before rising above (t, t). Repeating this idea for getting above $(2t, 2t)$, $(3t, 3t)$, etc. will successfully get us to the last gridline, at which point we'll be able to make an unobstructed beeline for $(1, 1)$, provided that we've made a sufficient amount of room for ourselves.



Note that the importance of t not "dividing 1 evenly" is so that the last segment can be drawn. Intuitively, since the last "square" in the upper-right corner is smaller than the others, we can plan the path so that the doppelganger is forced to go under $(1, 1)$.

A more explicit construction is given by Edward Hou: We can take

$$f(x) = x + \frac{t}{4}\left(\sin^2\left(\frac{\pi x}{t}\right) - x\sin^2\left(\frac{\pi}{t}\right)\right).$$

See https://www.desmos.com/calculator/ijwp5vbmyq for an interactive plot.

■

Source: Stolen from AoPS

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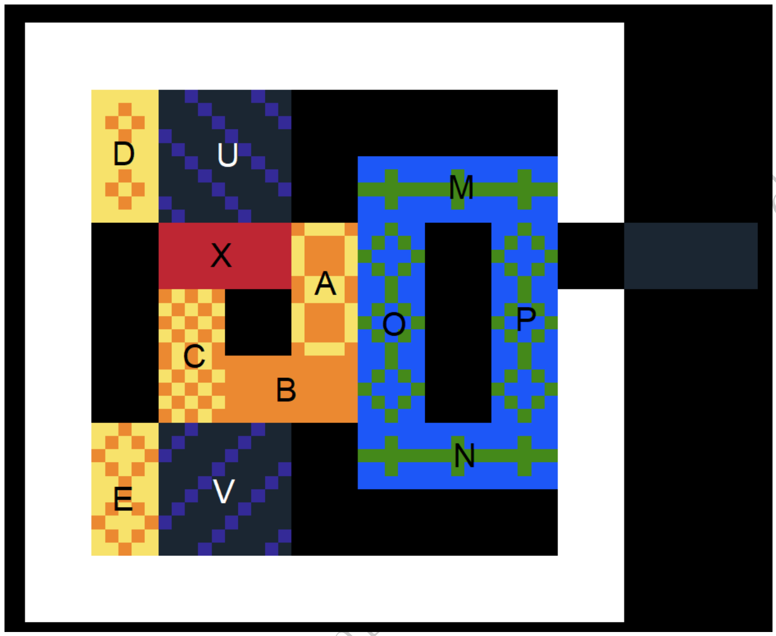
Solution 100

Let's label the trains with the following letters.



Let's make space for N to move all the way to the left via the following sequence:

1. A up 2
2. X right 1
3. C up 1
4. E up 3
5. B left 2
6. V right 1, up 2

Once we move N to the left, we can move O and P down so that X is free.

7. N left 3
8. O down 1
9. P down 1

10. X out!

■

Source: Scott Kim, ThinkFun Railroad Rush Hour

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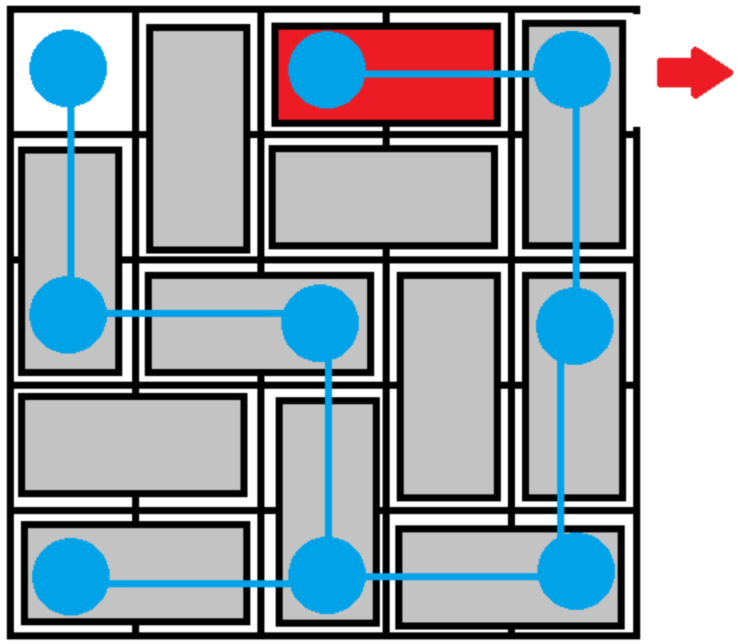
Solution 101

Let the squares at "odd coordinates" be the vertices of a graph, and connect two adjacent such squares with an edge if and only if there is some train running between them.

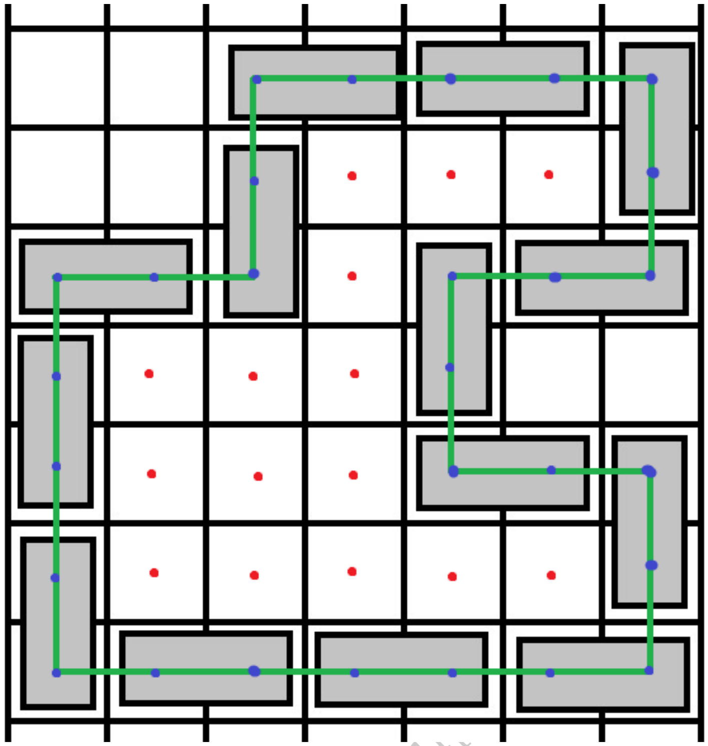


This forms a graph with 1011^2 vertices and $1011^2 - 1$ edges. This is because there is a 1-to-1 correspondence formed by taking any of the $1011^2 - 1$ vertices whose square is occupied (i.e. all except the one with the empty cell — hence why it is important that a corner is empty!) and corresponding it to the train running through it.

Suppose we can prove that the graph is a tree. Then there is a unique path from the empty cell to the top-right cell where the red block is. By construction of the graph we must be able to push each train along the edges of the path until we free up the square in front of the red train, solving the puzzle.

To show that the graph is a tree, note that since we've already shown that the number of vertices exceeds the number of edges by exactly 1, it remains to prove that the graph has no loops.

A loop would look something like this.



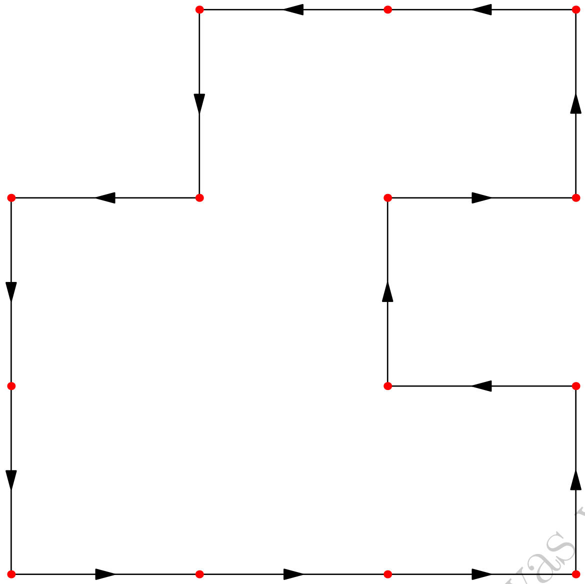
The inside of the loop must be filled with 1×2 trains, and since the single empty cell lies in a corner cell, that empty cell won't be in this interior. So the area enclosed within the loop must be even. We claim that this cannot be the case.

Consider the lattice grid formed by the centers of all cells. The lattice points that lie on the trains connect to form a polygon whose sides lengths are all even. By Pick's Theorem applied to this polygon, we have

$$A = I + \frac{B}{2} - 1,$$

where A is the area enclosed by the polygon, I is the number of lattice points strictly inside the polygon (colored red), and B is the number of lattice points on the boundary of the polygon (colored blue). Our goal is to show that I is odd.

To that end, we simply must argue that A is even and B is divisible by 4. That A is even follows quickly from the fact that all side lengths of the polygon are even (which entails that it can be subdivided into 2×2 squares). To show that B is divisible by 4, start by scaling down the polygon by a factor of 2.

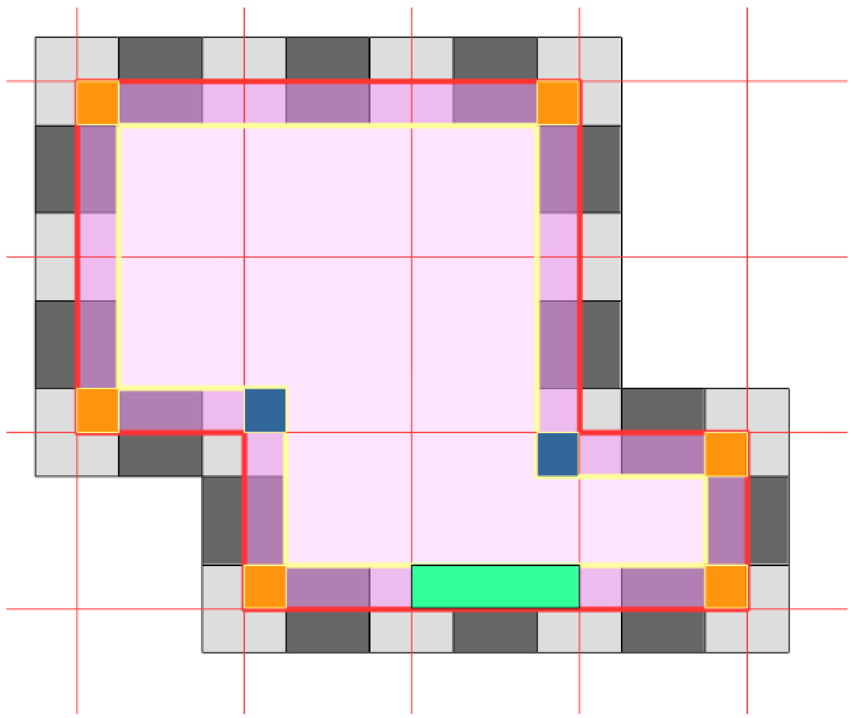


We're done if we can prove that the scaled-down polygon's perimeter is divisible by 2. At last, we can show this without further reduction. As suggested by the arrows, we see that if we were to traverse the boundary of the polygon, then the number of times we move upwards is the same as the number of times we move downwards, so the total vertical length must be even. Similarly, the total horizontal length must be even. This concludes the proof.

■

Remarks: A similar problem that uses the same key claim (that a loop of dominoes must enclose a region of odd area) appeared in the Math Hour Olympiad. Thanks to Dr. Jonah Ostroff for kindly providing a snippet of the official solution that proves this crucial result without Pick's Theorem.

> The interior is built from 2×2 squares and so its area is a multiple of 4. The original interior area is smaller, because it does not include the "inner half" of the cycle dominoes. Assuming a clockwise cycle, if the cycle contains N dominoes, has T CCW turns and $T + 4$ CW turns, then each of the N line segments contributes area 1 to this "inner half" (half the area of a 2×1 domino), except when a CW turn double-counts area $1/4$ or CCW turn under-counts area $1/4$. Thus, the total "inner half" area is
>
> $$2M - \frac{T+4}{4} + \frac{T}{4} = 2M - 1.$$
>
> Therefore, the original interior area is $4K - (2M - 1)$, which is odd.



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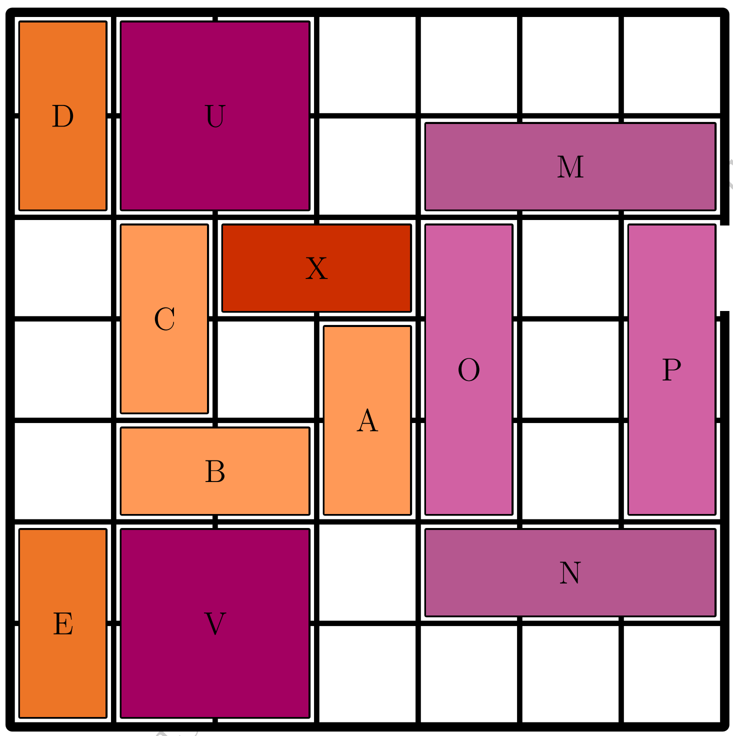
Solution 102

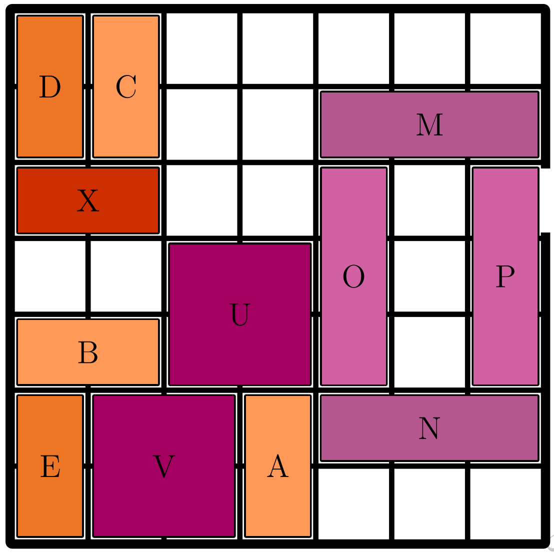
Label the trains as follows.



The X, A, B, and C trains form a "cycle". The difference between this problem and Problem 100 is that the cycle is "oriented" the other way. The key insight is that **the orientation of the cycle determines whether U or V can be moved out easily**. In Problem 100, when the cycle was oriented the other way, V could easily move upwards. Thus, by symmetry, it must be the case that U can easily be moved downwards in the current problem. With this observation, we are motivated to try moving U out of the way so that we have room to shuffle the X, A, B, and C trains around and **reverse the cycle**.

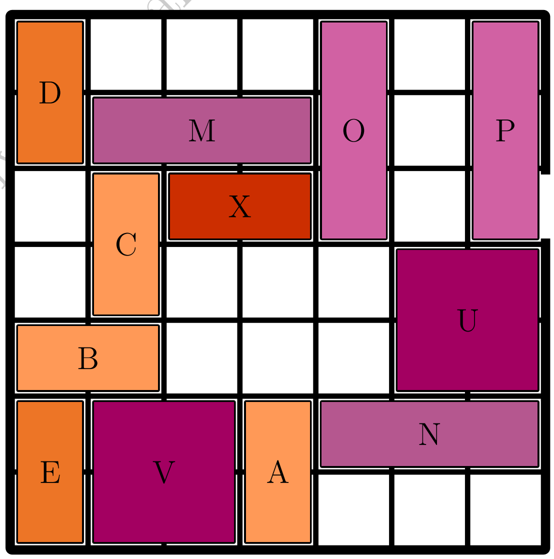
Let's start by getting U deeper down.

1. U right 1 2. C up 2 3. X left 2 4. B left 1
5. A down 2 6. U right 1 down 3

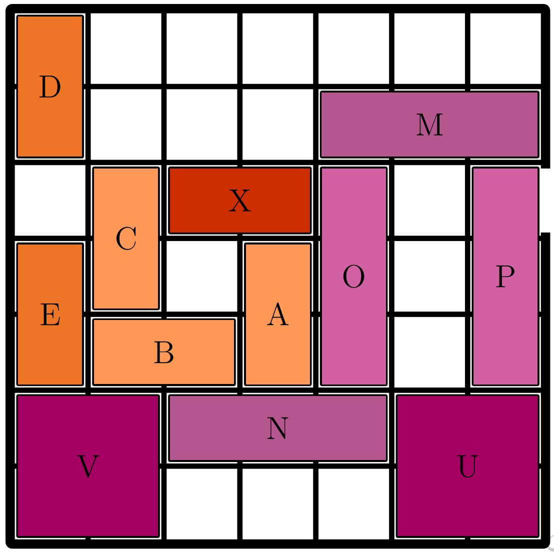


Now, notice that we can drive U into the bottom-right corner.

7. X right 2 8. C down 2 9. M left 3 10. O up 2
11. P up 2 12. U right 3

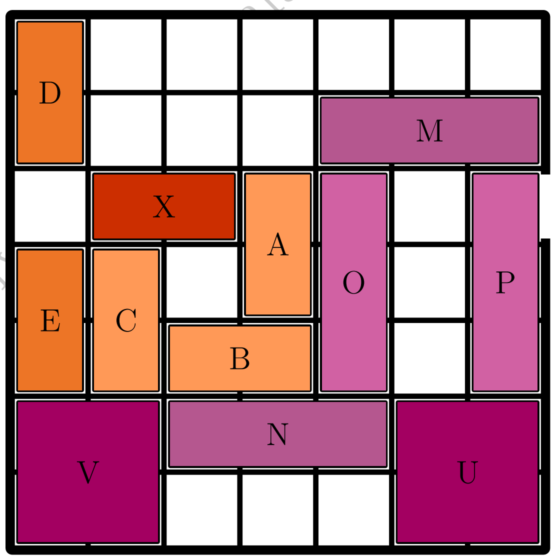


13. A up 2 14. B right 1 15. E up 2 16. V left 1
17. N left 2 18. U down 2 19. O down 2 20. P down 2
21. M right 3



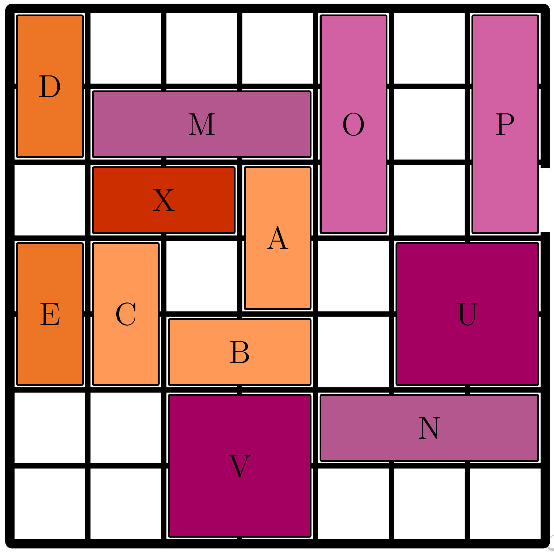
We've successfully made room to reverse the cycle!

22. C up 2 23. X left 1 24. A up 3 25. X right 1
26. B right 1 27. C down 3 28. X left 1 29. A down 2

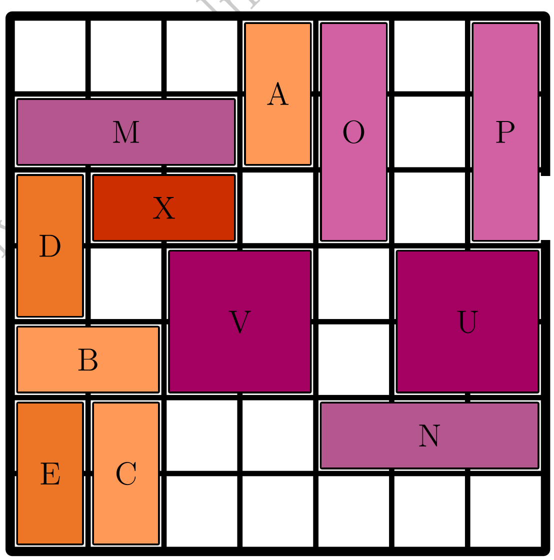


From the intuition outlined at the start, we now hope that V can be moved around more easily. This is indeed true — we can move V upwards enough to let it join U in the bottom-right corner.

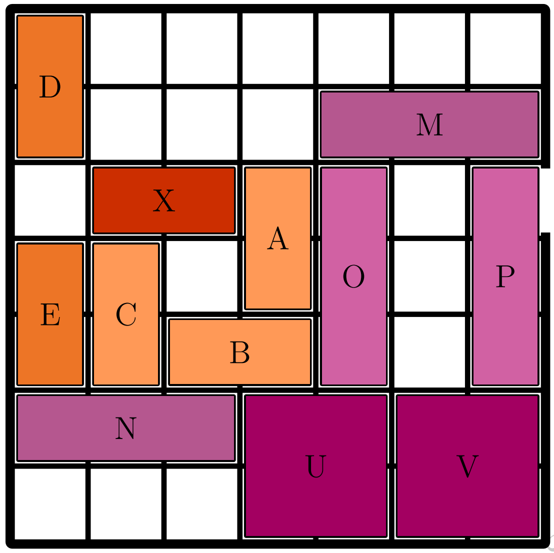
30. M left 3 31. O up 2 32. P up 2 33. U up 2
34. N right 2 35. V right 2



36. *E* down 2 37. *C* down 2 38. *B* left 2 39. *D* down 2
40. *M* left 1 41. *A* up 2 42. *V* up 2

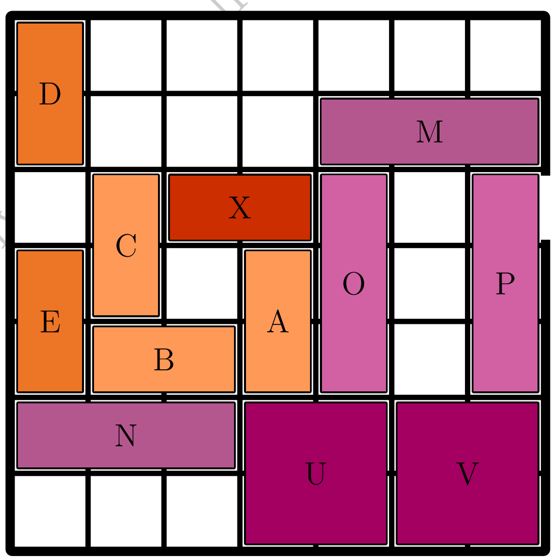


43. *N* left 2 44. *U* down 2 45. *V* right 3 46. *A* down 2
47. *B* right 2 48. *C* up 2 49. *O* down 2 50. *M* right 3
51. *D* up 2 52. *E* up 2 53. *N* left 2 54. *U* left 2
55. *V* down 2 56. *P* down 2 57. *M* right 1



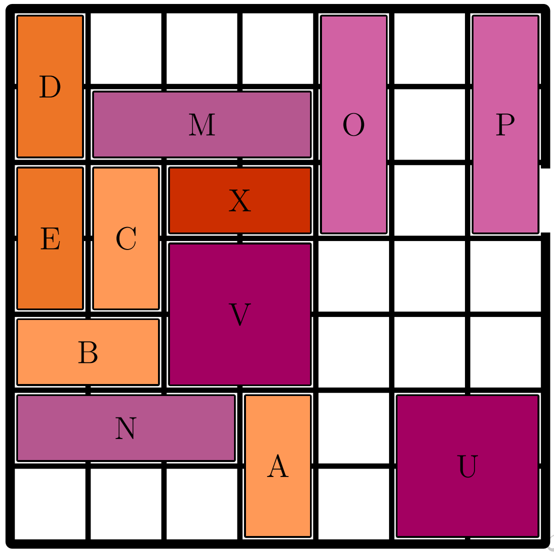
We arrive at what seems to be a dead end. Amazingly, the key to making more progress is to reverse the cycle *again*!

58. *A* up 2 59. *X* right 1 60. *C* up 3 61. *X* left 1
62. *B* left 1 63. *A* down 3 64. *X* right 1 65. *C* down 2

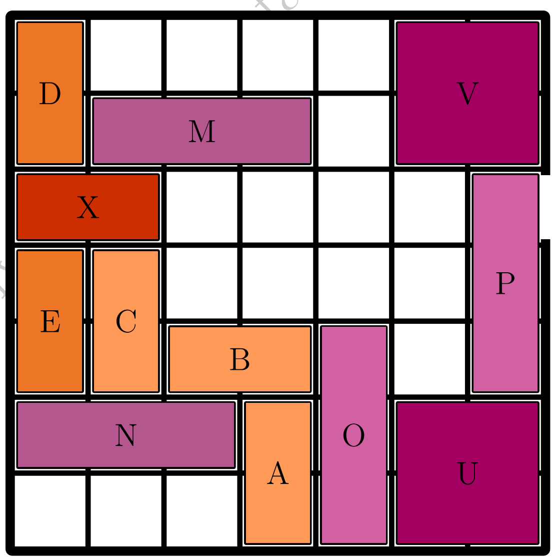


This allows us to move *V* to new places.

66. *M* left 3 67. *O* up 2 67. *P* up 2 68. *V* up 2
69. *U* right 2 70. *N* right 2 71. *E* up 1 72. *N* left 2
73. *A* down 2 74. *B* left 1 75. *V* left 3

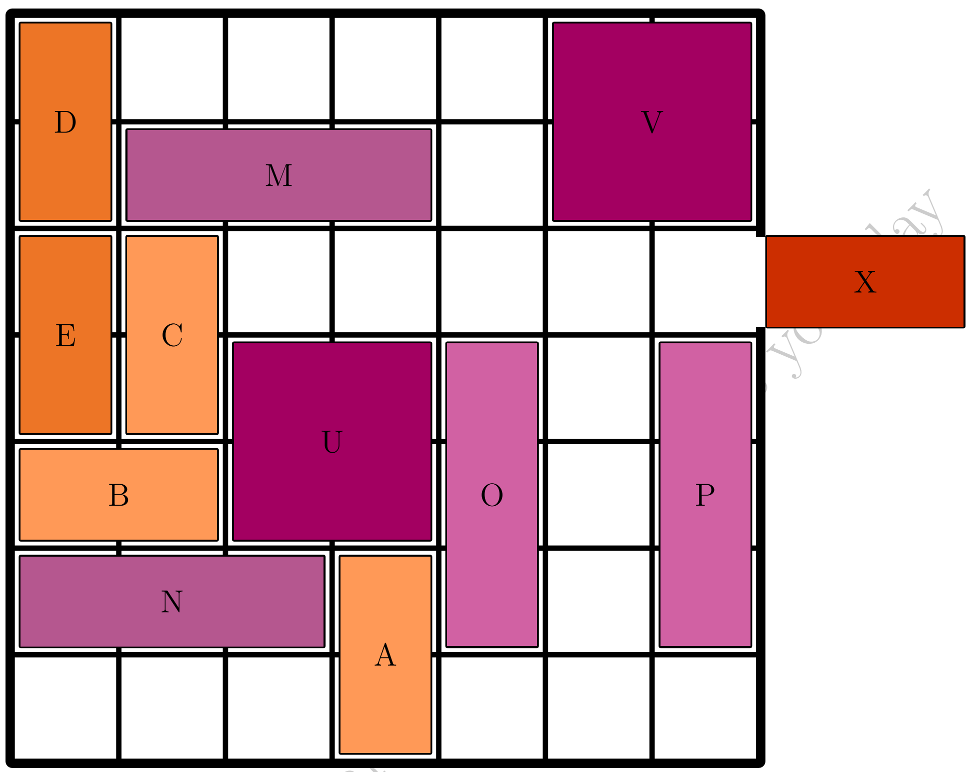


76. *O* down 4
77. *P* down 2
78. *X* right 2
79. *M* right 3
80. *V* up 3
81. *B* right 2
82. *E* down 1
83. *C* down 1
84. *X* left 4
85. *V* down 2
86. *M* left 3
87. *V* right 2 up 2 right 1



At this point, we can conclude that a solution exists as follows: If we were to move *O* up 2, *X* right 2, *C* up 1, *B* left 1, *A* up 2, *N* right 1, and *E* down 2, then we reach a symmetrical position except for the cycle. However, it's easy to reverse the cycle without changing anything. So by an argument of symmetry, we can reduce to Problem 100. The solution this generates is quite long, but fortunately there is a short finish from the current position.

88. O up 4 89. X right 2 90. E up 1 91. C up 1
92. B left 2 93. U left 1 up 2 left 2 94. O down 3
95. P down 1 96. X out!



■

Remarks: I was lucky enough to have encountered this puzzle as a toddler. Today, I still think this is the best sliding puzzle to ever exist. Who could possibly expect that the subtle difference between Problem 100 and Problem 102 could make such a devilish disparity in difficulty?

Sadly, the product in which these puzzles appear is no longer in stores. Thus, even though this isn't a math problem, I wanted to include this beautiful creation in the POTD collection to help give it the attention and renown that it deserves. I hope you'll forgive me.

Video Solution: https://youtu.be/UrJShUaJvpQ

Source: Scott Kim, ThinkFun Railroad Rush Hour

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Solution 103

The answer is 150. One can arrive at this easily by focusing on just two bottles of beer. We claim that we can get exactly three drinks from these bottles. Indeed, we can drink these two bottles of beer, take an empty glass from our friend, trade the three empty bottles in for a full beer, drink it (that's the 3rd drink!), and then return that empty bottle to our friend. We're left with nothing! Doing this for every pair, we end up with

$$\frac{3}{2} \times 100 = 150$$

drinks that have been drunk.

To see that we cannot do better, let's begin by making a small simplification: View our friend as a mechanism for allowing us to have a negative number of empty bottles, as long as in the end the number of empty bottles is non-negative.

Suppose that over the course of the alcohol-fueled night, we drink A beers and execute the trading operation B times. Every time we drink a beer, the number of empty bottles increases by 1, and every time we do the trading operation we lose 3 empty bottles. So we have $A - 3B$ empty bottles. To pay back the friend, this must be a non-negative quantity, so $A \geq 3B$.

On the other hand, we cannot drink more beers than 100 plus the number of beers obtained from trading, so $A \leq 100 + B$. Hence

$$3B \leq A \leq 100 + B,$$

which gives $B \leq 50$. Hence $A \leq 100 + B \leq 150$.

■

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Solution 104

The procedure can be done if and only if n is even.

It is possible when n is even

It suffices to prove that we can take any two pancakes and have them flipped in-place without changing anything else. Consider two pancakes A and B. Then, ignoring all other pancakes, we can notate the configuration as

$$A\ B\ \_,$$

where $\_$ denotes an empty pan. Specifically it is currently my friend's empty pan. We first flip A into the 3rd pan (use overhead bars to denote the flipped state).

$$\_\ B\ \overline{A}$$

Then we flip B into the first pan.

$$\overline{B}\ \_\ \overline{A}$$

Then we flip A back into the second pan. And, well, you kinda just keep going since there's only one move at each step that makes any progress.

$$\overline{B}\ A\ \_$$

$$\_\ A\ B$$

$$\overline{A}\ \_\ B$$

$$\overline{A}\ \overline{B}\ \_$$

Done! Repeating this procedure, we can flip two pancakes in-place at a time.

n *must* be even for the procedure to be possible

The key observation is that the number of flipped pancakes has the same parity as the executed permutation, where we view a pancake flip as a swapping of two "pancakes": one actual pancake and one "phantom pancake" that always resides on the empty pan. Indeed, with every move, the number of (not-phantom) flipped pancakes changes by 1, and the parity of the permutation changes from even to odd and vice versa.

If we are able to reach the goal, then since the permutation of the pancakes in the goal situation is the identity, which is even, it must follow that the number of flipped pancakes needs to be even by the observation. All pancakes are flipped, so n needs to be even.

∎

Remarks: The original problem's setting consisted of a tape-recorder, $n+1$ reels and n tapes. Unfortunately I'm not quite ancient enough to make sense of this. I hope you agree that pancakes are a much more fun and tastier context for the mathematics at hand.

Source: Mathematical Circles (Russian Experience)

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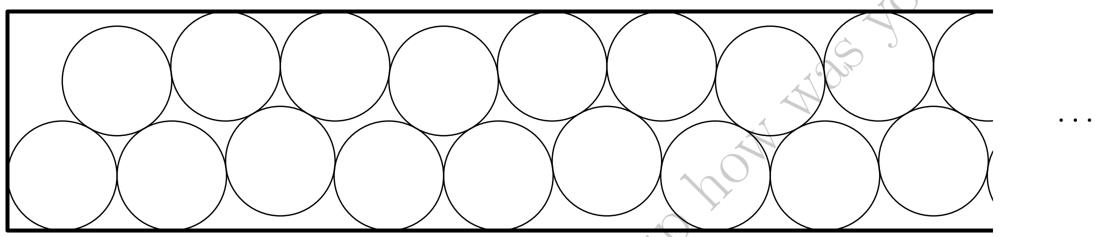
Solution 105

Part (a)

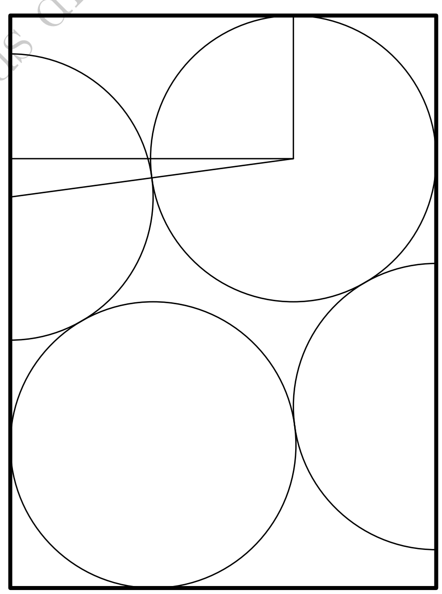
The naive "square packing" works just fine.

Part (b)

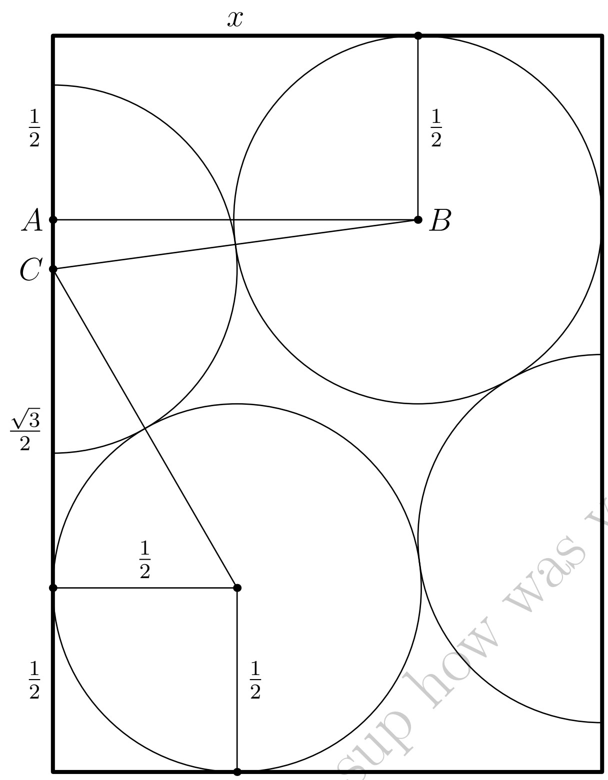
As ludicrous as it sounds, you can fit 401 coins in the box. The scheme involves alternating between "upright" and "upside-down" triangles of three coins, as shown.



Computing the number of coins this scheme allows us to fit is an instructive exercise in elementary geometry. First, we determine the period of the packing, i.e. how long it takes for the pattern of coins to repeat. The following diagram shows half a repeating segment.



Connecting centers with points of tangency and labelling various lengths gives us the following diagram.



We have that $AB = x$ and $BC = \frac{1}{2} + \frac{1}{2} = 1$, so $AC = \sqrt{1 - x^2}$. Using the fact that the height of the box is 2, we can write the equation

$$2 = \frac{1}{2} + \sqrt{1 - x^2} + \frac{\sqrt{3}}{2} + \frac{1}{2}.$$

Solving for the value of x gives $x = \frac{1}{2}\sqrt{4\sqrt{3} - 3}$. So the length of one period of the packing is $2\left(x + \frac{1}{2}\right)$, or $\sqrt{4\sqrt{3} - 3} + 1$.

It turns out that 6 coins every $\sqrt{4\sqrt{3} - 3} + 1$ is every so slightly denser than 2 coins every 1. Indeed, a quick calculator computation shows that $\frac{6}{\sqrt{4\sqrt{3}-3}+1}$ exceeds $\frac{2}{1}$ by about 0.0121. It remains to do some housekeeping to prove that exactly 401 coins can be fit in the box with this scheme, but I'll spare you the details.

■

Remarks: See https://www.desmos.com/calculator/eujaifhsmd for a Desmos visualization.

As far as I am aware, nobody knows whether or not 401 is the most number of coins we can fit. If it is, a proof of this seems quite hard.

Source: I saw this on Puzzling Stack Exchange

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Solution 106

Beth wins.

In general, for an $m \times n$ bar, Beth wins if and only if m and n are both odd. Beth's winning strategy is to do whatever the **** she wants. This is because the game always ends in $mn - 1$ moves. Indeed, this is due to the fact that the number of pieces goes up by 1 with every move.

■

Remarks: It's very common to use induction, but this very clearly isn't necessary. Another form of this problem is as follows: Prove that a jigsaw takes the same number of moves to complete no matter what you do, where a move consists of joining two pieces together.

Source: Folklore

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Solution 107

Every positive integer either starts with 1 or will gain a digit when multiplied by 5. Starting with $5^0 = 1$, we will perform 2023 multiplications by 5. 1414 of these multiplications gain a digit, so 1414 of the obtained powers of 5 will not start with 1. Thus $2023 - 1414 = \boxed{609}$ of them *do* start with 1.

■

Source: I stole the idea from a Mildorf Mock AIME

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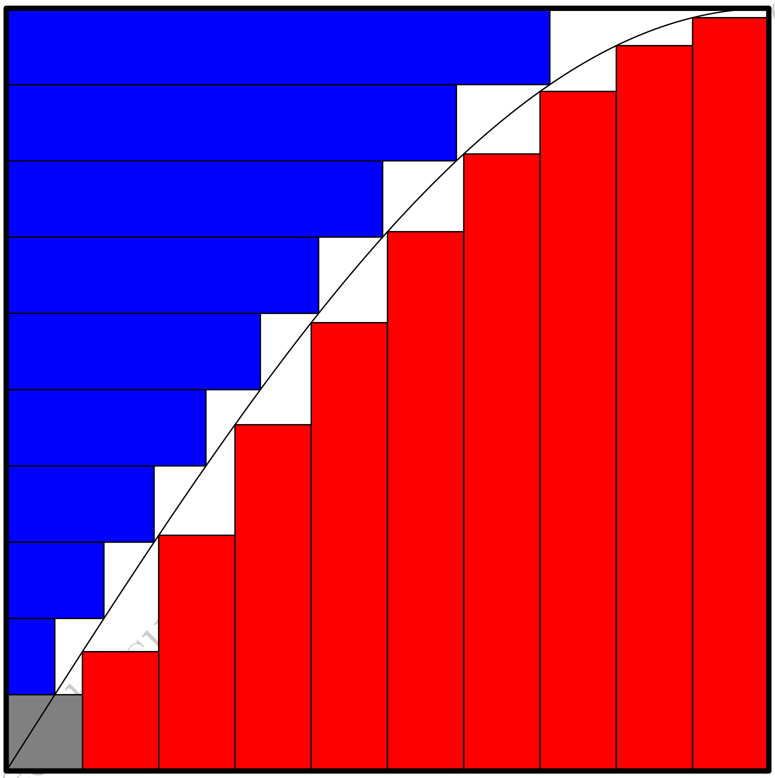
Solution 108

We'll instead prove that

$$\left(\sum_{n=1}^{9}\frac{1}{10}f(n/10)\right)+\left(\sum_{n=1}^{9}\frac{1}{10}f^{-1}(n/10)\right)\leq\frac{99}{100}.$$

This is proven pictorially with the following diagram.



The black curved line is the graph of f. The red rectangles each have width $\frac{1}{10}$, so the sum of their areas is $\sum_{n=1}^{9}\frac{1}{10}f(n/10)$. Likewise, the blue rectangles' areas sum to $\sum_{n=1}^{9}\frac{1}{10}f^{-1}(n/10)$. Their total area is bounded by the area of the square minus the uncovered gray square in the bottom-left corner. This gives the upper bound $1-\frac{1}{100}=\frac{99}{100}$, as needed.

■

Remarks: The bound cannot be attained, but it is tight.

Source: Leningrad Math Olympiad

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Solution 109

Clearly true for $n = 1$. Now let n be the first integer for which the tens digit of 3^n is odd. Since the tens digit of 3^{n-1} is even, it follows that the ones digit of 3^{n-1} must be a digit d for which $10 \leq 3d < 19$. Thus $d = 4, 5, 6$. However it is not hard to find that the ones digit of a power of 3 can only be 1, 3, 7, or 9, contradiction.

■

Source: I saw this on Brilliant back in the stone age

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Solution 110

I present the proof by Burns and Hasselblatt (https://www.math.arizona.edu/~dwang/BurnsHasselblattRevised-1.pdf). Denote by $f^{(i)}$ the i-fold composition of f and assume that $n \geq 2$ (the case $n = 1$ is not so interesting). We begin with the following lemma.

Lemma 1

Suppose $I_0, I_1, \cdots, I_{n-1}$ are subintervals of I such that $f(I_i)$ covers I_{i+1} for $i = 0, 1, 2, \cdots, n-2$, and $f(I_{n-1})$ covers I_0. Then we can find $x \in I_0$ such that $f^{(i)}(x) \in I_i$ for all i and $f^{(n)}(x) = x$.

Proof. Essentially, you just start with I_0 and pull it back n times via f to get a smaller interval.

We have that $f(I_{n-1}) \supseteq I_0$, so $f^{-1}(I_0) \subseteq I_{n-1}$. $f^{-1}(I_0)$ is a subset of I_{n-1} that gets mapped onto I_0, but before we pull back further, we want to make this subset an interval. $f^{-1}(I_0)$ is not necessarily an interval, but it's certainly a union of intervals! Pick one such interval that gets mapped onto I_0, and call it J_{n-1}. (*Why does J_{n-1} exist?*)

We now have a subinterval $J_{n-1} \subseteq I_{n_1}$ with $f(J_{n-1}) = I_0$. Pull back J_{n-1} to get a subset $f^{-1}(J_{n-1}) \subseteq I_{n-2}$. Again, $f^{-1}(J_{n-1})$ is a union of intervals, and we can pick one of them that gets mapped onto J_{n-1} and call it J_{n-2}.

This gives a subinterval $J_{n-2} \subseteq I_{n-2}$ with $f(J_{n-2}) = J_{n-1}$, and we can repeat this to get a subinterval $J_{n-3} \subseteq I_{n-3}$ with $f(J_{n-3}) = J_{n-2}$, and so on! In the end, we find $J_0 \subseteq I_0$, $J_1 \subseteq J_1$, $\cdots$, and $J_{n-1} \subseteq I_{n-1}$ such that the restrictions

$$f : J_0 \to J_1$$
$$f : J_1 \to J_2$$
$$f : J_2 \to J_3$$
$$\cdots$$
$$f : J_{n-2} \to J_{n-1}$$
$$f : J_{n-1} \to I_0$$

are all surjective! Hence the n-fold composition $f^{(n)} : J_0 \to I_0$ is also surjective. Recalling that $J_0 \subseteq I_0$ and that J_0 and I_0 are both intervals, a simple application of the Intermediate Value Theorem gives the existence of some $x \in J_0$ for which $f^{(n)}(x) = x$. Due to how we chose the J_i intervals, this point x satisfies all the desired properties. □

The way we use this lemma to solve the problem is extremely cool. Let's suppose that the point $a \in I$ has period 3. Let $b = f(a)$ and $c = f(b)$. Without loss of generality, we can assume that $a < b < c$. Then:

- $f((a,b))$ covers (b,c), and
- $f((b,c))$ covers (a,c).

In particular, it is both true that $f((b,c))$ covers (a,b) and $f((b,c))$ covers (b,c). Thus, if we make the following choices for $I_0, \cdots, I_{n-1}$:

- $I_0 := (a,b)$
- $I_1 := (b,c)$
- $I_2 := (b,c)$
- $I_3 := (b,c)$
- ...
- $I_{n-2} := (b,c)$
- $I_{n-1} := (b,c)$

then $f(I_i)$ covers I_{i+1} for all $0 \le i \le n-2$ and $f(I_{n-1})$ covers I_0.

So by the lemma, $f^{(n)}$ has a fixed point x in I_0 with $f(x) \in I_1$. But $I_0 = (a,b)$ and $I_1 = (b,c)$ are disjoint, so clearly x and $f(x)$ cannot be the same. In fact, $f^{(i)}(x) \in I_i = (b,c)$ for all $1 \le i \le n-1$, so $x \ne f^{(i)}(x)$ for any such i! So n is the first time that x gets sent back to itself. That is, we found a point of period n.

■

Source: This is a special case of Sharkovsky's Theorem

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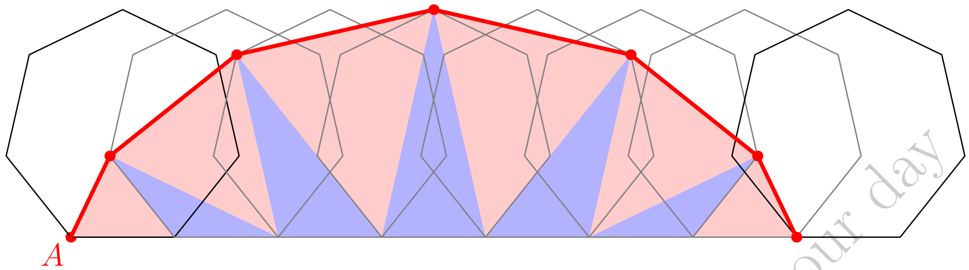
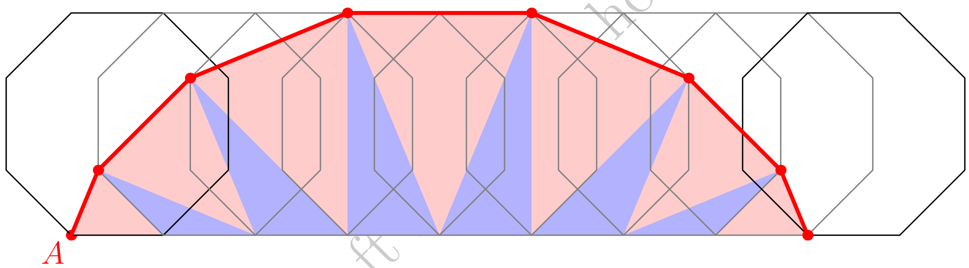
Solution 111

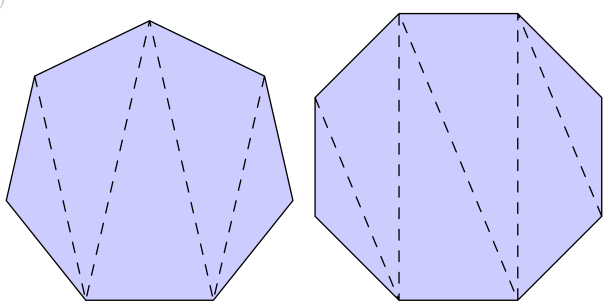
Thanks to Alan Abraham for the following approach. Begin with the following dissection.



The dissection also works when n is even, and would look something like this.



The utility of this dissection is that the blue triangles can be joined to form a copy of P. Both the odd and even cases are shown below.



What remains is $n-1$ red isosceles triangles. We aim to show that the sum of their areas is twice that of P.

Scale the diagram so that the radius of P (the distance from the center of P to a vertex of P) is 1. Then the area of P is given by $n \cdot \frac{1}{2} \sin\left(\frac{2\pi}{n}\right)$.

As for the $n-1$ red isosceles triangles, they are all similar. The lengths of their legs are given by $\{|z-1| : z^n = 1, z \neq 1\}$, and the angle formed by the legs is $\frac{2\pi}{n}$. So the sum of their areas is

$$\begin{aligned}
\sum_{z^n=1, z\neq 1} \frac{1}{2}|z-1|^2 \sin\left(\frac{2\pi}{n}\right) &= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) \sum_{z^n=1, z\neq 1} |z-1|^2 \\
&= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) \sum_{z^n=1} |z-1|^2 \\
&= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) \sum_{z^n=1} (z-1)(\overline{z}-1) \\
&= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) \sum_{z^n=1} 2 - z - \overline{z} \\
&= \frac{1}{2}\sin\left(\frac{2\pi}{n}\right) \sum_{z^n=1} 2 \qquad \text{(Roots of unity sum to zero)} \\
&= 2 \times \frac{n}{2}\sin\left(\frac{2\pi}{n}\right),
\end{aligned}$$

which is indeed twice the area of P.

■

Remarks: A purely dissective proof can be found on this page.

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Solution 112

As a consequence of the looping condition, we see that the amount of money gained or lost between any two rooms is independent of the path taken.

Suppose I start from the northwest room with no money. Note that any room that is not the southeast room can be reached in 7 steps, so I could not possibly have more than \$7 in any of these rooms since my money can go up by at most \$1 with each step. Thus I could only have observed having \$8 in the southeast room.

Considering the 8-step path $EESSSSEE$, I must end up with \$8 via this path, since any path that reaches the southeast room must do so. So on each step of this 8-step path, I must gain a dollar. It follows that I would lose a dollar if I exit the center room via its north door. ■

Remarks: The ideas in this problem bear a resemblance to the arguments used with *conservative vector fields.*

Source: Math Hour Olympiad

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Solution 113

I pick a degree-$(k-1)$ polynomial $P(x)$ such that $P(0)$ is my favorite number. For each integer x with $1 \leq x \leq n$, I give friend x the ordered pair $(x, P(x))$. Any k of these points is enough to uniquely identify P, and hence let my friends deduce $P(0)$. But knowing $k-1$ of the points is never enough. In fact, it won't give *any* information regarding $P(0)$.

■

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Solution 114

Part (a)

As mentioned in the hint, the key insight is that if you start moving in the direction perpendicular to the segment connecting you and the center of the circle, then the lion can never catch you. In fact, you could change directions as much as you want, as long as when you change direction, you begin moving in the direction perpendicular to the segment connecting you and the center.

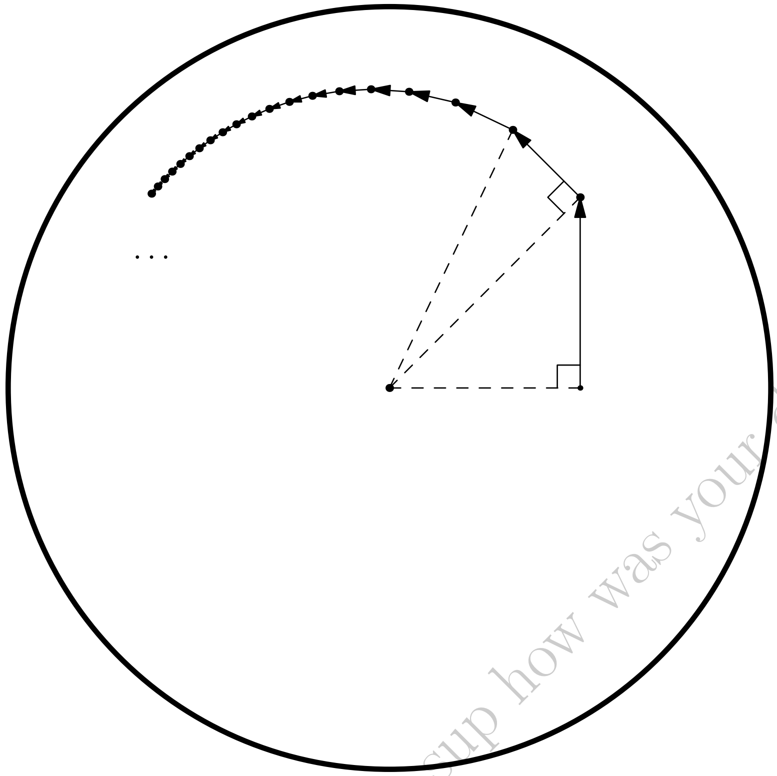
This motivates the following strategy:

- Orient yourself so that you're perpendicular to the segment connecting you and the center.
- Run r_1 units forwards.
- Orient yourself so that you're perpendicular to the segment connecting you and the center.
- Run r_2 units forwards.
- Orient yourself so that you're perpendicular to the segment connecting you and the center.
- Run r_3 units forwards.
- etc.

Provided that we can keep running in this way forever, this strategy will work. To ensure that this strategy works forever, the numbers $r_1, r_2, \cdots$ must be chosen so that we never exit the cage and that we run indefinitely. Assume for convenience that the radius of the cage is 10.

- To ensure that we never exit the cage, note that by iteratively applying the Pythagorean Theorem, our squared distance to the center of the cage is given by $\sum_{i=1}^{\infty} r_i^2$. So we require that $\sum_{i=1}^{\infty} r_i^2 < 10^2$.
- To ensure that we run indefinitely, we must plan to run an infinite distance. That is, we require that $\sum_{i=1}^{\infty} r_i = +\infty$.

From these conditions, we choose $r_i = \frac{1}{i}$, which works!



Part (b)

Impose coordinate axes centered at the center of cage. The first lion chases your projection unto the x axis, and the second lion chases your projection unto the y axis. Once each lion has caught up with these projections, they move towards you in such a way that the first lion's x-coordinate always the matches yours, and the second lion's y-coordinate always matches yours.

I'll leave it to you to convince yourself that this works.

■

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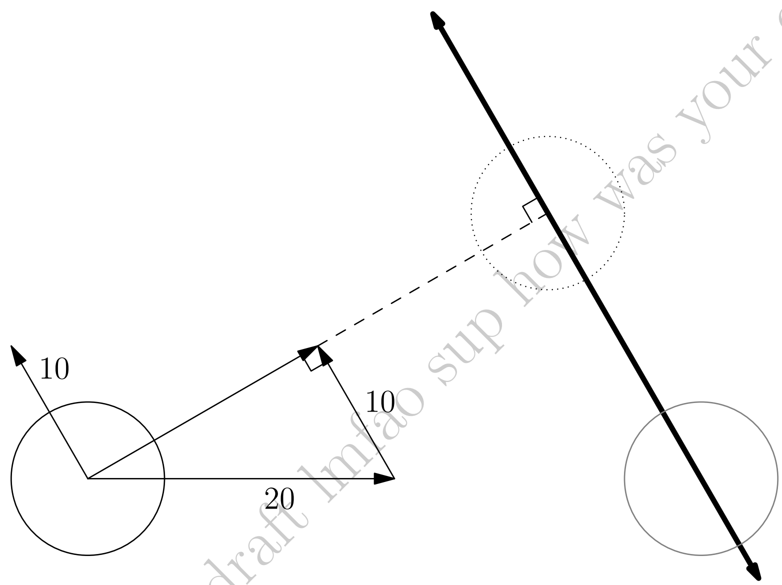
Solution 115

If we change the frame of reference so that the rain is stationary, then our velocity has a component of 10 in a 60° direction, and our goal position moves along a 60°-sloped line at the same speed of 10. We see that, provided these restrictions, reaching the goal is equivalent to reaching the line. Minimizing the rain encountered is equivalent to finding the shortest path to this line. This is given by orthogonal projection, and some vector arithmetic shows that we should run at a speed of 20.



∎

Source: Someone posted this in a Discord server and I stole it.

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Solution 116

We begin by working under the assumption that an even number of servers are on.

1. Press all four buttons. (If not successful, we now know exactly two servers are on.)
2. Press two diagonally-opposite buttons.
3. Press all four buttons. (If not successful at this point, we now know that there are exactly two servers on, and that they are adjacent.)
4. Press two adjacent buttons.
5. Press all four buttons. (If not successful at this point, we now know that there are exactly two servers on, and that they are diagonally opposite.)
6. Press two diagonally-opposite buttons.
7. Press all four buttons.

If not successful at this point, then our assumption that there were an even number of servers was wrong. Thus we may win in 8 more steps as follows:

8. Press any button. (We now know that there are an even number of servers that are on.)
9. Repeat steps 1-7.

Hence we may guarantee success within 15 steps.

■

Remark: Apparently you can still win if there instead are 2^n servers for positive integer n.

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Solution 117

We claim that everyone is equally likely to try the grape juice last (!).

> **Lemma 1**
>
> Imagine a frog at the integer 0. It repeatedly hops left and right until it either reaches $-a$, where it wins, or reaches b, where it loses. Then the probability that the frog wins is $\frac{b}{a+b}$.

Proof. There are a good number of ways to approach this. Here is a short one. Let p be the desired probability. Let M_n be the martingale representing the frog's location after n steps (and it is indeed a martingale because the expectation of its change at each step is 0). We endow it with the stopping time $\tau := \inf\{k \in \mathbb{N} : M_k \in \{-a, b\}\}$. It is now easy to justify the application of Doob's Optional Stopping Theorem, which entails that $\mathbb{E}M_0 = \mathbb{E}M_\tau$. But $\mathbb{E}M_0 = 0$, and $\mathbb{E}M_\tau = p(-a) + (1-p)b$. Solving for p gives $p = \frac{b}{a+b}$. □

We can now solve the original problem. Let us index my friends and I via the integers from 0 to 2023, where I am labelled with 0, and we will consider the friend at some $1 \leq n \leq 2023$. We will show that friend n tries the grape juice last with probability $1/2023$.

There are exactly two ways in which friend n could try the grape juice last. Either

- the grape juice reaches friend $n-1$, then goes around the other way to friend $n+1$, without ever reaching friend n, or
- the grape juice reaches friend $n+1$, and then goes around the other way to friend $n-1$, without ever reaching friend 0.

We compute the probability of the first case. For convenience, allow negative indices, taking all indices mod 2024. The probability that the grape juice reaches friend $n-1$ before it reaches friend $n+1 \equiv -(2023-n)$ is given by $\frac{2023-n}{2022}$ by the lemma. From here, the probability of the grape juice reaching friend $n+1 \equiv -(2023-n)$ before reaching friend n is $\frac{1}{2023}$, by the lemma again. Thus the probability of the first case occurring is $\frac{2023-n}{(2022)(2023)}$.

Analogously, the probability of the second case occurring is given by $\frac{n-1}{(2022)(2023)}$. Summing the cases, we conclude that the probability that friend n tries the grape juice last is

$$\frac{2023-n}{(2022)(2023)} + \frac{n-1}{(2022)(2023)} = \frac{1}{2023}$$

as claimed.

■

Remarks: See https://math.stackexchange.com/a/2390627/372663 for what appears to be a clean, computation-less solution.

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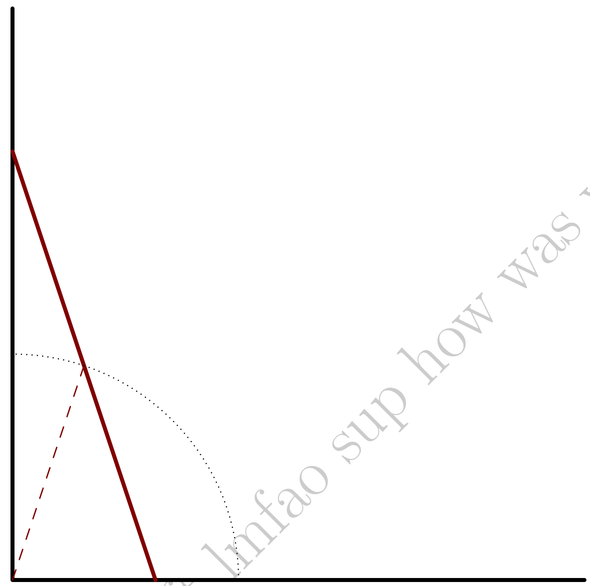
Solution 118

From classical right triangle geometry, the length of the segment connecting the corner of the wall and the ladder's midpoint is always half the ladder's length. So it's constant. Hence the shape traced is a circle. Specifically, it is a quarter circle centered at the corner of the wall.



■

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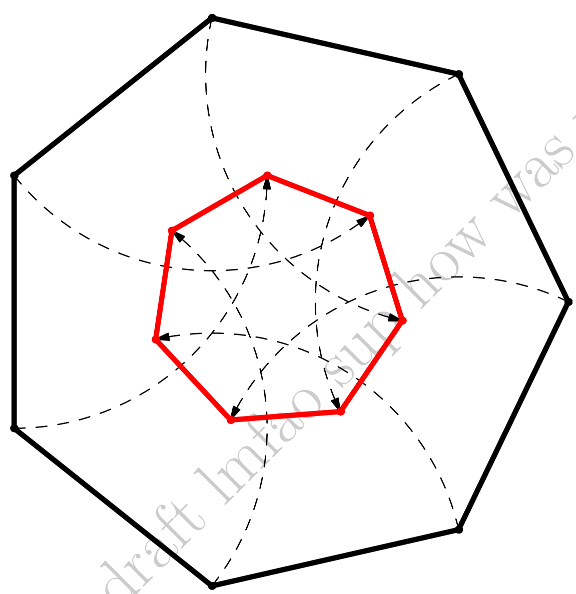
Solution 119

We claim that such a regular n-gon exists only for $n = 4$. Obviously squares exist, so $n = 4$ certainly works. Now let us rule out $n \geq 5$.

Suppose that $n \geq 5$ and that there is a regular n-gon with lattice vertices. Rotating each vertex 90° *inwards* about the previous vertex, we form a smaller regular n-gon with lattice vertices, which is a contradiction since we may descend in this way infinitely.



Finally, we rule out $n = 3$. Suppose there were an equilateral triangle $\triangle ABC$ where A, B, C are lattice points. Multiply all the coordinates of A, B, and C by 3. Then, take the two trisection points on each of the three sides of $\triangle ABC$. These are lattice points and they form a regular hexagon. But we ruled out $n = 6$ from the previous analysis, contradiction.

■

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Solution 120

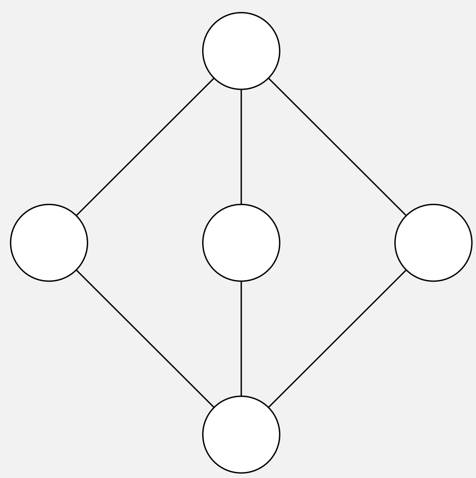
This approach seems novel enough to justify crediting myself for it. We claim that 6 is the best we can do.

To see that it is obtainable, take the 6 lines that pass through two opposite vertices of a given icosahedron.

We now show that 7 is impossible. First we prove the following weird lemma that does not seem to have any relevance to the problem whatsoever.

Lemma 1

Color the edges of K_7 graph red and blue. Consider a "Lights Out"-type game in which we may "press" any vertex to toggle the color of all edges emanating from that vertex (from red to blue and vice versa).
Then there exists a sequence of moves that will result in there existing a monochromatic



subgraph.

Proof. Pick any vertex v. Press some of the other vertices so that all edges from v are red.

Pick another vertex w. Since there 5 other vertices, there exist 3 of them, x, y and z, such that edges wx, wy, and wz are the same color.

If this common color is red, then w, x, y, z, and v form the desired monochromatic subgraph, which will be all red. If otherwise the common color is blue, then these vertices will form a blue such monochromatic subgraph after we press v. □

With this totally irrelevant lemma proven, we may proceed to solve the original problem.

Suppose there exist 7 distinct lines through the origin that form the same angles with each other. Pick unit vectors $v_1, v_2, \cdots, v_7$, one in the direction of each of these 7 lines (there will be two legal choices for each). By the equal angle condition, the quantity $|v_i \cdot v_j|$ is the same constant c for all distinct i and j.

Form a graph on these 7 vectors, coloring the edge between v_i and v_j red if $v_i \cdot v_j = c$, and coloring it blue if otherwise $v_i \cdot v_j = -c$.

Note that if we were to replace v_i with $-v_i$, then all the edges emanating from v_i in this graph will toggle colors. By the irrelevant lemma, we may make a sequence of such replacements such that among the vectors $\{v_1, v_2, \cdots, v_7\}$, there will exist five distinct vectors v, x, y, z, and w for which

$$v \cdot x = v \cdot y = v \cdot z = w \cdot x = w \cdot y = w \cdot z = \pm c.$$

Note that from $v \cdot x = v \cdot y = v \cdot z$, we have that v is perpendicular to the plane formed by x, y, and z. Indeed, this is because $v \cdot (x - y) = v \cdot (x - z) = 0$. Similarly, w is also perpendicular to the plane formed by x, y, and z. This can only happen if v and w lie on the same line through the origin (because the space of vectors orthogonal to both $x - y$ and $x - z$ has dimension 1), contradicting how we chose the vectors.

■

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Solution 121

Whenever the question mentions "*the* apple that's X...", this implies that there is exactly one apple that satisfies the condition X. From this, we can work backwards to get some information on the apples, starting from the green apple.

- "...the apple that's cheaper than the apple that's green" implies that there is only one apple that is cheaper than the green apple, so the green apple is \$2 and "the" apple in question is \$1.
- "...the apple that's smaller than [the \$1 apple]" implies that the \$1 apple is the second-smallest, and "the" apple in question is the smallest.
- "the apple that that costs more than [the smallest apple]" implies that the smallest apple is \$4 and "the" apple in question is \$5.
- "the apple that's bigger than [the \$5 apple]" implies that the \$5 apple is the second-largest (i.e. fourth-smallest) and "the" apple in question is the largest apple. Moreover, the "it" in "given that it is red" refers to this apple so the largest apple is red.

We may collate this information into the following "logic grid".

	\$1	\$2	\$3	\$4	\$5
smallest	X	X	X	O	X
2nd-smallest	O	X	X	X	X
3rd-smallest	X			X	X
4th-smallest	X	X	X	X	O
5th-smallest	X			X	X

To resolve the ambiguity, we use the colors of the apples! The largest apple is given to be red whereas the green apple is \$2, so the largest apple and the \$2 apple are different apples. This lets us place one more X in the grid, and we conclude that the red apple (i.e. the largest apple) is **\$3**.

■

Source: Jack Lance

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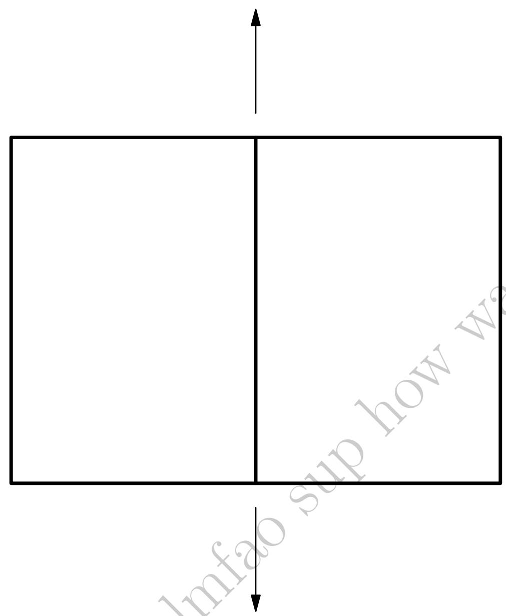
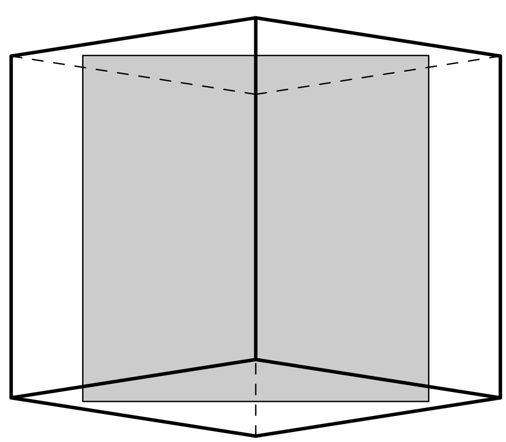
Solution 122

Here is one possible argument: Suppose that the cube is $1 \times 1 \times 1$. We can tilt the cube 45° (any less is also perfectly fine) so that its shadow looks like a $1 \times \sqrt{2}$ rectangle, as shown below.



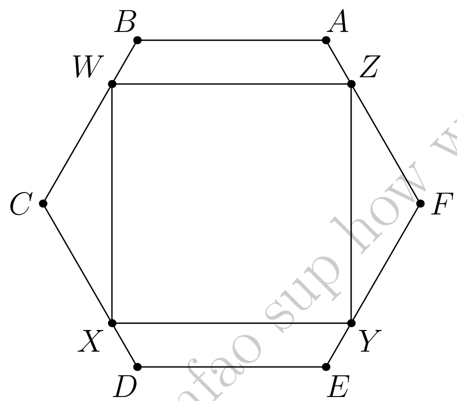
Rotating the cube very slightly in the direction of either of the pictured arrows, the shadow becomes slightly vertically elongated. Any amount of vertical elongation will be enough to be able to fit a 1×1 square with room to spare.



Since there is room to spare, the 1×1 square can be enlarged to some $(1+\varepsilon) \times (1+\varepsilon)$ square (still with room to spare), and drilling a hole through this square gives the desired hole, through which we can pass through a cube of side length $1+\varepsilon$.

For a more constructive/explicit approach, let's tilt the cube so that one vertex lies directly above the opposite vertex. Then the cube's shadow will be a regular hexagon $ABCDEF$. But what is the side length of this hexagon? You can find that the segment $\overline{AC}$ (and ditto for $\overline{BD}$, $\overline{CE}$, etc.) lies directly under a *face diagonal* of the cube, which is actually parallel to the shadow's plane. So $AC = \sqrt{2}$ and using 30-60-90 triangles will tell us that the side length is $AB = \sqrt{\frac{2}{3}}$.

Now to geometrically represent the hole we plan to drill, let us inscribe a square $WXYZ$ into $ABCDEF$. To maximize the area of this square, a good guess is to inscribe it so that $\overline{WZ}$ is parallel to a side of the hexagon. See the diagram.



If the side length of the square is x, then $CW = \frac{x}{\sqrt{3}}$. So $BW = \sqrt{\frac{2}{3}} - \frac{x}{\sqrt{3}}$. But now

$$x = WZ = AB + BW\cos 60^\circ + AZ\cos 60^\circ = AB + BW = 2\sqrt{\frac{2}{3}} - \frac{x}{\sqrt{3}}.$$

So $(1+\sqrt{3})x = 2\sqrt{2}$ which solves as $x = \sqrt{6} - \sqrt{2} \approx 1.035$, which is very slightly larger than 1, so it is indeed barely possible to drill a square hole into a unit cube that will fit a larger cube.

■

Remark: We can actually do better than $\sqrt{6} - \sqrt{2}$. You can think about how to achieve this or look it up.

Remark 2: It is known that the problem still holds for any other platonic solid! Mathematicians have no idea if this extends to all *convex* polyhedra.

Source: Classic, known as Prince Rupert's Cube

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Solution 123

Let the positive integer be n, and suppose it has k digits. Assume for contradiction that all digits of n are at least 6, and all digits of n^2 are at most 4.

A natural inequality to write is $n \geq 666\ldots66$, but it will be helpful for later if we can improve this bound. Indeed, we can do better by examining the units digit: If n ends in a 6, then n^2 will end in a 6 as well, which is not less than 5, so this cannot be. Similarly, n could not end in 7 since otherwise n^2 will end in 9. So $n \geq 666\ldots68$.

More mathematically, this entails that

$$10^k \geq n \geq 6 \cdot \frac{10^k - 1}{9} + 2 = \frac{2}{3} \cdot 10^k + \frac{4}{3},$$

so

$$10^{2k} \geq n^2 \geq \frac{4}{9} \cdot 10^{2k} + \frac{16}{9} \cdot 10^k + \frac{16}{9} > 4 \cdot \frac{10^{2k} - 1}{9}.$$

The number $4 \cdot \frac{10^{2k}-1}{9}$ is simply the $2k$-digit number $444\ldots4$. So n is somehow strictly greater than this, and cannot exceed the $(2k+1)$-digit number $100\ldots0$, while also having all of its digits between 0 and 4. This is evidently impossible.

■

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Solution 124

I present two solutions.

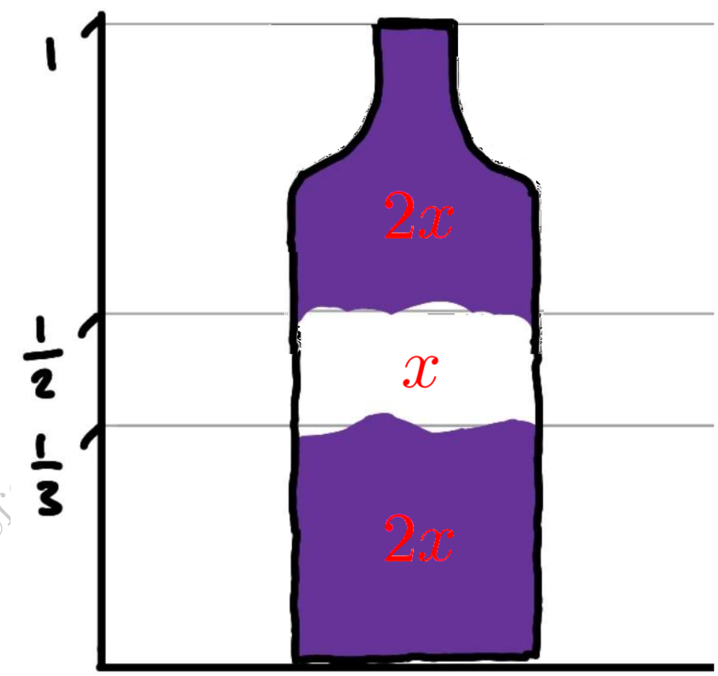
Solution 1

Let's say the width of the bottle is 1. Then the bottle on the left shows that the amount of wine in the bottle is $1 \times \frac{1}{3} = \frac{1}{3}$. Now, instead of finding the total area of the bottle, notice that the amount of *empty space* must be the same on the left and on the right, and in the bottle on the right, the empty space is a $1 \times \frac{1}{2}$ rectangle whose area is $\frac{1}{2}$.

Thus, the total area of the bottle is $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$, and so the bottle is $\frac{1/3}{5/6} = \boxed{\frac{2}{5}}$ full.

■

Solution 2



■

Source: I saw this on the MindYourDecisions channel.

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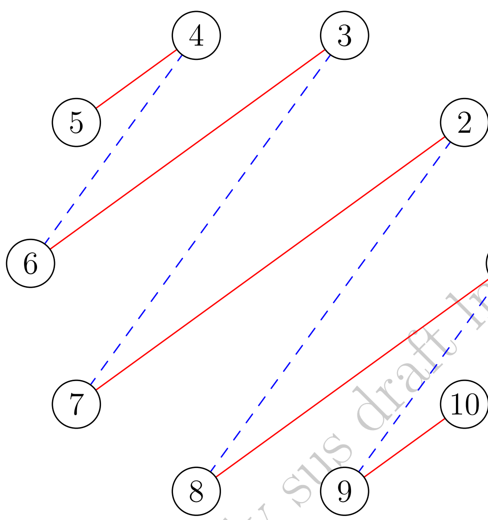
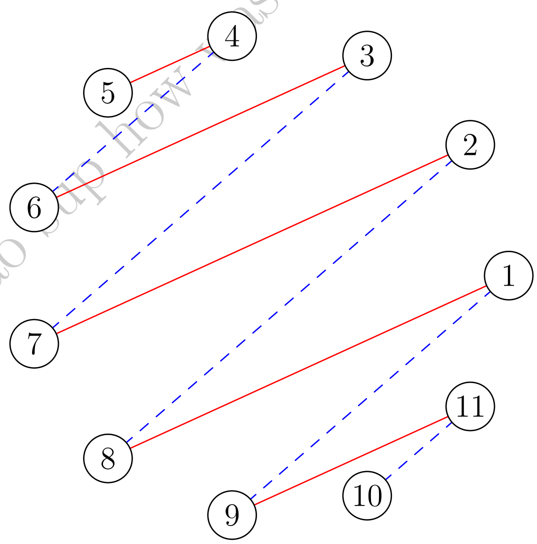
Solution 125

Consider the permutation which maps each student to the student that holds their midterm exam. Since each permutation can be decomposed into disjoint cycles, it suffices to solve the problem under the assumption that this permutation is just a cycle. That is, we may assume that the students are $S_1, S_2, \ldots, S_n$, and that S_k holds S_{k+1}'s exam (where we identify $S_{n+1} := S_1$).

The procedure for resolving this case is best described graphically. There are two cases depending on the parity of the number of students in the cycle, n. In either case (depicted as $n = 10$ and $n = 11$ below), we first swap along the solid red lines and then swap along the dashed blue lines.





∎

Source: Leningrad Olympiad

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Solution 126

Solution 1

This proof follows one of the official solutions from the contest in which this problem appeared. We will use the notation $[XYZ]$ to denote the area of $\triangle XYZ$. First we prove the following lemma.

Lemma 1

Let $\triangle X'Y'Z'$ be a congruent copy of $\triangle XYZ$ which is rotated by 180° and then translated. Then

$$\text{Area}(\triangle XYZ \cap \triangle X'Y'Z') \le \frac{2}{3}\text{Area}(\triangle XYZ).$$

More succinctly, any centrally symmetric subset of a triangle takes up at most $\frac{2}{3}$ of the triangle's area.

Proof. Without loss of generality we may assume $\triangle XYZ$ is equilateral with side length 1. There are two cases.

First Case: $\triangle XYZ \cap \triangle X'Y'Z'$ is a parallelogram, two of whose vertices are X and X' (without loss of generality). In the interest of maximizing the area of intersection we may assume that X' lies on side $\overline{YZ}$. Let $a = |YX'|$ and $b = |X'Z|$, so that $a + b = 1$. The area of the parallelogram is now

$$[XYZ] - |YX'|^2[XYZ] - |X'Z|^2[XYZ] = (1 - a^2 - b^2)[XYZ],$$

and from $a^2 + b^2 \ge \frac{1}{2}(a+b)^2 = \frac{1}{2}$ we obtain an upper bound of $\frac{1}{2}[XYZ]$, which is certainly $\le \frac{2}{3}[XYZ]$.

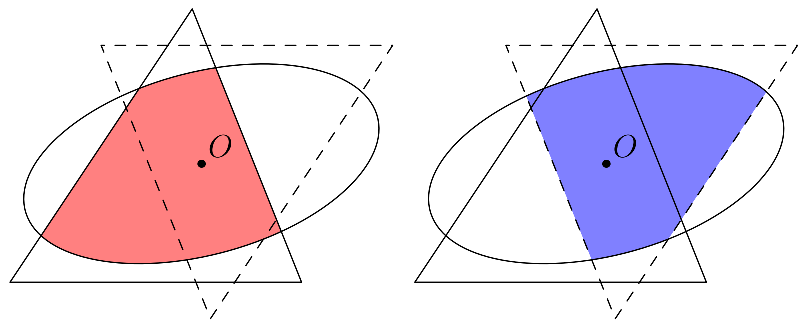
Second Case: $\triangle XYZ \cap \triangle X'Y'Z'$ is a hexagon. If this hexagon is removed from $\triangle XYZ$, then we are left with three equilateral triangles of side lengths a, b, and c. Using the fact that the hexagon is centrally symmetric, it can be seen that $a + b + c = 1$. The area of the hexagon is thus

$$[XYZ] - a^2[XYZ] - b^2[XYZ] - c^2[XYZ] = (1 - a^2 - b^2 - c^2)[XYZ].$$

From the QM-AM inequality, $a^2 + b^2 + c^2$ is minimized subject to $a + b + c = 1$ exactly when $a = b = c = \frac{1}{3}$. So the area of the hexagon is at most $\left(1 - \frac{3}{3^2}\right)[XYZ] = \frac{2}{3}[XYZ]$. □

Now we return to the original problem. Let the triangle be $\mathcal{T}$, and let the ellipse be $\mathcal{E}$ with center O. Rotate $\mathcal{T}$ 180° about O to obtain a triangle $\mathcal{T}'$.

Consider the regions $\mathcal{R} := \mathcal{T} \cap \mathcal{E}$ and $\mathcal{R}' := \mathcal{T}' \cap \mathcal{E}$. These regions are shown below.



Note that $\mathcal{R}$ and $\mathcal{R}'$ are congruent, and each have area A. Moreover $\mathcal{R} \cup \mathcal{R}'$ is a subset of the ellipse, hence

$$\text{Area}(\mathcal{R} \cup \mathcal{R}') \leq E.$$

The area of the union may be expressed as

$$\begin{aligned}\text{Area}(\mathcal{R} \cup \mathcal{R}') &= \text{Area}(\mathcal{R}) + \text{Area}(\mathcal{R}') - \text{Area}(\mathcal{R} \cap \mathcal{R}') \\ &= 2A - \text{Area}(\mathcal{R} \cap \mathcal{R}').\end{aligned}$$

But by the lemma, $\text{Area}(\mathcal{R} \cap \mathcal{R}') \leq \frac{2}{3}T$. We conclude that

$$2A - \frac{2}{3}T \leq E,$$

which rearranges to $\frac{T}{3} + \frac{E}{2} \geq A$, as needed.

■

We present an alternate solution starting on the next page.

Solution 2 (Sketch)

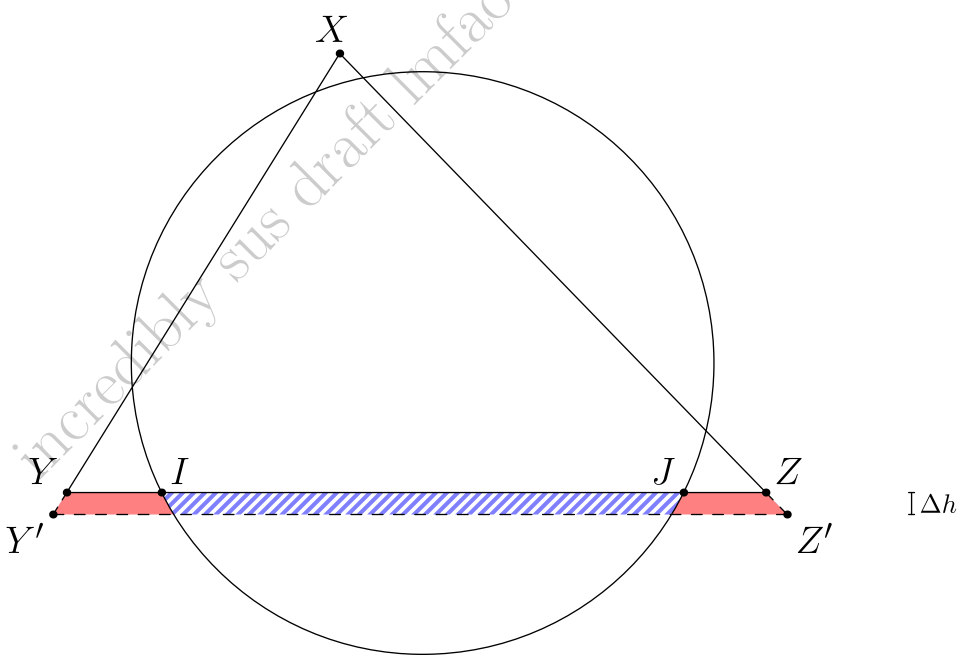
This solution is on the lengthy side, but it showcases a nice technique known as *visual calculus*.

To set up, let us first assume without loss of generality that the ellipse is a circle. We will show that the minimum value for $\frac{T}{3} + \frac{E}{2} - A$ is 0. To avoid technicalities we will take the following facts for granted:

- There exists a minimum value for $\frac{T}{3} + \frac{E}{2} - A$.
- In a configuration which achieves this minimum value, the circle intersects the triangle 6 times — twice per side.

Take a configuration which achieves the minimum value of $\frac{T}{3} + \frac{E}{2} - A$. Then it follows that any perturbation to the configuration cannot decrease the value of $\frac{T}{3} + \frac{E}{2} - A$. This principle is the basis for the following deductions.

Call the triangle $\triangle XYZ$. One way to perturb the configuration is to expand the triangle slightly by a dilation centered at a vertex (say, X), as depicted below.



Let's start with a slightly informal argument using Calculus. Label two of the intersections as I and J as shown.

If we perturb the triangle in this way *continuously in time* so that the perturbation in height Δh increases with rate $\frac{d\Delta h}{dt} = 1$, then by the Fundamental Theorem of Calculus (seen more easily by rotating the above diagram 90°), we have $\frac{dT}{dt} = YZ$ and $\frac{dA}{dt} = IJ$, and so

$$\frac{d}{dt}\left(\frac{T}{3} + \frac{E}{2} - A\right) = \frac{YZ}{3} + 0 - IJ.$$

But $\frac{d}{dt}\left(\frac{T}{3} + \frac{E}{2} - A\right) = 0$ because $\frac{T}{3} + \frac{E}{2} - A$ is minimized, so we conclude that $\frac{YZ}{3} - IJ = 0$.

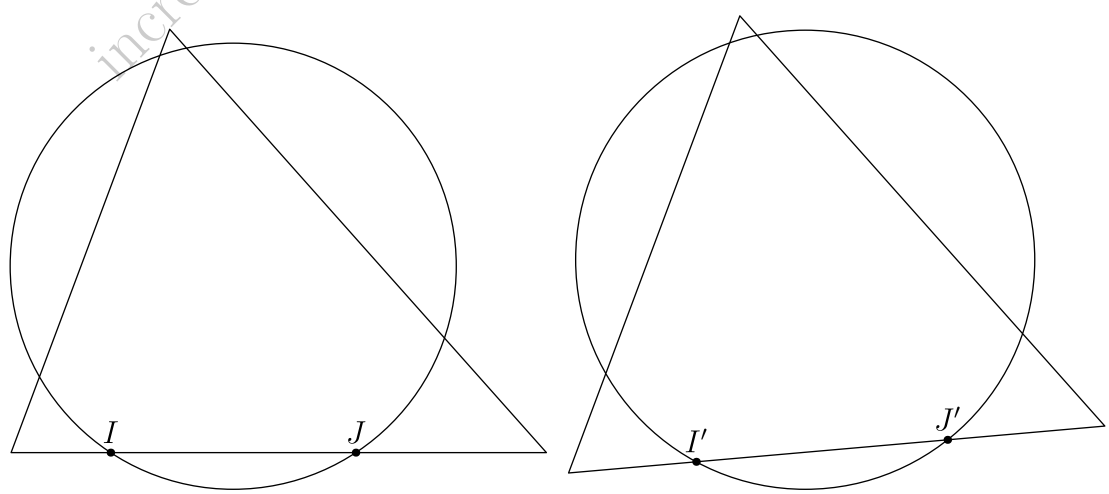
If the reader finds this argument suspicious, we can obtain the same result with elementary arguments: Suppose it were the case that $IJ > \frac{YZ}{3}$. As shown in the diagram from before, we expand side YZ slightly to $Y'Z'$, increasing the triangle's height by a small Δh. Δh should be chosen to be so small that it is negligible compared to the difference $IJ - \frac{YZ}{3}$. Then:

- The quantity T increases by the area of trapezoid $YZZ'Y'$, which is essentially $YZ \cdot \Delta h$ for small Δh.
- E does not change.
- A increases by just the area of the striped blue region, which is essential $IJ \cdot \Delta h$.

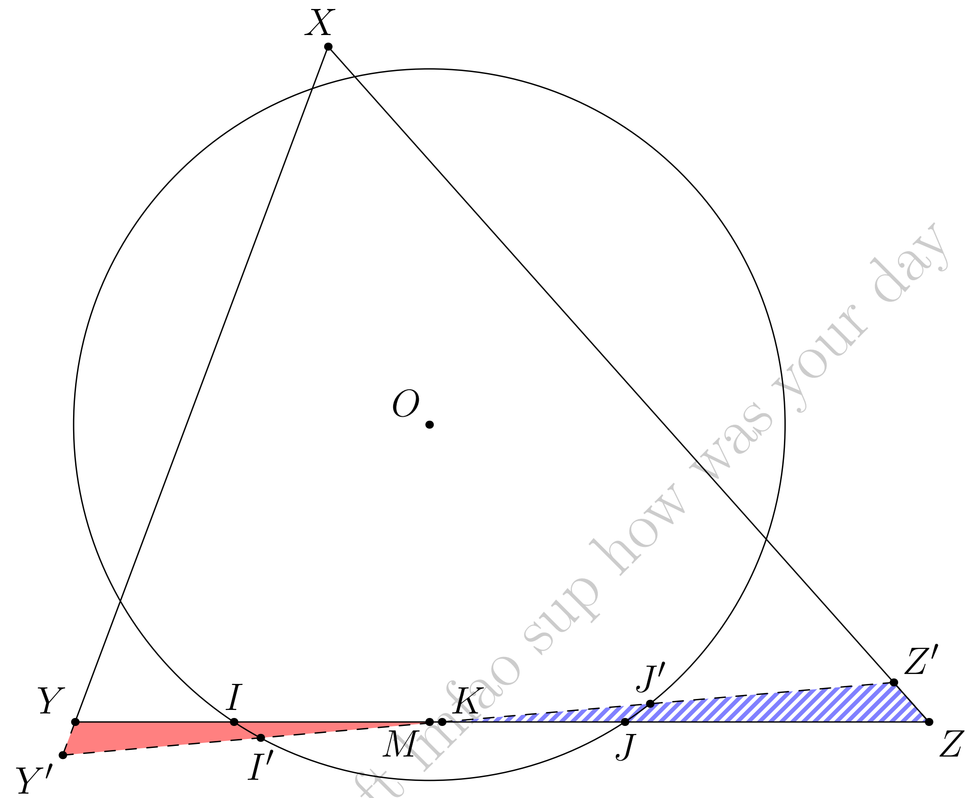
So the quantity $\frac{T}{3} + \frac{E}{2} - A$ changes by $\left(\frac{YZ}{3} - IJ\right) \cdot \Delta h$. But since $\frac{T}{3} + \frac{E}{2} - A$ is minimized, this can be possible only if $\frac{YZ}{3} - IJ \geq 0$, contradicting the assumption that $IJ > \frac{YZ}{3}$. We similarly can show that $IJ < \frac{YZ}{3}$ is impossible, so $IJ = \frac{YZ}{3}$.

Running the same logic for the other sides, we conclude that exactly a third of each of the triangle's sides must lie inside the circle.

A second way to perturb the configuration is to take a side and *rotate* it slightly around the center of the circle. We can think of this as moving the chord IJ around the circle, as shown below.



It is clear that E doesn't change. It turns out that A doesn't change either (why?), thus only T changes. The change in T is depicted below.



Denote the intersection of $\overline{IJ}$ and $\overline{I'J'}$ as K, and let the midpoint of segment $\overline{IJ}$ be M. In the perturbation, the area is increased by the area of $\triangle YY'K$ (in red, shaded) and decreased by the area of $\triangle ZZ'K$ (in blue, striped).

If the angle of rotation $\Delta\theta$ for the perturbation is negligibly small, then M and K are basically the same point, and so the lengths $YK, Y'K, YM$ are morally indistinguishable. Hence

$$\text{Area}(\triangle YY'M) = \frac{1}{2}(YK)(Y'K)\sin(\Delta\theta) \approx \frac{1}{2}|YM|^2 \sin(\Delta\theta)$$

and, similarly,

$$\text{Area}(\triangle YY'M) = \frac{1}{2}(ZK)(Z'K)\sin(\Delta\theta) \approx \frac{1}{2}|ZM|^2 \sin(\Delta\theta).$$

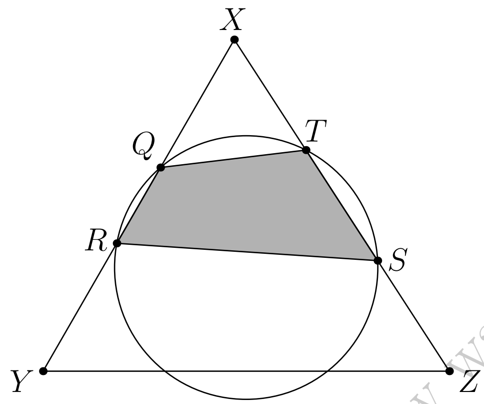
(The estimate $\sin(\Delta\theta) \approx \Delta\theta$ is applicable but unnecessary.) It follows that T changes by

$$\approx \frac{1}{2}(|YM|^2 - |ZM|^2)\sin(\Delta\theta),$$

which must be non-negative by the hypothesis that $\frac{T}{3} + \frac{E}{2} - A$ is minimized, thus $YM \geq ZM$. By a completely symmetrical argument, $YM \leq ZM$. Hence M is the midpoint of side $\overline{YZ}$.

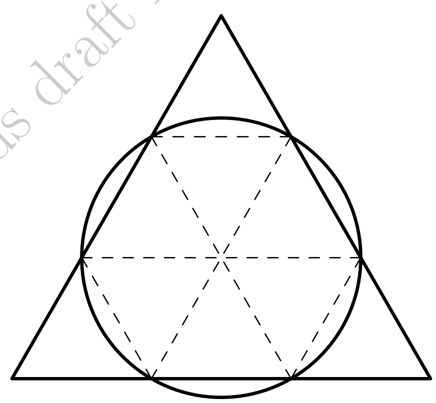
By applying this argument to the other two sides, we conclude that the center of the circle, O, is the circumcenter of $\triangle XYZ$ (!). Moreover, in view of the deduction from the first perturbation, we see that the circle must divide each side of $\triangle XYZ$ into thirds.

To finish, consider the below quadrilateral $QRST$.



Since the circle divides each side into thirds, we have $XQ = QR$ and $XT = TS$. Thus $QRST$ is a trapezoid. But the only cyclic trapezoids are isosceles trapezoids, so $QR = ST$ and thus $XY = XZ$. Analogously, $YX = YZ$, so $\triangle XYZ$ is equilateral.

The below diagram is hence the minimal configuration.



Computing T, E, and A, we find that $\frac{T}{3} + \frac{E}{2} - A = 0$, thus 0 is the minimum value, as desired.

■

Remark: In the official solution, the ellipse (scaled to be a circle) was perturbed instead, but making this work is trickier.

Source: HMIC

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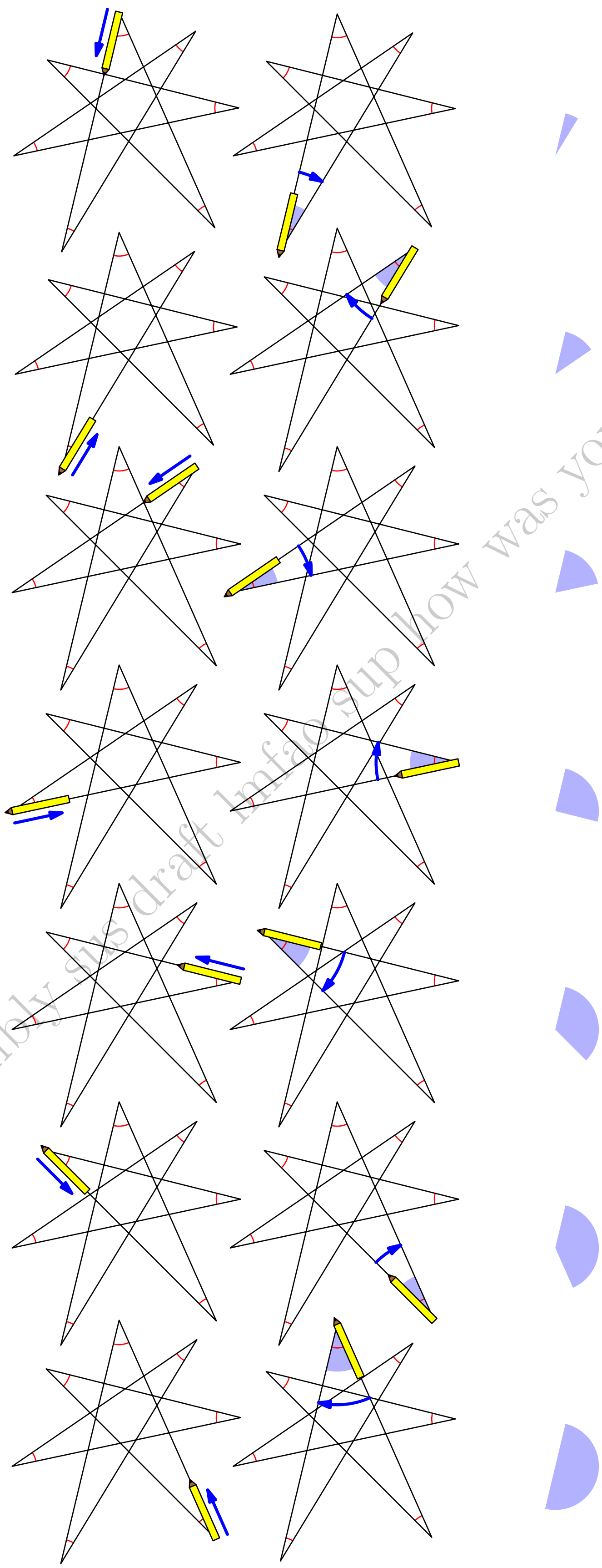
Solution 127

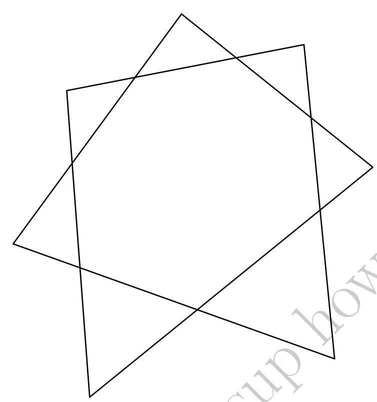
Move a pencil along the edges of the star in the manner described by the diagram on the next page.



Note that the pencil's orientation captures the sum of the angles of the star that the pencil has rotated through. At the end of the procedure, we see that the pencil's orientation has reversed, completing exactly half a rotation. Thus the sum of the angles is $\boxed{180°}$.

■

Remark: This result holds for any "thin" star, no matter how many vertices it has. For "thicker" stars, like the one shown below, the same methodology can be applied to quickly compute the sum of the angles.



There is also a similar (and possibly more well-known) procedure for showing that, in any convex polygon, the sum of the external angles is 360°.

Source: Classic

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Solution 128

Suppose we found such a partition into n sequences, $n > 1$. Let their initial terms be $a_1, \cdots, a_n$ and their common differences be $d_1, \cdots, d_n$. Since the progressions partition the positive integers, we have for all complex $|x| < 1$ that

$$\sum_{k=1}^{\infty} x^k = \sum_{j=1}^{n} \sum_{k=1}^{\infty} x^{a_j + k d_j},$$

or

$$\underbrace{\frac{x}{1-x}}_{LHS} = \underbrace{\sum_{j=1}^{n} \frac{x^{a_j}}{1-x^{d_j}}}_{RHS}.$$

Assume without loss of generality that d_n is the largest common difference. Then, by the condition that all common differences are distinct, we have in particular that $\frac{x^{a_n}}{1-x^{d_n}}$ is the only term in RHS with a pole at $x = e^{2\pi i/d_n}$. (All other terms will be continuous at $x = e^{2\pi i/d_n}$.) So RHS has a pole at $x = e^{2\pi i/d_n}$. In other words,

$$\lim_{x \to e^{2\pi i/d_n}} |RHS| = +\infty.$$

However, the only pole of LHS is at $x = 1$, so

$$\lim_{x \to e^{2\pi i/d_n}} |LHS| \neq +\infty.$$

The two limits are in contradiction.

■

Source: This is an exercise in Stein and Shakarchi's complex analysis textbook.

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Solution 129

The key property of centroids that we require is as follows: For distinct points $x_1, x_2, \cdots, x_n$ in the plane, the minimum value for the sum of squared distances

$$\sum_{i=1}^{n} |x - x_i|^2$$

occurs exactly when x is the centroid of $\{x_1, x_2, \cdots, x_n\}$. We will prove this afterwards.

Let $r > 0$ be Yohane's visual radius. Let the footprints be located at the points $x_1, x_2, \cdots, x_n$. Assume for contradiction that Yohane never stops moving. We claim that the quantity

$$f(x) := \sum_{i=1}^{n} \min\left(|x - x_i|^2, r^2\right)$$

is a strictly decreasing monovariant.

To see this, suppose that Yohane is currently at the point $x = a$, and that the footprints in Yohane's visual radius at this point are $x_1, \cdots, x_k$, relabeling the indices as necessary. We'll split the sum for $f(x)$ into two parts,

$$f(x) = \sum_{i=1}^{k} \min\left(|x - x_i|^2, r^2\right) + \sum_{i=k+1}^{n} \min\left(|x - x_i|^2, r^2\right),$$

and analyze the change in each part separately as Yohane moves from $x = a$ to the centroid of $\{x_1, \cdots, x_k\}$, which we will call g.

For the first part, we use the aforementioned key property of centroids to find that

$$\sum_{i=1}^{k} |a - x_i|^2 > \sum_{i=1}^{k} |g - x_i|^2, \tag{1}$$

where the inequality is strict because g is the unique minimizer of the sum of squared distances to the points in $\{x_1, x_2 \cdots, x_k\}$. Now, on one hand, it is plain to see that

$$\sum_{i=1}^{k} |g - x_i|^2 \geq \sum_{i=1}^{k} \min\left(|g - x_i|^2, r^2\right). \tag{2}$$

On the other hand, since all the points in $\{x_1, x_2, \cdots, x_k\}$ are in Yohane's visual radius at $x = a$, we have $|a - x_i|^2 \leq r$ for all $1 \leq i \leq k$. Hence

$$\sum_{i=1}^{k} \min\left(|a - x_i|^2, r^2\right) = \sum_{i=1}^{k} |a - x_i|^2. \tag{3}$$

Combining (1), (2), and (3), we deduce that

$$\sum_{i=1}^{k} \min\left(|a - x_i|^2, r^2\right) > \sum_{i=1}^{k} \min\left(|g - x_i|^2, r^2\right), \tag{$*$}$$

which shows that the first part must (strictly) decrease.

As for the other part of the sum, we see that $x = a$ maximizes

$$\sum i = k + 1^n \min\left(|x - x_i|^2, r^2\right)$$

since all terms are exactly r^2, which is as large as possible, so Yohane's movement cannot increase this part of the sum. That is,

$$\sum_{i=k+1}^{n} \min\left(|a - x_i|^2, r^2\right) \geq \sum_{i=k+1}^{n} \min\left(|g - x_i|^2, r^2\right). \tag{$**$}$$

Summing $(*)$ and $(**)$ gives $f(a) > f(g)$. This proves that $f(x)$ is a strictly decreasing monovariant, as claimed.

To finish, note since $f(x)$ is *strictly* decreasing, it follows that it takes on infinitely many values while Yohane moves, which in turn implies that Yohane visits infinitely many points (as opposed to revisiting certain points). But there are only finitely many centroids of subsets of $\{x_1, x_2, \cdots, x_n\}$, which are the only possible locations that Yohane can move to. This is a contradiction.

∎

For completeness we now prove the key property of centroids that we've used. In fact, we will prove the following more general statement.

Lemma 1

Let $x_1, x_2, \cdots, x_n$ be points in an inner product space. Let g be the centroid of these points,

$$g = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Then the value of

$$\sum_{i=1}^{n} |x - x_i|^2$$

depends only on $|x - g|$, and in particular it is a strictly increasing function of $|x - g|$.

Proof. This can be proven by a direct computation, but we'll avoid the mess that comes with this via a more refined approach. Write

$$\begin{aligned}\sum_{i=1}^{n}|x-x_i|^2 &= \sum_{i=1}^{n}|x-g+g-x_i|^2\\ &= \sum_{i=1}^{n}\langle x-g+g-x_i, x-g+g-x_i\rangle\\ &= \sum_{i=1}^{n}\left(|x-g|^2+2\langle x-g, g-x_i\rangle+|g-x_i|^2\right)\\ &= n|x-g|^2+2\left\langle x-g, \sum_{i=1}^{n} g-x_i\right\rangle+\sum_{i=1}^{n}|g-x_i|^2,\end{aligned}$$

and note that $\sum_{i=1}^{n} g-x_i = 0$. It follows that

$$\sum_{i=1}^{n}|x-x_i|^2-\sum_{i=1}^{n}|g-x_i|^2 = n|x-g|^2,$$

which proves the lemma. □

As a remark, the same computation shows that

$$\int_E |x-y|^2\,d\mu(y)-\int_E |g-y|^2\,d\mu(y) = \mu(E)|x-g|^2$$

for any measure μ on $\mathbb{R}^N$ and any measurable set $E \subseteq \mathbb{R}^N$, where the centroid g of μ is defined as

$$g := \frac{1}{\mu(E)}\int_E y\,d\mu(y).$$

When μ is "mass", this is the *parallel axis theorem* from physics.

Source: I stole this from someone, but I do not know of an original source.

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Solution 130

Let the triangle be $\triangle ABC$ and let the unique lattice point in its interior be P. By Pick's Theorem, the triangles $\triangle ABP, \triangle BCP$, and $\triangle CAP$ all have an area of $\frac{1}{2}$. In particular, their areas are all equal. This is a defining property of the centroid, so P is the centroid.

■

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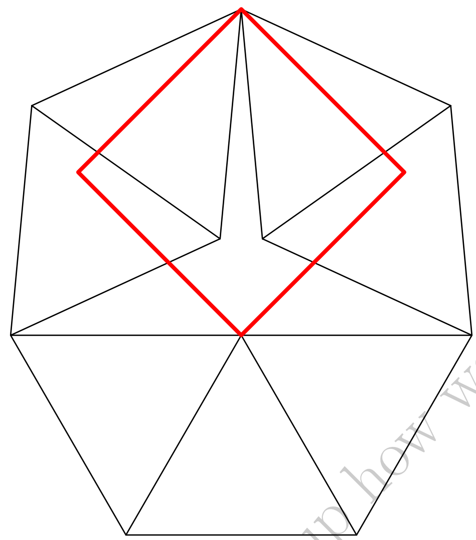
Solution 131

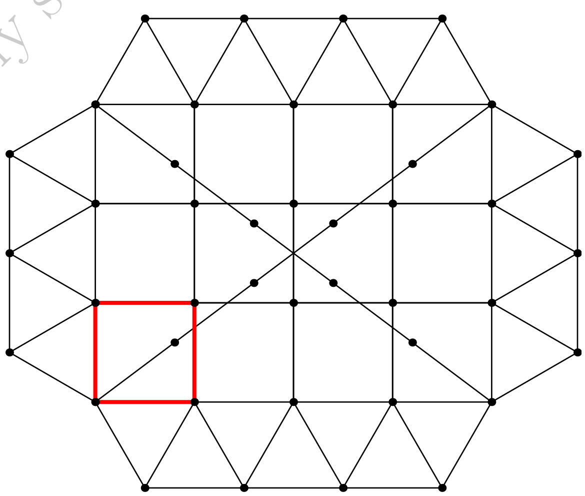
Solution 1



∎

Solution 2

Thanks to "InductionEnjoyer" for the following hilarious construction.



∎

Remark: The original problem can be solved without making any triangles, and can also be solved without any rods intersecting. See https://mathworld.wolfram.com/BracedPolygon.html for the constructions.

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Solution 132

The first series diverges for $\theta = 2\pi/3$. For this value of θ, we have that

$$\sin n^2\theta \cos n\theta = \begin{cases} 0, & n \equiv 0 \pmod 3 \\ \frac{-\sqrt{3}}{4}, & \text{otherwise} \end{cases},$$

so that $\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n} = -\infty$.

The second series converges. Consider the partial sum $\sum_{n=1}^{N} \frac{\cos n^2\theta \sin n\theta}{n}$ and apply the product-to-sum formula to see that

$$\sum_{n=1}^{N} \frac{\cos n^2\theta \sin n\theta}{n} = \sum_{n=1}^{N} \frac{\sin(n(n+1)\theta) - \sin((n-1)n\theta)}{n}.$$

Now split into two sums and do an index shift:

$$= \sum_{n=1}^{N} \frac{\sin(n(n+1)\theta)}{n} - \sum_{n=0}^{N-1} \frac{\sin(n(n+1)\theta)}{n+1}$$

$$= \frac{\sin(N(N+1)\theta)}{N} + \sum_{n=1}^{N-1} \frac{\sin(n(n+1)\theta)}{n^2+n}$$

Sending $N \to +\infty$ we find that

$$\sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n} = \sum_{n=1}^{\infty} \frac{\sin(n(n+1)\theta)}{n^2+n},$$

which converges uniformly by using the bound $|\sin(n(n+1)\theta)| \leq 1$.

■

Source: AMM, C.E. Stanaitis

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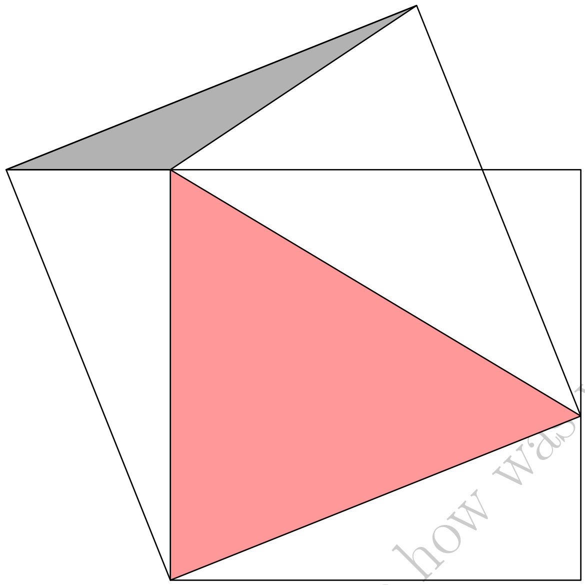
Solution 133



If the smaller square has area 16, then the shaded red triangle has area 8. The red triangle, together with the gray triangle, takes up half the area of the larger square (!), thus the larger square has area $2(1+8) = \boxed{18}$.

■

Source: Math Kangaroo

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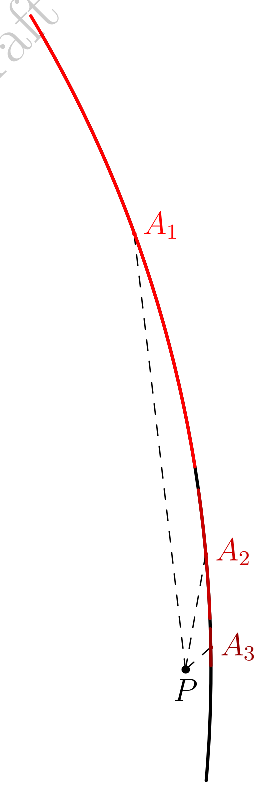
Solution 134

No (finite) number of pirates is sufficient!

Say there are n pirates. My strategy is as follows:

1. Move towards the shore until I am ε away from the edge of the lake. (Here, $\varepsilon > 0$ is a very small distance that will depend on n.)
2. Pick $n + 1$ disjoint arcs each satisfying the following property: If the interval has no pirate in it, then I am guaranteed to escape by moving straight to the interval's midpoint.
3. Since $n + 1 > n$, one of these intervals must be empty. This is the only time I need to check where the pirates are.
4. Move towards that interval's midpoint to escape!

All we need to do is pick $\varepsilon > 0$ so that Step 2 is possible. This entails selecting n disjoint arcs $A_1, A_2, \ldots, A_{n+1}$ such that for all j, my distance to the midpoint of A_j is at most the length of A_j.



A procedure for choosing ε and constructing these arcs is as follows. For ease, take the convention that arcs include their endpoints (and are thus closed).

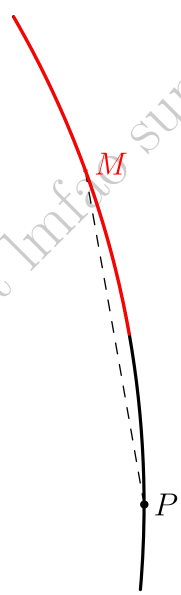
Step 1

The idea here is to "take $\varepsilon = 0$". Take a point P on the boundary of the circle, and we call an arc A *safe* if

$$\text{dist}(P, \text{midpoint}(A)) \leq \text{length}(A).$$

We claim that safe arcs can be disjoint from P and contained in an arbitrarily small neighborhood of P. *(To be more precise: Any arc with endpoint P, no matter how small, will contain a safe arc.)*

To see that this claim is true, choose a point M on the circle that is as close to P as you desire. Then, since **any chord is shorter than the minor arc that it subtends**, the arc with midpoint M and length MP will not contain P.



This arc is safe. Moreover it is evident that this arc "shrinks" to P as we move M towards P, which shows that the safe arcs can indeed be arbitrarily small and arbitrarily close to P. This entails the claim.

Step 2

From Step 1, we can pick a safe arc A_1 disjoint from P. But by the claim from Step 1, we can fit another safe arc A_2 in the gap between P and A_1. We can do this as much as we want, thus we can generate $n + 1$ safe arcs $A_1, A_2, \ldots, A_{n+1}$, all of which are disjoint!

Step 3

Now we move P a very small distance $\varepsilon > 0$ away from the boundary. The danger in doing so is that the condition we desire on the arcs,

$$\text{dist}(P, \text{midpoint}(A_j)) \leq \text{length}(A_j),$$

may no longer be satisfied.

However, by taking ϵ to be sufficiently small, the amount that each of the distances $\text{dist}(P, \text{midpoint}(A_j))$ change by, over all j, can be made to be arbitrarily small! Thanks to this, the condition will be satisfied if we simply expand each arc A_j ever so slightly, to become a new arc A'_j centered at the same midpoint. If these arc expansions are small enough, then the new arcs $A'_1, A'_2, \ldots, A'_{n+1}$ will still be disjoint. This completes the proof.

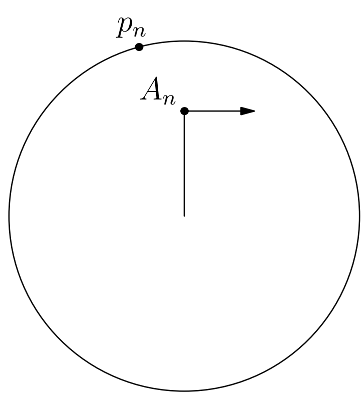
■

Remark: We have shown that no finite number of pirates is sufficient. What about a countably infinite number of pirates? Shockingly, this is still not sufficient!

Showing this requires a more technical argument. Let the radius of the lake be 1. Let O be the center of the lake. Enumerate the pirates via the sequence $\{p_1, p_2, p_3, \ldots\}$. Then follow this procedure:

1. Move towards the edge of the lake until you are x_1 away from the edge of the lake, where $x_1 > 0$ is to be chosen later. Let your current location be the point A_1. Set $n = 1$.

2. We will now "dodge pirate p_n". To do so, draw the chord through A_n which is perpendicular to $\overline{OA_n}$, and move in the direction along this chord which faces away from pirate p_n.



Move in this direction until you are $x_{n+1} > 0$ away from the boundary, where $x_{n+1} > 0$ is to be chosen later. Let the point you arrive at be A_{n+1}.

3. Increment n and loop back to the previous step, repeating infinitely.

This procedure constructs a continuous path. To verify that it has finite length, observe, using power of a point, that

$$|A_nA_{n+1}|^2 < (2 - x_n)x_n < 2x_n. \tag{$*$}$$

Then we have the upper bound

$$|OA_1| + \sum_{n=1}^{\infty} |A_nA_{n+1}| \le (1 - x_1) + \sum_{n=1}^{\infty} \sqrt{2x_n}$$

on the length of the path. For this to converge, we can impose that $x_n \le \frac{1}{4^n}$.

With this, the path must converge to a point, and since $x_n \to 0$, the limit point lies on the boundary of the lake. In other words, the path will reach the edge of the lake at a point A_∞. Fixing n, we claim that pirate p_n cannot have reached A_∞ at this time.

To show this, note that when we reach A_n, the "worst case" position for p_n will be the point M_n, defined as the point on the boundary such that O, A_n, and M_n are collinear in that order. Now:

- Since the remainder of our path has length $\sum_{k=n}^{\infty} |A_kA_{k+1}|$, the set of points that p_n can reach before we reach the shore is the arc with length $2\sum_{k=n}^{\infty} |A_kA_{k+1}|$ and midpoint M_n.
- Once we reach A_{n+1}, the remainder of our path has length $\sum_{k=n+1}^{\infty} |A_kA_{k+1}|$, so a superset of the set of points that the path can end at is given by the disk with center A_{n+1} and radius $\sum_{k=n+1}^{\infty} |A_kA_{k+1}|$.

Now rotate the diagram so that ray $\overrightarrow{A_nA_{n+1}}$ points in the direction of the positive x-axis. For the two sets described in the above two bullet points to be disjoint, it is sufficient to require that their sets of x-coordinates are disjoint.

- When we are at A_n, the x-coordinate of p_n is exactly 0. Hence, the maximum possible x-coordinate that can be reached by p_n by the time we reach A_{n+1} is given by $\sin(|A_nA_{n+1}|)$. After that, a loose upper bound on the maximum possible x-coordinate reached by p_n when we reach A_∞ is

$$\sin(|A_nA_{n+1}|) + \sum_{k=n+1}^{\infty} |A_kA_{k+1}|.$$

- When we are at A_{n+1}, our x-coordinate is exactly $|A_nA_{n+1}|$. Hence a loose lower bound on the minimum possible x-coordinate that we could end up at is given by

$$|A_nA_{n+1}| - \sum_{k=n+1}^{\infty} |A_kA_{k+1}|.$$

Hence it suffices to achieve the inequality

$$\sin(|A_nA_{n+1}|) + \sum_{k=n+1}^{\infty} |A_kA_{k+1}| < |A_nA_{n+1}| - \sum_{k=n+1}^{\infty} |A_kA_{k+1}|,$$

or

$$2\sum_{k=n+1}^{\infty} |A_kA_{k+1}| < |A_nA_{n+1}| - \sin(|A_nA_{n+1}|), \qquad (**)$$

for all n. The next goal is to convert this to another sufficient inequality that is written in terms of the sequence $\{x_n\}_n$.

First we tame the left hand side of $(**)$. Thanks to the earlier observation $(*)$, we have that

$$2\sum_{k=n+1}^{\infty} |A_kA_{k+1}| \leq 2\sqrt{2}\sum_{k=n+1}^{\infty} \sqrt{x_k}.$$

Now we tame the right hand side of $(**)$. From Taylor expansion it is not too hard to verify the bound

$$x - \sin x \geq \frac{x^3}{7}$$

for all $0 \leq x \leq 1$. It follows that

$$|A_nA_{n+1}| - \sin(|A_nA_{n+1}|) \geq \frac{1}{7}|A_nA_{n+1}|^3.$$

To continue, observe by $n-1$ applications of the Pythagorean Theorem that

$$|OA_1|^2 + \sum_{k=1}^{n-1} |A_kA_{k+1}|^2 = |OA_n|^2 = (1-x_n)^2,$$

and similarly

$$|OA_1|^2 + \sum_{k=1}^{n} |A_kA_{k+1}|^2 = |OA_{n+1}|^2 = (1-x_{n+1})^2.$$

Subtracting these two equations and some factoring gives

$$|A_nA_{n+1}|^2 = x_n(2-x_n) - x_{n+1}(2-x_{n+1}).$$

Since $x_n, x_{n+1} \in (0,1)$, it follows that

$$|A_nA_{n+1}|^2 \geq x_n - 2x_{n+1}.$$

Hence

$$\frac{1}{7}|A_nA_{n+1}|^3 \geq \frac{1}{7}(x_n - 2x_{n+1})^{3/2}.$$

Gathering the pieces, we finally arrive at the sufficient inequality

$$2\sqrt{2}\sum_{k=n+1}^{\infty}\sqrt{x_k} \leq \frac{1}{7}(x_n - 2x_{n+1})^{3/2}.$$

I leave it to the reader to verify that taking

$$x_n = \frac{1}{100^{n2^n}}$$

works.

So we have proven that for this choice of $\{x_n\}_n$, p_n could not reach A_∞ by the time we do. But n was arbitrary, so the constructed path will successfully avoid the infinitely many pirates and escape.

This leads to a rather uncomfortable conclusion! On one hand, $|\mathbb{N}|$ pirates is insufficient. On the other hand, $|\mathbb{R}|$ pirates is clearly enough. So the *minimum* number of pirates is either $|\mathbb{R}|$ or some cardinality strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$. I shall let you discover which is true.

Source: Puzzling Stack Exchange

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Solution 135

We prove the contrapositive. Suppose that all subset sums of $S = \{a_1, a_2, \cdots, a_n\}$ are distinct. Then

$$\prod_{i=1}^{n}(1 + x^{a_i}) < \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

for all $0 < x < 1$. Thus

$$\sum_{i=1}^{n} \log(1 + x^{a_i}) < -\log(1-x).$$

Dividing by x and integrating, we find that

$$\sum_{i=1}^{n} \int_0^1 \frac{\log(1 + x^{a_i})}{x}\, dx < \int_0^1 \frac{-\log(1-x)}{x}\, dx.$$

The RHS evaluates to $\frac{\pi^2}{6}$ (Sketch: Taylor expand!), so we focus on the LHS. Using the substitution $u = x^{a_i}$ we have $du = a_i x^{a_i - 1}\, dx = a_i u^{1-\frac{1}{a_i}}\, dx$, so

$$\int_0^1 \frac{\log(1 + x^{a_i})}{x}\, dx = \int_0^1 \frac{\log(1+u)}{u^{\frac{1}{a_i}}} \cdot \frac{du}{a_i u^{1-\frac{1}{a_i}}} = \frac{1}{a_i} \int_0^1 \frac{\log(1+u)}{u}\, du.$$

This evaluates to $\frac{1}{a_i} \cdot \frac{\pi^2}{12}$ (Sketch: Taylor expand!). All in all we have

$$\sum_{i=1}^{n} \frac{1}{a_i} \cdot \frac{\pi^2}{12} < \frac{\pi^2}{6},$$

hence $\sum_{i=1}^{n} \frac{1}{a_i} < 2$ as needed.

■

Remark: The 2 in the problem is tight, by taking S to be $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}\}$ for increasingly large values of n.

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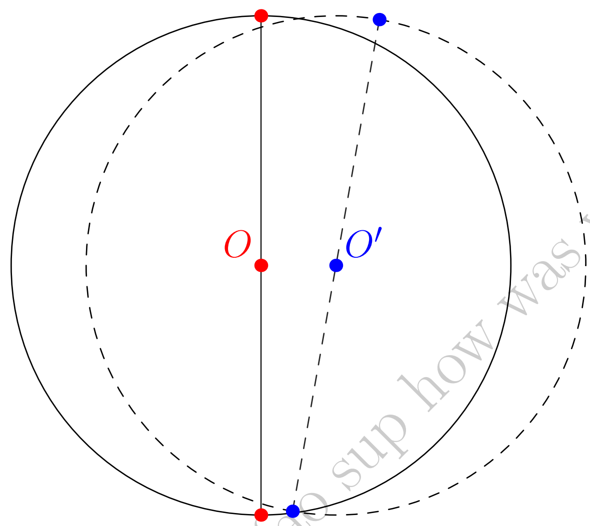
Solution 136

Let the disk have radius 1 and have center O. If the disk is partitioned into two congruent sets A and B, then O lies in one of these sets, say A. Let O' be the point in the *other* set, B, that corresponds to O under the congruency.



Consider the diameter perpendicular to OO'. The endpoints of this diameter cannot belong to B because they lie outside the unit circle centered at O', so they instead belong to A. That is, there exists a segment of length 2 whose endpoints are in A and whose midpoint is O.

By the congruency, it follows that there must be a segment of length 2 whose endpoints are in B and whose midpoint is O'. But every segment of length 2 is a diameter, and the midpoint of every diameter is O, so certainly it could never be O'.

■

Source: Putnam

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Solution 137

Let us call the required condition on the cake slices the *fairness condition.*

We start by cutting the cake into a 2:3 ratio. This ratio is far from tight — it just needs to be somewhere strictly in-between 1:2 and 1:1.

For every next cut, we always bisect the largest piece, but miss slightly so that (1) all piece sizes are distinct, and (2) the fairness condition holds. (Ensuring that all piece sizes are distinct is the key idea!)

Our very first cut satisfied these two criteria, so it remains to show that this condition keeps holding inductively. Suppose that at a certain point, the sizes of the smallest and largest pieces are m and M, respectively. By inductive hypothesis, we know two crucial facts:

(a) The largest piece is the only piece with size M *(because all sizes are distinct!).*

(b) $M/2 < m$

If we were to bisect the piece of size M, then we get two new pieces of sizes $M/2$. By (b), these are the two new smallest pieces. It remains to transfer a small amount of mass from one of these pieces to the other to get sizes of $M/2 - \epsilon$ and $M/2 + \epsilon$, while satisfying the required properties.

We must first ensure that double the size of the smallest piece, which is $2(M/2 - \epsilon)$ or $M - 2\epsilon$, is greater than the size of the new largest piece. By (a), the new largest piece's size cannot be M, so it is strictly smaller than M. Hence we can choose ϵ small enough so that $M - 2\epsilon$ is larger. Thus we can satisfy the fairness condition for all ϵ small enough.

Now it remains to ensure all piece sizes are distinct. Since all piece sizes were distinct before the cut, I need only ensure that $M/2 + \epsilon$ is not equal to the size of any existing piece. Indeed, since I have finitely many pieces, I can always ensure this by a careful choice of ϵ. This completes the induction.

■

Remarks: I found this problem on Math Stack Exchange (https://math.stackexchange.com/questions/2882265/optimal-strategy-for-cutting-a-sausage). The link contains some more interesting discussion on the problem.

For example, the above solution shows that the ratio r between the sizes of the largest and smallest pieces can be ensured to always satisfy $r < 2$. But can we do better than 2? That is, can we find a different strategy so that the ratio r can be ensured to always satisfy

$r < \alpha$ where α is a constant number that's even smaller than 2? The answer is no — 2 is the best you can do.

The accepted answer in the link also provides an interesting explicit construction for the cuts: If we view the cake as the interval $[0, 1]$, then you can make the nth cut at $\{\log_2(2n+1)\}$, where the curly brackets denote the fractional part, $\{x\} := x - \lfloor x \rfloor$.

Source: Math Stack Exchange

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Solution 138

Call my friends Emily and Sydney, number the gold pieces from 1 to 101, and for each piece i pick a distribution of pieces so that Emily and Sydney get the same weight of gold. Now construct a 101×101 matrix M as follows:

$$M_{i,j} = \begin{cases} 0, & i = j \\ 1, & \text{Emily gets piece } j \text{ if I take piece } i \\ -1, & \text{Sydney gets piece } j \text{ if I take piece } i \end{cases}$$

Visually, row i depicts the pieces that would be in Emily's share with 1's if I choose piece i.

Let v be the 101×1 column vector that stores the weights of the gold pieces. Then by construction of M, $Mv = 0$. We would like to show that v is a scalar multiple of 1, the column vector consisting only of ones. Since $M1 = 0$, we have that 1 is in the null space of M, so it is sufficient to prove that the null space of M has dimension 1. This will force v, which is also in the null space, to be in the space spanned by 1, which is what we need.

By rank-nullity, we need only show that M has rank 100. To do this, take the upper 100×100 submatrix of M, and call it M'. We are done if M' has full rank. We show this by proving that $\det M' \neq 0$. Since $\det M'$ is an integer, it is further sufficient to show that $\det M'$ is odd. In the pursuit of this, we may view the elements of M' as elements of $\mathbb{F}_2$, so that we hence need to show that $\det M' = 1$. Over $\mathbb{F}_2$, M' is a matrix with 0's on the diagonal and 1's everywhere else. By a standard formula for determinant, we see that $\det M'$ is equivalent to the number of derangements in S_{100}, which is odd because 100 is even.

■

Source: I forgot where I stole this from but it does seem to be relatively well-known.

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Solution 139

No matter the strategy, the odds will be 1/13. One way to convince yourself of this is to observe that if we stop at any time, then the probability that the top is an ace is always equal to the probability that the bottom card is an ace. So we may reformulate the game to an equivalent one as follows: Deal cards until you say stop, and then you win if the bottom card is an ace.

Well, the bottom card never changes, so the probability of victory will always be the same.

■

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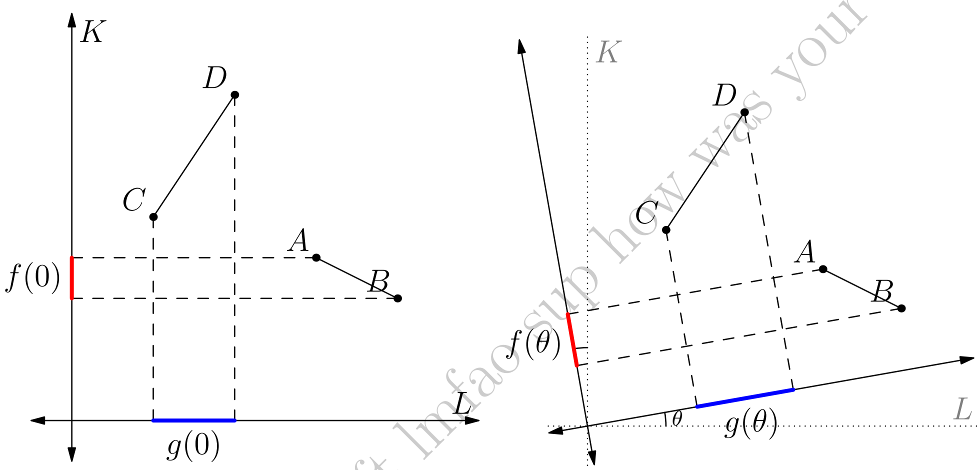
Solution 140

It suffices to prove that $\overline{AB} \perp \overline{CD}$, for if so, then a completely symmetrical argument shows that $\overline{AC} \perp \overline{BD}$ and $\overline{AD} \perp \overline{BC}$, which will entail that any one of the four points will be the orthocenter of the triangle formed by the other three.

Let K and L be two perpendicular lines in the plane, and rotate them counter-clockwise by θ. Let $f(\theta)$ be the length of the orthogonal projection of $\overline{AB}$ unto the rotated K, and let $g(\theta)$ be the length of the orthogonal projection of $\overline{CD}$ unto the rotated L.



Notice that if there exists an angle θ for which $f(\theta) = g(\theta)$, then there will exist a square whose sides will pass through the four points A, B, C, and D. *(This is not quite an equivalent condition as it only accounts for one possible assignment of the four points to the square's four sides.)* Since the hypothesis asserts the contrary, we have $f(\theta) \neq g(\theta)$ for all θ.

If we let α be the angle at which $\overline{AB}$ intersects K, and β be the angle at which $\overline{CD}$ intersects L, then it is not hard to discover that

$$f(\theta) = |AB| \cdot |\cos(\alpha + \theta)|$$

and

$$g(\theta) = |CD| \cdot |\cos(\beta + \theta)|.$$

Since these two quantities are never equal, we may write

$$\frac{|AB|}{|CD|} \neq \left| \frac{\cos(\beta + \theta)}{\cos(\alpha + \theta)} \right|$$

for all θ, which in turn implies that the function $\theta \mapsto \frac{\cos(\beta+\theta)}{\cos(\alpha+\theta)}$ cannot have full range (image) over the domain on which it is defined. If the zeroes of $\theta \mapsto \cos(\beta+\theta)$ and $\theta \mapsto \cos(\alpha+\theta)$ do not coincide, then this function will have zeroes and attain the limits $\pm\infty$ at singularities, which will give it full range by continuity. Thus $\cos(\alpha + \theta)$ and $\cos(\beta + \theta)$ must have the same zeroes, which can occur only when α and β differ by a multiple of π.

This implies that the angle formed between $\overline{AB}$ and $\overline{CD}$ will be $\frac{\pi}{2}$ plus some multiple of π, thus they are perpendicular, as needed.

■

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Solution 141

Solution 1

For each $1 \le i \le n$, let $v_i \in \mathbb{R}^n$ be the vector whose kth component is $\sqrt{\varphi(k)}$ if $k \mid i$, and 0 otherwise. Let A be the matrix whose ith column is v_i. For example, when $n = 5$,

$$A = \begin{bmatrix} \sqrt{\varphi(1)} & \sqrt{\varphi(2)} & \sqrt{\varphi(3)} & \sqrt{\varphi(4)} & \sqrt{\varphi(5)} \\ 0 & \sqrt{\varphi(2)} & 0 & \sqrt{\varphi(4)} & 0 \\ 0 & 0 & \sqrt{\varphi(3)} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\varphi(4)} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\varphi(5)} \end{bmatrix}.$$

Then

$$\begin{aligned} v_i \cdot v_j &= \sum_{k=1}^{n} 1_{(k \mid i \text{ and } k \mid j)} \cdot \sqrt{\varphi(k)} \\ &= \sum_{k=1}^{n} 1_{k|\gcd(i,j)} \cdot \varphi(k) \\ &= \sum_{k|\gcd(i,j)} \varphi(k) = \gcd(i,j), \end{aligned}$$

where in the last line we applied the identity $m = \sum_{k|m} \varphi(k)$ with $\gcd(i,j)$ in place of m.

It follows that the matrix in the original question, whose (i,j) entry was $\gcd(i,j)$, is given by $A^T A$. Its determinant is thus the square of the determinant of A. But A is upper triangular with diagonal entries $\sqrt{\varphi(1)}, \sqrt{\varphi(2)}, \ldots, \sqrt{\varphi(n)}$, so

$$\det A = \sqrt{\varphi(1)\varphi(2)\ldots\varphi(n)}$$

and the desired determinant is the square of the right hand side, which was what was sought.

■

Solution 2

Here I present the proof given in the original paper by Smith which proved this determinant identity.

Let $n = \prod_{i=1}^{k} p_i^{k_i}$. Then

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \sum_{S \subseteq [k]} (-1)^{|S|} n \prod_{j \in S} \frac{1}{p_j}.$$

Let $a_S := n\prod_{j\in S}\frac{1}{p_j}$, so

$$\varphi(n) = \sum_{S\subseteq[k]}(-1)^{|S|}a_S. \tag{$*$}$$

Now, for each set $\emptyset \subset S \subseteq [k]$ we add $(-1)^{|S|}$ times column a_S to the last column. Focus on what happens to the last row. From this column operation, we see from $(*)$ that the (n,n) entry will become

$$\sum_{S\subseteq[k]}(-1)^{|S|}\gcd(n,a_S) = \sum_{S\subseteq[k]}(-1)^{|S|}a_S = \varphi(n).$$

It remains to prove that (m,n) entry is 0 for each $1 \le m < n$, so that we may conclude by argument of induction.

The (m,n) entry is given by

$$\sum_{S\subseteq[k]}(-1)^{|S|}\gcd(m,a_S).$$

We must show that this sum is 0. Since $m < n$, there exists a prime p_j such that $v_{p_j}(m) < v_{p_j}(n)$. This prime is the catalyst for causing cancellations in the sum: We claim that

$$(-1)^{|S|}\gcd(m,a_S) + (-1)^{|S\cup\{j\}|}\gcd(m,a_{S\cup\{j\}}) = 0$$

for each $S \subseteq [k]$ with $j \notin S$. This will complete the proof.

We need only show that $\gcd(m,a_S) = \gcd(m,a_{S\cup\{j\}})$, or

$$\gcd\left(m, n\prod_{i\in S}\frac{1}{p_i}\right) \stackrel{?}{=} \gcd\left(m, n\cdot\frac{1}{p_j}\prod_{i\in S}\frac{1}{p_i}\right). \tag{?}$$

If $p_j \nmid m$, evidently they are equal, since removing a factor of p_j from n will not change the gcd. If $p_j \mid m$, then we need only check that the exponent of p_j is the same on both sides of (?). Indeed, note that

$$v_{p_j}(m) < v_{p_j}(n) = v_{p_j}\left(n\prod_{i\in S}\frac{1}{p_i}\right) \tag{$**$}$$

and thus

$$v_{p_j}(m) \le v_{p_j}\left(n\prod_{i\in S}\frac{1}{p_i}\right) - 1 = v_{p_j}\left(n\cdot\frac{1}{p_j}\prod_{i\in S}\frac{1}{p_i}\right) \tag{$***$}$$

From both $(**)$ and $(***)$ we conclude that there are $v_{p_j}(m)$ factors of p_j on both sides of (?), as needed.

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Solution 142

Call the gloves Glove A and Glove B. To mix all the three bowls, I can use the following procedure. All gloves are to be put on my dominant hand, which shall do all the mixing.

1. Wear Glove A, and then wear Glove B on top of that. Then I'll mix the the first bowl with this.
2. I'll take off Glove B and then mix the second bowl using only Glove A.
3. Lastly, I turn Glove B inside-out and wear it on top of Glove A. I mix the third bowl with this.

Tada!

■

Remark: Those who recognize the problem's premise have likely inferred that I've deemed its original context quite unsavory for a general audience. This is true, and rewriting the problem has proven to be quite bothersome. Indeed, it took me more than a year to come up with a more family-friendly setting!

Here are two highly mathematical problems relying on the same premise, for your perusal.

- m employees from company A are meeting with n companies from company B for a company merger. Every employee from A must shake hands with every employee from company B. Assuming that all employees are germaphobes, what is the least number of gloves needed to accomplish this?
- n germaphobic graduate students are getting to know each other. Every pair of graduate students must shake hands. What is the least number of gloves needed to accomplish this?

Source: Classic

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Solution 143

Let Leilani be the first guard to complete their loop around the museum (or, any such guard that completes their loop at the first time in which a guard's loop is completed). Consider a point T in time at which Leilani is in the final room in her tour of the museum before she returns to her assigned room.

We claim that at time T, no guard is watching their assigned room. Indeed, Leilani is not watching her assigned room. Now consider any other guard, and suppose for contradiction that this guard is watching their assigned room. Then there are two possibilities: Either

(1) the guard has not yet started their tour, or

(2) the guard has finished their tour.

Since Leilani was the first guard to complete their tour, (2) is impossible. So (1) must be the case. That is, the guard has stayed put up to time T. But Leilani visited this guard's assigned room without encountering them, a contradiction.

■

Remark: You can also consider the last guard to start their tour, and then study a point in time when this guard has just exited their assigned room.

Source: Leningrad Mathematical Olympiad, abridged slightly by me

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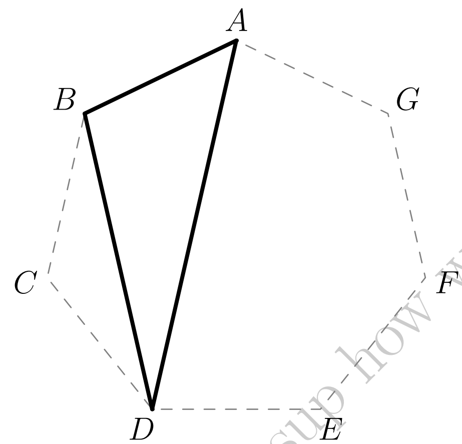
Solution 144

Part (a)

The answer is yes. Take T to be the following triangle formed by three vertices of a regular heptagon H.



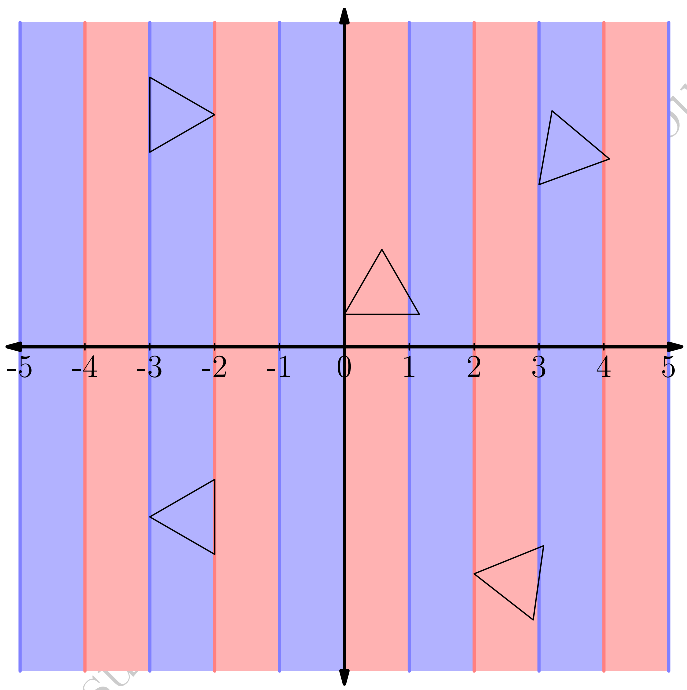
To win with this triangle, Amber only needs to choose 7 points that form a regular heptagon congruent to H. This works because any 2-coloring of the vertices of H must contain a congruent copy of T.

One way to reason this is as follows: Since 7 is odd, two adjacent vertices must have the same color. Without loss of generality we may assume that these vertices are D and E (following the labels in the diagram) and that they are red. If B is red then $\triangle BDE$ is a red triangle congruent to T, so we may assume that B is blue. Similarly we may assume G is blue. Now if C is blue then $\triangle BGC$ is a blue triangle congruent to T, so we may assume C (and, symmetrically, F) are red. With this, $\triangle CFD$ is a red triangle congruent to T.

Part (b)

I thank "wen" for finding this construction. The answer is yes. Take T to be an equilateral triangle with **height** 1 (hence a side length of $\frac{2}{\sqrt{3}}$). Then, to prevent Amber from winning, Beth employs the following simple strategy: If Amber chooses the point (x, y), then Beth colors (x, y) red if $\lfloor x \rfloor$ is even, and otherwise colors (x, y) blue if $\lfloor x \rfloor$ is odd.

In this way, Beth essentially colors the xy-plane in alternating "stripes" of red and blue.



Beth's strategy, with some sample copies of T

Amber can never win because T is "too wide" to fit inside one stripe, but "too thin" for its vertices to land in two different same-color stripes.

More concretely, a copy of T can fit inside a red stripe if and only if there exists a line such that the orthogonal projection of T unto this line has length less than 1. It is not too hard to see that the minimum possible value for the length of this projection (the "minimum width" of T) is given by the length of the shortest height of T, which is 1. So this case is not possible.

On the other hand, for a copy of T to have vertices across two different red stripes, say, A in one stripe and B, C in the other, we have that the height from A to $\overline{BC}$ crosses the width of a blue stripe. So this height has length greater than 1, which is impossible since the heights of $\triangle ABC$ have lengths exactly 1.

The same reasoning holds for the blue stripes, exhausting all reasonable possibilities for a monochromatic copy of T to exist, completing the proof. In fact, this argument shows that any triangle for which such a coloring will result in a win for Beth must have all of its heights to have length both ≥ 1 and ≤ 1, and it is easy to see that the equilateral triangle of height 1 is the only such triangle.

■

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Solution 145

Since each side is 2π-periodic and is invariant under a reflection about $x = \pi$, it is sufficient to consider $0 \le x \le \pi$. In fact, since a reflection over $x = \pi/2$ negates $\sin(\cos x)$ but does nothing to $\cos(\sin x)$, it is sufficient to consider $0 \le x \le \pi/2$, where $\sin(\cos x)$ is non-negative.

For such x, we may write

$$\sin(\cos x) \le \cos x \le \cos(\sin x), \qquad (*)$$

where we have applied the inequality $\sin y \le y$ for $y \ge 0$, twice, and have used the fact that cos is decreasing over $[0, \pi/2]$.

Since the equality case of $\sin y \le y$ occurs only when $y = 0$, we could only have equality in $(*)$ if $\cos x = 0$ and $x = 0$ hold simultaneously, which cannot be the case, so it is strict.

■

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Solution 146

There are six points of tangency, distributed symmetrically over the surface of the sphere, and so they form the vertices of a regular octahedron with edge length $\sqrt{2}$. The answer is the circumradius of one of its sides, which is $\sqrt{\frac{2}{3}}$.

■

Source: Me, for CMIMC

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Solution 147

This video by *polylog* (https://www.youtube.com/watch?v=-64UT8yikng) does a very nice job at showcasing a construction for the dice. Give it a watch if you'd prefer a more visual explanation.

It suffices to construct a finite sequence of elements in $\{1, 2, ..., n\}$ such that the subsequence $\pi(1), \pi(2), \ldots, \pi(n)$ appears equally many times over all permutations $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$.

We go by induction. Obviously, by taking the sequence $1, 2, \cdots, n$, we can construct a sequence such that the one-element subsequence (i) appears equally many times over all i.

Now assume that for some $1 \leq k < n$, we have constructed a sequence A for which the sequence $a_1, a_2, \cdots, a_k$ appears as a subsequence of A equally many times over all possible selection of k distinct elements $a_1, a_2, \cdots, a_k \in \{1, 2, \cdots, n\}$.

Let $\pi_1, \pi_2, \cdots, \pi_{n!}$ be the $n!$ permutations on $\{1, 2, \cdots, n\}$. Denote by $\pi_i(A)$ the sequence whose jth element is $\pi_i(A_j)$. That is, it is simply the sequence A but with its elements relabeled according to π_i. Now we take the sequence

$$B := \pi_1(A)^\frown \pi_2(A)^\frown \cdots ^\frown \pi_{n!}(A),$$

where $^\frown$ denotes concatenation of sequences. We claim that the sequence $b_1, b_2, \cdots, b_{k+1}$ appears as a subsequence of B equally many times over all possible selection of $k+1$ distinct elements $b_1, b_2, \cdots, b_{k+1} \in \{1, 2, \cdots, n\}$.

It is easier to see this by reverting back to the probabilistic view: Let us select a random subsequence b of B which consists of $k+1$ distinct integers. Then we claim that every selection and ordering of these $k+1$ integers is equally likely.

There are two ways to pick such a subsequence b.

- The first way is that all elements of b fall into the same $\pi_i(A)$ sequence for some i. If we must choose b in this way, then it is equally likely to fall into any $\pi_i(A)$ for any $1 \leq i \leq n!$. Since $\pi_1, \cdots, \pi_{n!}$ runs through all permutations, this forces every selection and ordering of b to be equally likely.

- The second way is that the first way does not occur. That is, each $\pi_i(A)$ contains at most k elements of b. Suppose we restrict the probability space to a specific way to distribute the elements in this way (e.g. consider only subsequences b with 3 elements in $\pi_7(A)$, 2 elements in $\pi_8(A)$, and $k-4$ elements in $\pi_9(A)$). By definition of A, every possible selection and ordering of the elements of b within some $\pi_i(A)$ is equally likely. This holds for all i, so every selection and ordering of b is equally likely.

This completes the induction.

■

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Solution 148

Consider a gridline segment. If it's on the border, then it can be claimed and counted towards the perimeter by claiming the square that it borders. Otherwise, the segment will either not count towards either player's perimeter (when both adjacent squares are claimed by the same player) or it will count equally towards both player's perimeters (when the adjacent squares are claimed by different players).

Thus, no interior gridline segment will help either player win. The only factor that contributes to the difference in the players' perimeters is the number of segments claimed on the boundary of the grid. Thus the best strategy entails claiming as many such segments as possible.

The corner squares are worth the most since they each contain two boundary segments. Ashley and Beth must first rush to claim as many of these as possible, and they will be tied in doing so because four is even. Then, since there are an even number of non-corner squares along the boundary (4×2021, to be exact), they will also be tied in claiming the number of such squares. So they will tie in the end. This fully decides the game since, as discussed, nothing that occurs in the interior of the board actually matters. So they will tie.

■

Source: Math Hour Olympiad

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Solution 149

Part (a)

We take the $\binom{n}{2}$ points that have exactly two components equal to 1, with all other components equal to 0.

For any two distinct such points, their "1-components" either "overlap" at one component, resulting in one possible distance ($\sqrt{2}$), or they do not "overlap", resulting in a second possible distance ($\sqrt{4}$).

Part (b)

We use the construction for Part (a) for the next dimension, $\mathbb{R}^{n+1}$! The key observation is that the $\binom{n+1}{2}$ points in the constructed set are coplanar (i.e. lie in a common n-dimensional subspace), and this is because they all lie in the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 2$. This hyperplane is a copy of $\mathbb{R}^n$, thus this corresponds to a two-distance set of size $\binom{n+1}{2}$ in $\mathbb{R}^n$. ■

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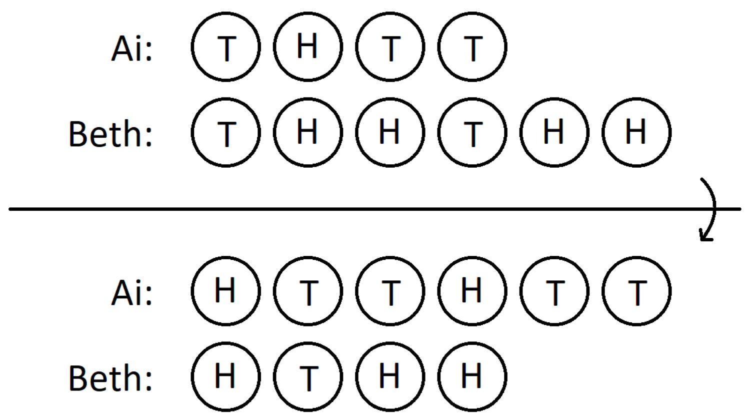
Solution 150

Part (a)

The answer is neither!

We will biject each way that Ai could go broke first to a way that Beth can go broke that has the same probability of occurring. This will prove that Ai and Beth are equally likely to have gone broke first.

The bijection is simple: For a sequence of coin flips where Ai goes broke first, flip every heads to a tails and vice versa, then switch Ai's and Beth's sequences of coinflips.

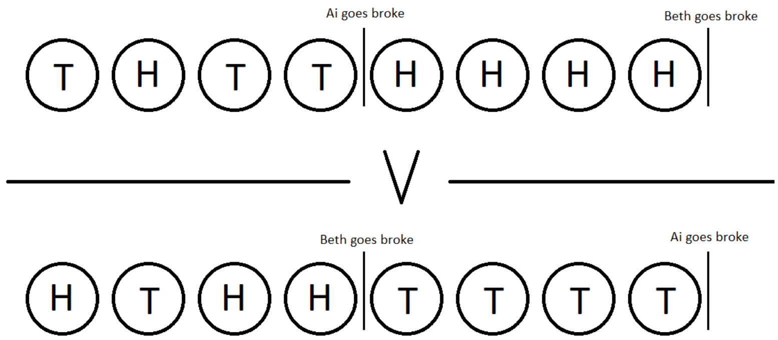


It is easy to see that this results in a game where Beth goes bankrupt first instead of Ai, and that this is a bijection.

To see that they have the same probability of occurring, observe that when Ai and Beth go bankrupt, the difference between the tails flipped and heads flipped by Ai is equal to the difference between the heads flipped and tails flipped by Beth, because they start with the same amount of money. Thus the total number of heads flipped is equal to the total number of tails flipped. Ergo, switching all heads to tails and vice versa does not change the total number of heads and tails.

Part (b)

We use the same bijection in which we flip every heads to a tails and vice versa. However, we now note that in the new sequence, the probability decreases.



This is because in the original sequence, Beth goes broke last, meaning there were more heads than tails. The transformed sequence will then have more tails than heads. Since heads are more likely, the original sequence will always be more likely than the transformed sequence.

We deduce that Beth is more likely to go broke *last*. So Ai is more likely to go broke *first*. ■

Source: Peter Winkler

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Solution 151

Beth needs to only spend two dollars.

Beth first obtains an upper bound on the largest coefficient by asking for $P(1)$. Let k be the number of digits of $P(1)$. Beth now knows that every coefficient has at most k digits. So, Beth now asks Angela to hand over the value of $P(10^k)$, and this forces Angela to quite literally write down all the coefficients plainly.

For example, if $P(x) = 31x^2 + 41x + 59$, then Angela will end up giving Beth the value of $P(1000)$, which is 31041059. The selection of k ensures that no two coefficients "collide" upon computing $P(10^k)$.

■

Remarks: Beth could have used any base in place of 10. If Beth were allowed to request $P(a)$ for a a *real* number, then in theory one dollar would be enough by simply asking or $P(\pi)$. Though, this would require infinitely precise computation to ensure victory.

Source: Classic

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Solution 152

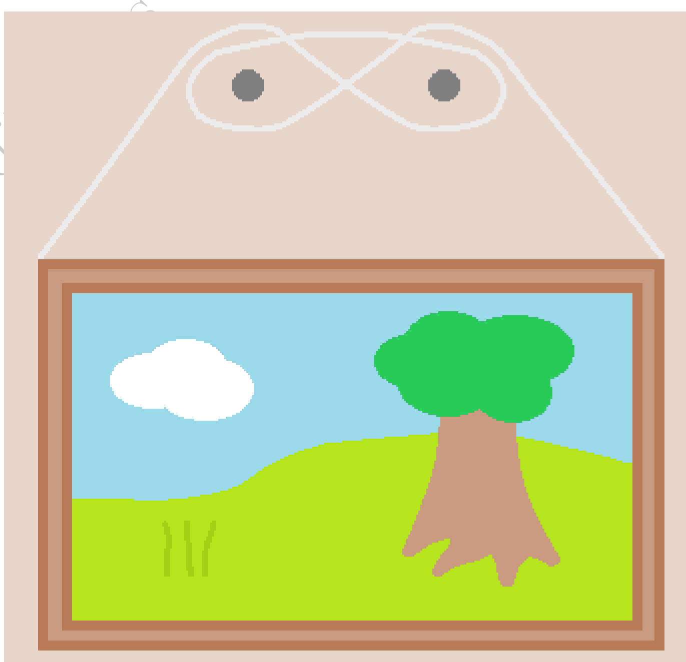
A clean approach is as follows. To tie a rope around two nails consists of a sequence of the following four actions:

- (L) Wrap it once counter-clockwise around the left nail.
- (L^{-1}) Wrap it once clockwise around the left nail.
- (R) Wrap it once counter-clockwise around the right nail.
- (R^{-1}) Wrap it once clockwise around the right nail.

A sequence of such actions can be simplified if inverse operations are adjacent. For example, $LR^{-1}RR$ simplifies to LR, and this represents the weight of the painting untying some of its loops. If a sequence can fully simplify into an empty sequence, such as $LRR^{-1}L^{-1}$, then this represents the knot failing and the painting falling.

We aim to generate such a sequence of these actions that cannot be simplified to an empty sequence, but would collapse to an empty sequence if either all "left" actions (L, L^{-1}) are removed or all "right" actions (R, R^{-1}) are removed, as this corresponds to removal of the left nail or right nail.

Indeed, the *commutator* $LR^{-1}L^{-1}R$ works! This corresponds to the following picture.



You can verify visually that this painting will indeed fall if either nail is removed.

■

Remarks: See the paper http://arxiv.org/pdf/1203.3602.pdf for more problems of this flavor. (It comes with nice pictures!)

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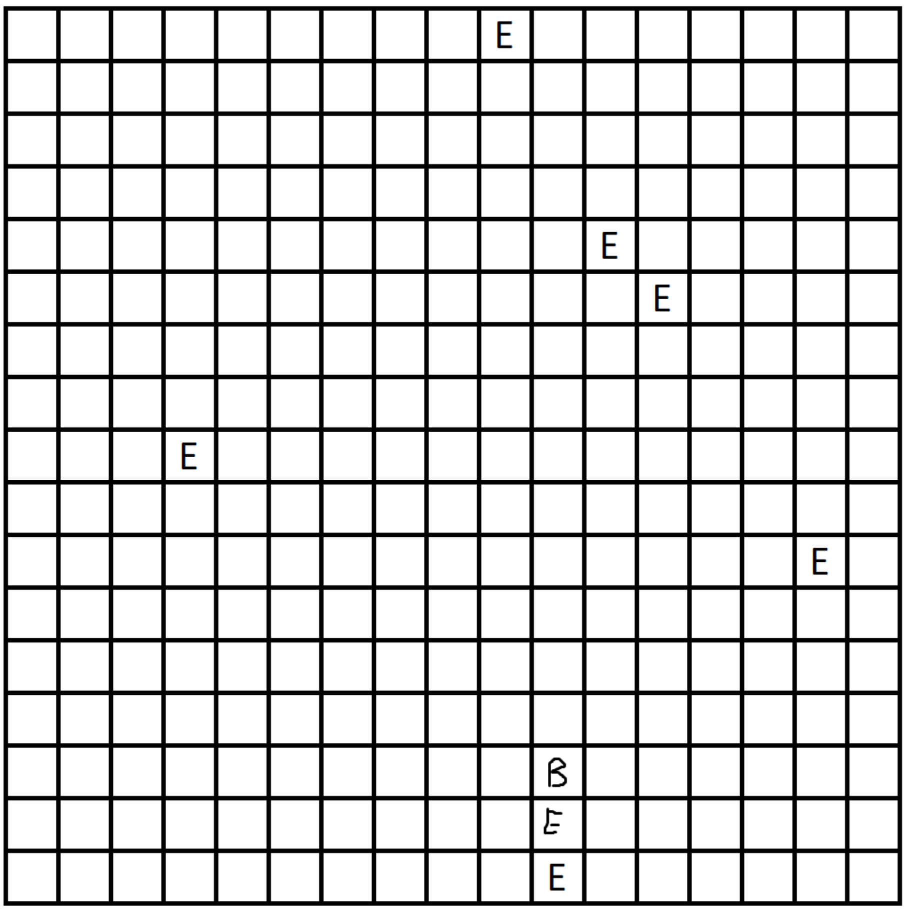
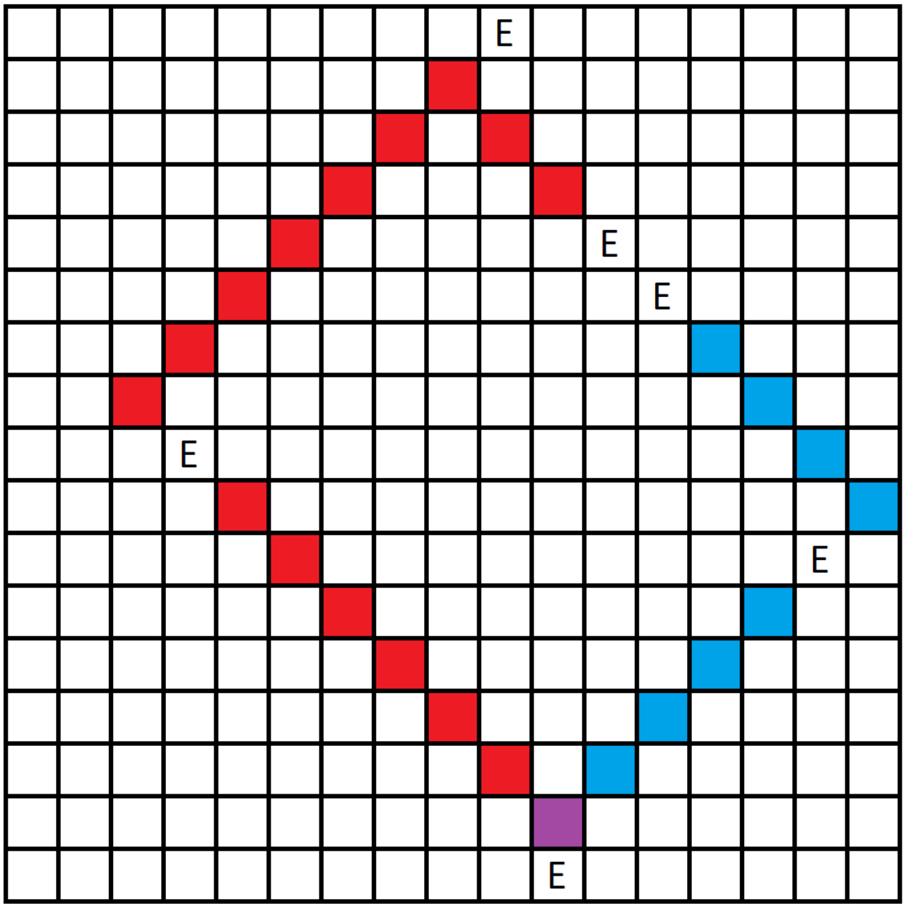
Solution 153

We claim BEE is located here:



Suppose for contradiction that BEE is located elsewhere. If the location of BEE is known, then for any sequence of cells of the form EE_ or _EE, we may deduce that the blank square is an E, since we are guaranteed that BEE cannot appear more than once. Call this principle "tripling".

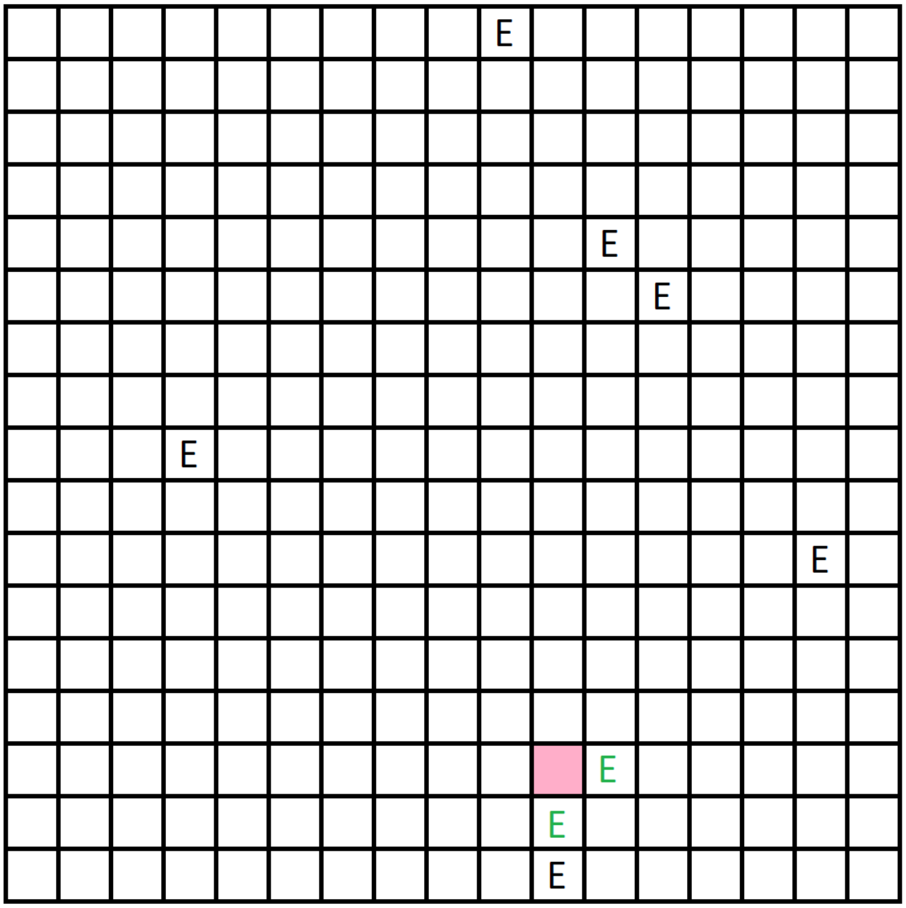
Consider the two "chains" of squares:



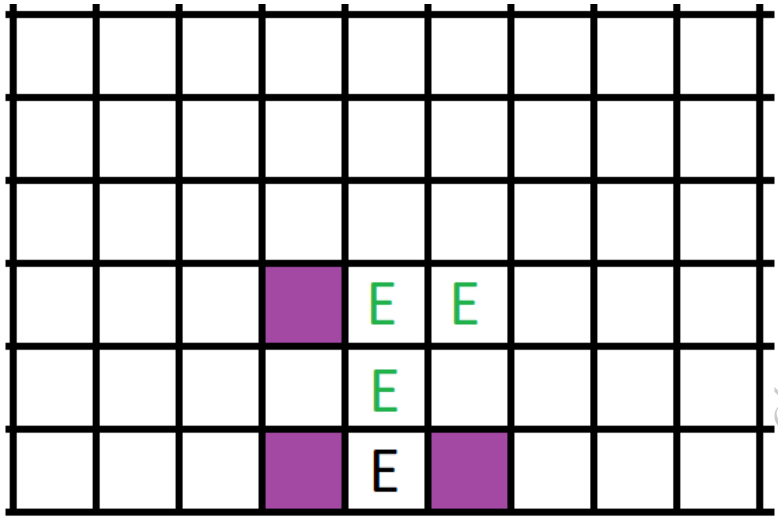
(The chain of shaded squares leading out left of the diagonally-adjacent pair of E's is colored red. The chain of shaded squares on the right is colored blue. The square above the bottom-most E is colored purple.)

We claim that the purple square is E. Suppose not. The BEE must be somewhere, and the BEE's B cannot be on both the red and blue chains at once. So one of the chains does not contain the BEE's B. Without loss of generality, suppose it is the red chain. Then by tripling, starting from the two diagonally-adjacent E's, we must keep placing E's along the red chain until we reach the purple square, which is a B, thus forming a second BEE, contradiction.

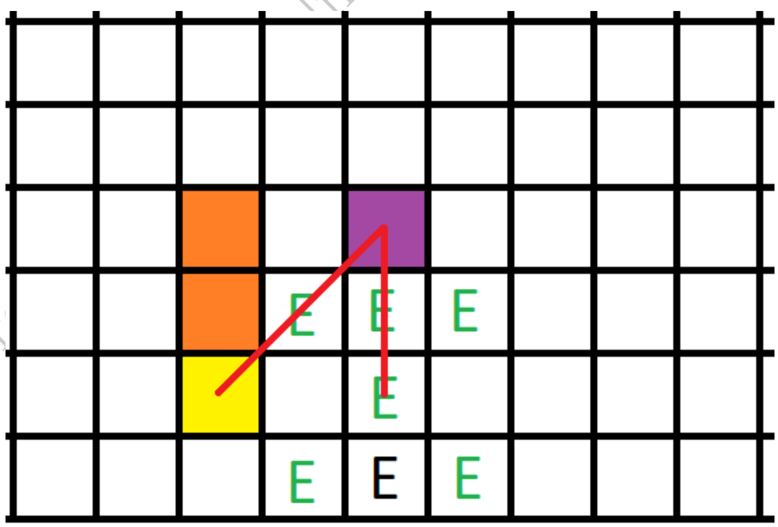
Using the same idea, it is not hard to deduce also that one of the red or blue squares diagonally adjacent to the purple square is also an E. Without much harm to the generality of the argument, let us place the E on the blue square.



Since we have assumed for contradiction that the BEE we're looking for is not where we claimed it is, we know that the pink square is an E. From here we need to do a bit of gruntwork to expand to a more workable structure. First, observe by a simple inspection that none of the three purples squares shown below can be a B, otherwise two BEEs will be created among these squares.

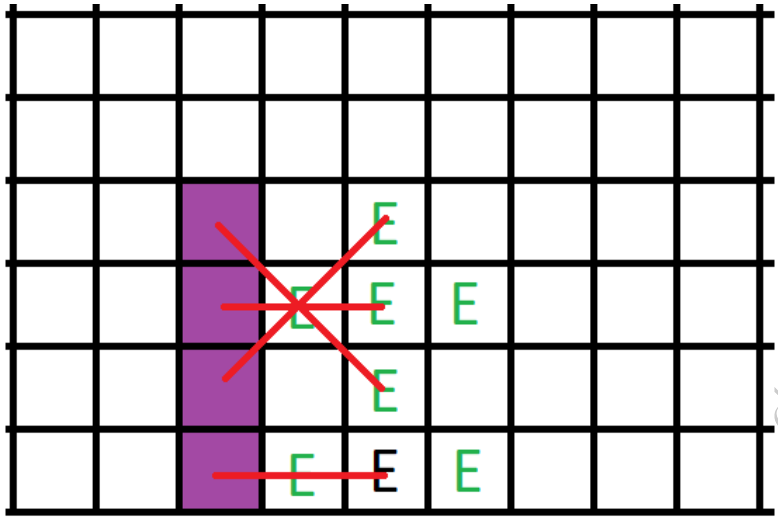


Next, the below purple square must be an E, otherwise by three triplings we see that the orange squares are E and then the yellow square must be an E, creating two BEEs.

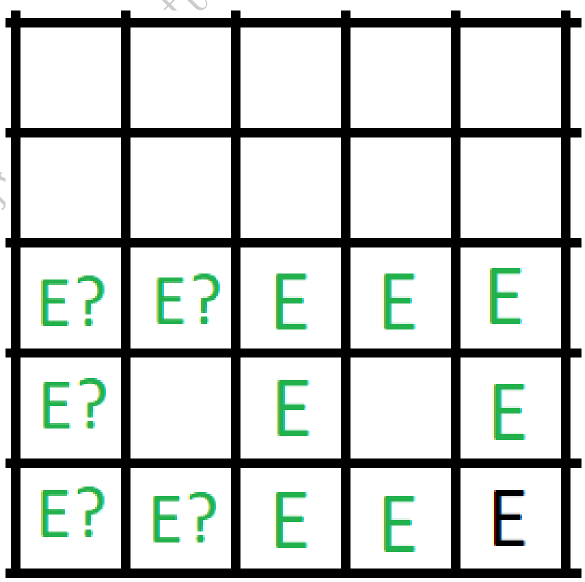


(The colors of the shaded squares, in "book order", are: orange, purple, orange, yellow)

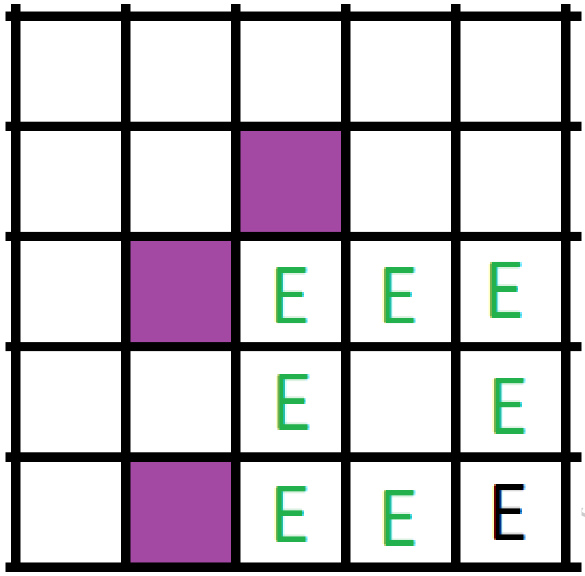
At this point, we see that if any of the four below purple squares are B, then the other three must be E, creating two BEEs. So in fact, all of them are E.



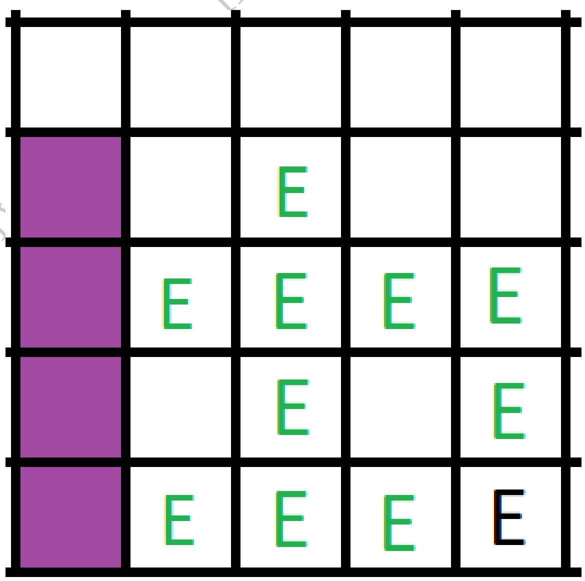
What we have accomplished with this work is the creation of a "square" of E's. The major claim is that we can always expand any square of E's in any direction. Note that in general it is safe to assume that the empty squares above exist.



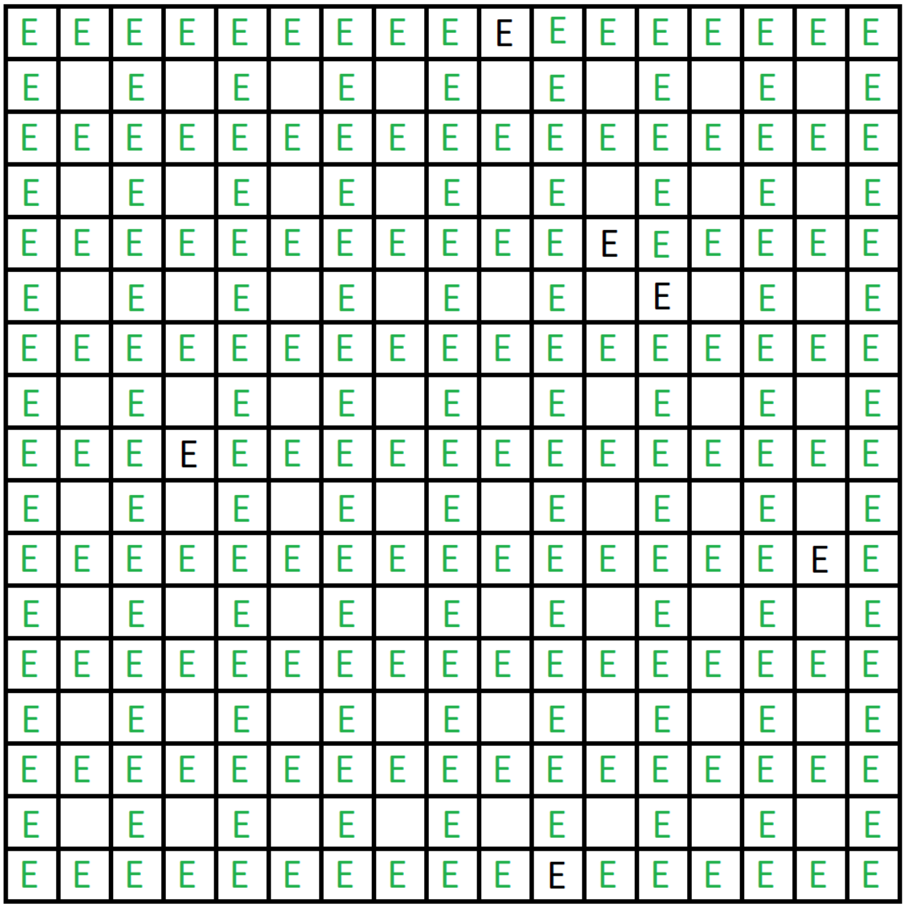
Indeed, we have for free that the three below purple squares are E,



and that, as before, all of the four purple squares below must simultaneously be E.



We have thus proven the major claim. By iteratively applying the major claim, we may fill the following squares with E's.



From here, it is now not difficult to argue that all of the remaining blank squares must be the same letter, i.e. either all B's or all E's. It follows that BEE does not appear in the grid, contradiction.

■

Source: Me!

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Solution 154

Let the quartic be $P(x) = ax^4 + bx^3 + cx^2 + dx + e$.

- Adding a linear function to $P(x)$ corresponds to an affine transformation of the plane, which preserves ratios along lines. Thus we may add $-dx - e$ to assume WLOG that $P(x)$ takes the form $ax^4 + bx^3 + cx^2$.
- By vertical scaling, which also preserves ratios along lines, we may divide by a to assume that $P(x)$ takes the form $x^4 + bx^3 + cx^2$.
- $-b$ is the sum of the roots of P, and by translation we can assume that this sum is zero. So we may assume that $P(x)$ takes the form $x^4 + cx^2$.
- By a horizontal scaling by a factor of $\frac{1}{\sqrt{|c|}}$, followed by another vertical scaling, we may assume that $P(x)$ is either $x^4 + x^2$ or $x^4 - x^2$.

The second derivatives of these two candidates are $12x^2 + 2$ and $12x^2 - 2$. Since there are two inflection points, we may eliminate the first candidate. The inflection points then occur at the roots of $x^2 - 1/6$. Now write

$$x^4 - x^2 + 5/36 = (x^2 - 1/6)(x^2 - 5/6).$$

This shows that the line connecting these inflection points is given by $y = -5/36$, and that the other intersections of this line with the graph of $x^4 - x^2$ occur at $x = \pm\sqrt{5/6}$. The desired ratio is then

$$\frac{\sqrt{5/6} - (-\sqrt{1/6})}{\sqrt{1/6} - (-\sqrt{1/6})} = \frac{\sqrt{5} + 1}{2} = \varphi$$

as needed.

■

Source: This is called "Lin McMullin's Theorem".

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Solution 155

Part (a)

We guess that $f(x)$ takes the form ax^b. If $f'(x) = f^{-1}(x)$, then

$$abx^{b-1} = a^{-1/b}x^{1/b}.$$

So we wish to solve the system

$$\begin{cases} ab = a^{-1/b} \\ b - 1 = 1/b \end{cases}.$$

The second equation immediately gives $b = \varphi$. So it remains to solve for a in $\varphi a = a^{-1/\varphi}$. Dividing by a gives $a^{-1/\varphi - 1} = \varphi$ or $a^{-\varphi} = \varphi$. So $a = \varphi^{\frac{-1}{\varphi}}$. We conclude that

$$\boxed{f(x) = \varphi^{-1/\varphi}x^{\varphi}}$$

is a solution.

■

Part (b)

Step 0

Since $x \in (0, \infty)$, we must have $f^{-1}(x) > 0$, and so $f'(x) > 0$ for all x. Moreover we must have $f(0^+) = 0$ and $f(\infty) = \infty$ in order for f^{-1} to be well-defined for all $x > 0$. In view of this we may treat f to be of type $f : [0, \infty] \to [0, \infty]$.

Step 1

The key tool is that if we know that $f(x) > ax^b$ for $x \in [0, T]$, then we have the following deductions:

$$\begin{aligned} f(x) > ax^b \text{ for } 0 \le x \le T &\implies f^{-1}(x) < \left(\frac{x}{a}\right)^{1/b} \text{ for } 0 \le x \le f(T) \\ &\implies f'(x) < \left(\frac{x}{a}\right)^{1/b} \text{ for } 0 \le x \le f(T) \\ &\implies f(x) < \int_0^x \left(\frac{t}{a}\right)^{1/b} dt \text{ for } 0 \le x \le f(T) \\ &\implies f(x) < \frac{b}{(b+1)a^{1/b}}x^{\frac{b+1}{b}} \text{ for } 0 \le x \le f(T) \end{aligned}$$

We similarly have that

$$f(x) < ax^b \text{ for } 0 \le x \le T \implies f(x) > \frac{b}{(b+1)a^{1/b}}x^{\frac{b+1}{b}} \text{ for } 0 \le x \le f(T).$$

For this to be of use, we need to select a T so that $[0,T] = [0, f(T)]$. This motivates locating a non-trivial fixed point of f.

Suppose for contradiction that f has no fixed point in $(0,\infty)$. Then either $f(x) > x$ or $f(x) < x$ for all $0 < x < \infty$. If $f(x) > x = 1x^1$ then $f(x) < \frac{1}{1+1}\left(\frac{x}{1}\right)^{\frac{1+1}{1}} = \frac{1}{2}x^2$ for all such x, and it is not too hard to see that this is contradictory with $f(x) > x$. Similarly the hypothesis $f(x) < x$ leads to a contradiction. So a fixed point $x_0 \in (0,\infty)$ exists.

Step 2

x_0 is, in fact, the unique fixed point in $(0,\infty)$. To see this, write

$$f''(x) = \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} > 0$$

to deduce that f is strictly convex for $x > 0$. Now, as $f(0) = 0$ and $f(x_0) = x_0$, it must follow that $f(x) < x$ for $0 < x < x_0$, and $f(x) > x$ for $x_0 < x < \infty$.

Step 3

We now repeatedly apply Step 1 to the inequality $f(x) < x$ over the interval $[0, x_0]$. Let $a_0 = 1$, $b_0 = 1$, and, recursively, define

$$a_n := \frac{b_{n-1}}{(b_{n-1}+1)a_{n-1}^{\frac{1}{b_{n-1}}}}$$

and

$$b_n := \frac{b_{n-1}+1}{b_{n-1}}$$

for $n \geq 1$. Then $f(x) < a_0 x^{b_0}$ for $x \in [0, x_0]$, and so Step 1 tells us that $f(x) > a_1 x^{b_1}$ for $x \in [0, x_0]$. Proceeding inductively, we discover that

$$a_{2k+1}x^{b_{2k+1}} < f(x) < a_{2k}x^{b_{2k}}, \qquad x \in [0, x_0]$$

for all k. In the next step we will see that as $k \to \infty$, the upper and lower bounds will squeeze f to the function $\varphi^{\frac{-1}{\varphi}} x^{\varphi}$.

Step 4

We will now show that the sequences $\{a_n\}_n$ and $\{b_n\}_n$ converge. It is classical that $\{b_n\}_n$ converges to φ and so I will omit the proof of this. As for $\{a_n\}_n$, we will apply the following lemma which I will prove later.

Lemma 1

Suppose that the sequences $\{r_n\}_n$ and $\{s_n\}_n$ converge to r and s respectively, and that $|r| < 1$. Then the sequence $\{u_n\}_n$ recursively defined by

$$u_n := r_{n-1}u_{n-1} + s_{n-1}$$

converges for any initial data u_0.

To apply this lemma, apply the logarithm to the definition of a_n to find that

$$\log a_n = \log\left(\frac{b_{n-1}}{(b_{n-1}+1)}\right) - \frac{1}{b_{n-1}} \log a_{n-1}.$$

We apply the lemma to the sequence $\{\log a_n\}_n$, noting that $-\frac{1}{b_{n-1}} \to -\frac{1}{\varphi}$ and that $\left|-\frac{1}{\varphi}\right| < 1$. This tells us that $\log a_n$ converges, hence so does a_n.

To find the value of the limit of a_n, we simply send $n \to \infty$ in the definition of a_n. This, combined with some algebra, will give us that $\lim_{n\to\infty} a_n = \varphi^{-\frac{1}{\varphi}}$, which is what we expected.

From the convergence of these sequences, we may conclude by the squeeze rule that $f(x) = \varphi^{-\frac{1}{\varphi}} x^{\varphi}$ for all $x \in [0, x_0]$.

Step 5

Since x_0 is a fixed point,

$$x_0 = f(x_0) = \varphi^{-\frac{1}{\varphi}} x_0^{\varphi}$$

and so we may now solve for x_0. Working out the algebra, we find that $x_0 = \varphi$.

Step 6

We know from strict convexity that $f(x) > x$ over $(x_0, \infty) = (\varphi, \infty)$. It follows that, for any large $T \gg \varphi$ of our choice, we have that

$$f(x) > \frac{1}{T}x^2$$

for all $0 < x < T$. Indeed, for $0 < x \leq \varphi$ you can verify that $f(x) = \varphi^{-1/\varphi}x^{\varphi} > \frac{1}{T}x^2$, and for $\varphi < x < T$ we have $f(x) > x > \frac{1}{T}x^2$.

Hence, if we define the sequences $\{a_n\}_n$ and $\{b_n\}$ as in Step 3, with initial data $a_0 = \frac{1}{T}$ and $b_0 = 2$, then by Step 1, we must have $f(x) < a_1 x^{b_1}$ for $0 < x < f(T)$. But $f(T) > \frac{1}{T}T^2 = T$, so $f(x) < a_1 x^{b_1}$ holds for $0 < x < T$.

Inductively, as in Step 3, we thus obtain

$$a_{2k}x^{b_{2k}} < f(x) < a_{2k+1}x^{b_{2k+1}}, \qquad x \in [0, T)$$

for all k. By Step 4, the upper and lower bounds converge and squeeze $f(x)$ so that we get $f(x) = \varphi^{-1/\varphi}x^{\varphi}$ for all $0 < x < T$. But T was arbitrary, so this conclusion holds for all $0 < x < \infty$. This completes the main proof.

Step 7

Finally, we prove the lemma. The proof I present is a bit weird, and it turns out to be slightly cleaner to re-index the sequences: Suppose $r_n \to r$ and $s_n \to s$ with $|r| < 1$, and recursively define $u_n = r_n u_{n-1} + s_n$. Then we claim u_n converges for any u_0.

We in fact claim that $u_n \to \frac{s}{1-r}$. Since $\frac{s_n}{1-r_n} \to \frac{s}{1-r}$, it is sufficient to show that $\left|u_n - \frac{s_n}{1-r_n}\right| \to 0$.

Before we begin to do this, it will be important for later to demonstrate that $\{u_n\}_n$ is bounded. First, pick some $R > 0$ strictly between $|r|$ and 1. Then there is some N large enough so that $|r_n| < R < 1$ for all $n \geq N$. For all such n we have

$$|u_n| \leq R|u_{n-1}| + S$$

where S is an upper bound on $|s_n|$. By induction, we have that $|u_n| \leq h^{n-N}(|u_N|)$ where $h(z) := Rz + S$, and since $|R| < 1$ the Banach Fixed Point theorem applied to h shows that $\lim_{n\to\infty} h^{n-N}(|u_N|)$ exists, proving that u_n is bounded.

We return to the proof that $\left|u_n - \frac{s_n}{1-r_n}\right| \to 0$. Fix $\varepsilon > 0$ and find N_ε such that for all $n > N_\varepsilon$, the following hold:

- $\left|\frac{s_n}{1-r_n} - \frac{s}{1-r}\right| < \frac{\varepsilon}{10}$ (In particular we seek $\left|\frac{s_n}{1-r_n} - \frac{s_{n-1}}{1-r_{n-1}}\right| < \varepsilon$)
- $N_\varepsilon \geq N$, so that $|r_n| < R < 1$.

Then, for $n > N_\varepsilon$, we have the following bound.

$$\begin{aligned}
\left|u_n - \frac{s_n}{1-r_n}\right| &= \left|r_n u_{n-1} + s_n - \frac{s_n}{1-r_n}\right| \\
&= \left|r_n u_{n-1} - \frac{r_n s_n}{1-r_n}\right| \\
&= |r_n| \cdot \left|u_{n-1} - \frac{s_n}{1-r_n}\right| \\
&\leq |r_n| \cdot \left|u_{n-1} - \frac{s_n}{1-r_n}\right| + |r_n| \cdot \left|\frac{s_n}{1-r_n} - \frac{s_{n-1}}{1-r_{n-1}}\right| \\
&\leq |r_n| \cdot \left|u_{n-1} - \frac{s_n}{1-r_n}\right| + |r_n|\varepsilon.
\end{aligned}$$

Inductively,

$$\left|u_n - \frac{s_n}{1-r_n}\right| \le |r_n r_{n-1} \dots r_{N_\varepsilon+1}| \cdot \left|u_{N_\varepsilon} - \frac{s_{N_\varepsilon}}{1-r_{N_\varepsilon}}\right| + |r_n|\varepsilon + |r_n r_{n-1}|\varepsilon + \cdots + |r_n r_{n-1} \dots r_{N_\varepsilon+1}|\varepsilon.$$

Now, we will demonstrate that this is small. For the first term, both u_n and $\frac{s_n}{1-r_n}$ are bounded sequences, hence so is $\left|u_n - \frac{s_n}{1-r_n}\right|$. Moreover $|r_k| \le R < 1$ for all $n \le k \le N_{\varepsilon+1}$. So, if M is an upper bound on $\left|u_n - \frac{s_n}{1-r_n}\right|$, then

$$|r_n r_{n-1} \dots r_{N_\varepsilon+1}| \cdot \left|u_{N_\varepsilon} - \frac{s_{N_\varepsilon}}{1-r_{N_\varepsilon}}\right| \le R^{n-N_\varepsilon} M.$$

For the other terms, we again apply the bound $|r_k| \le R$ to find that

$$|r_n|\varepsilon + |r_n r_{n-1}|\varepsilon + \cdots + |r_n r_{n-1} \dots r_{N_\varepsilon+1}|\varepsilon \le \varepsilon \sum_{k=1}^{n-N_\varepsilon} R^k \le \varepsilon \sum_{k=0}^{\infty} R^k \le \frac{\varepsilon}{1-R}.$$

In all, we have for all $n > N_\varepsilon$ that

$$\left|u_n - \frac{s_n}{1-r_n}\right| \le R^{n-N_\varepsilon} M + \frac{\varepsilon}{1-R}.$$

Sending $n \to \infty$,

$$\limsup_{n\to\infty} \left|u_n - \frac{s_n}{1-r_n}\right| \le \frac{\varepsilon}{1-R}.$$

But ε was arbitrary, so $\left|u_n - \frac{s_n}{1-r_n}\right| \to 0$.

■

Remark: I've taken the time to write this argument in detail because most popular sources which mention this problem do not seem to bother with proving the uniqueness of the solution. This is perfectly understandable since any proof of uniqueness will likely be quite technical.

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Solution 156

The answer is $\frac{1}{\varphi}$ where φ is the golden ratio. Let $\sum_{k=0}^{n} a_k x^k$ be our polynomial, so that $a_k \in \{0,1\}$ for all k and

$$0 = \sum_{k=0}^{n} a_k z^k.$$

Since $z \neq 0$, we may divide by a power of z so that the constant term is 1. That is, we may assume WLOG that $a_0 = 1$. Thus

$$0 = 1 + \sum_{k=1}^{n} a_k z^k.$$

There are now two cases.

Case 1: $a_1 = 0$, i.e. there is no z^1 on the RHS.

Then

$$-1 = \sum_{k=2}^{n} a_k z^k,$$

and so we may use the rough bound

$$1 \leq \sum_{k=2}^{n} a_k |z|^k \leq \sum_{k=2}^{\infty} |z|^k = \frac{|z|^2}{1-|z|}.$$

So $|z|^2 + |z| - 1 \geq 0$ which directly implies $|z| \geq 1/\varphi$.

Case 2: $a_1 = 1$

Then we have

$$0 = 1 + z + \sum_{k=2}^{n} a_k z^k.$$

Multiplying by $1 - z$ on each side gives

$$0 = 1 - z^2 + \sum_{k=3}^{n} b_k z^k$$

where $b_k \in \{-1, 0, 1\}$. Now there is no z term, and the coefficients of -1 are not an issue for the argument in Case 1, so we may repeat the argument in Case 1 to deduce again that $|z| \geq 1/\varphi$.

This shows that $1/\varphi$ is a lower bound. To "obtain" $1/\varphi$, note that $z = -1/\varphi$ is a solution to

$$0 = 1 + z + z^3 + z^5 + z^7 + \dots.$$

So we can expect that for large n, there is a root of $1 + z + z^3 + z^5 + z^7 + \ldots + z^{2n+1}$ that is quite close to $-1/\varphi$.

If you want the murky details, here you go. Fix $\varepsilon > 0$. Let $f_n(z) = 1 + \sum_{k=0}^{n} z^{2k+1}$ and $g_n(z) = \sum_{k=n+1}^{\infty} z^{2k+1}$, so that $z = -1/\varphi$ is a root of $f_n + g_n$. Since $|-1/\varphi| < 1$, we have that $f_n + g_n$ is holomorphic around a neighborhood $D_r(-1/\varphi)$ of $-1/\varphi$ by studying the radius of convergence. For a choice of $0 < r < \varepsilon$ small enough we can guarantee that $f_n + g_n$ does not vanish on $\partial D_r(-1/\varphi)$, so $|f_n + g_n| \geq \delta > 0$ over $\partial D_r(-1/\varphi)$. Moreover we see that $g_n \to 0$ uniformly on $\overline{D_r(-1/\varphi)}$ so we may pick an n so large that $2|g_n| < \delta$ on $\partial D_r(-1/\varphi)$. For this n we have

$$2|g_n| \leq \delta < |f_n + g_n| \leq |f_n| + |g_n|$$

or $|g_n| < |f_n|$, over $\partial D_r(-1/\varphi)$. Of course, both f_n and g_n are holomorphic in $\overline{D_r(-1/\varphi)}$, so by Rouché's theorem f_n and $f_n + g_n$ have the same number of roots in $D_r(-1/\varphi)$. We conclude that f_n, which is a polynomial, has at least one root in $D_r(-1/\varphi) \subseteq D_\varepsilon(-1/\varphi)$. But ε was arbitrary.

■

Source: Putnam, modified slightly

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Solution 157

We first take $u = 1 + x^{\varphi}$ to convert to

$$I = \frac{1}{\varphi}\int_1^{\infty} \frac{1}{u^{\varphi}(u-1)^{\frac{\varphi-1}{\varphi}}}\, du$$

In order to make the integrand's structure more symmetrical, we now take $v = 1/u$ to write I as

$$I = \frac{1}{\varphi}\int_0^{1} \frac{v^{\varphi} v^{\frac{\varphi-1}{\varphi}}}{v^2(1-v)^{\frac{\varphi-1}{\varphi}}}\, dv.$$

Miraculously, using, $\varphi^2 - \varphi - 1 = 0$, this simplifies as

$$= \frac{1}{\varphi}\int_0^{1} \frac{1}{(1-v)^{\frac{\varphi-1}{\varphi}}}\, dv.$$

This evaluates to $\frac{1}{\varphi} \cdot \frac{1}{\varphi^{-1}}$, which is just 1, as needed.

■

Remark: We are quite confident that $p = \varphi$ is the unique value of p that solves

$$\int_0^{\infty} \frac{1}{(1+x^p)^p}\, dx = 1.$$

This can be seen visually using Desmos by either graphing the function $f(x) = \int_0^{50} \frac{1}{(1+t^x)^x}\, dt$ (which is computationally expensive) or using a few substitutions to discover that, for $p > 1$,

$$\int_0^{\infty} \frac{1}{(1+x^p)^p}\, dx = \frac{\Gamma\left(p - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma(1+p)},$$

and then graphing $\frac{\Gamma\left(x-\frac{1}{x}\right)\Gamma\left(\frac{1}{x}\right)}{\Gamma(1+x)}$ (by implementing the Gamma function as $(x-1)!$). This is equal to the original integral only on $(1, \infty)$ due to convergence issues on $(0, 1)$. I do not know of an elegant way to rigorously demonstrate the uniqueness of $p = \varphi$.

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Solution 158

Let $n \geq 1$ be the degree of P, and let $x_1, \ldots, x_n$ be the roots of P.

Black Magic Claim: For all real $x \in [0, 3]$, we have

$$|x(x-1)(x-2)(x-3)| \leq 1,$$

with equality iff $x \in \{\frac{3\pm\sqrt{5}}{2}\}$.

Proof. Write

$$x(x-1)(x-2)(x-3) = (x^2-3x)(x^2-3x+2) = (x^2-3x+1)^2 - 1.$$

So the inequality is equivalent to showing that $|x^2 - 3x + 1| \in [0, \sqrt{2}]$. An analysis of the extreme values of the quadratic $x^2 - 3x + 1$ over $[0, 3]$ reveal that, in fact, $0 \leq |x^2 - 3x + 1| < \sqrt{2}$, and the equality case $|x^2 - 3x + 1| = 0$ occurs exactly at the roots of the quadratic, $x = \frac{3\pm\sqrt{5}}{2}$. □

With the claim, the proof is amazingly short: note that by the claim applied to each x_i,

$$|P(0)P(1)P(2)P(3)| = \prod_{i=1}^{n} |x_i(x_i-1)(x_i-2)(x_i-3)| \leq 1.$$

On the other hand, since P has no integer root, we know $|P(0)P(1)P(2)P(3)| \geq 1$.

Thus equality holds everywhere. In particular, equality holds in the black magic claim for each x_i, so $x_i \in \{\frac{3\pm\sqrt{5}}{2}\}$ for all i. The end is simple: in order for P to have integer coefficients, there must be an equal amount of $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$ among its roots, so in particular $\frac{3+\sqrt{5}}{2}$ *is* a root. Thus $P\left(\frac{3+\sqrt{5}}{2}\right) = 0$.

■

Remarks: To say that this problem is shrouded in mystery would be an understatement. Back in my high school years, I put this in my personal collection of interesting problems, but I didn't mark its source. To this day, despite my best efforts to track down its origins, I haven't the slightest clue where it came from, much less who invented it.

Unfortunately, I also didn't know how to solve it. In several layers of outsourcing, some very dedicated solvers came up with a variety of convoluted but fascinating methodologies. It was only after quite some time that someone on AoPS sent me the “”true” solution given above: a completely elementary proof of the statement that was not more than half a page. Yet, almost comically, they did not know the source of the proof — a black magic solution from nowhere to a problem that came from nowhere.

Source: I have no idea.

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Solution to the Grand Finale

After reading all the problems, it is evident that there is something funky going on. The 12 problems appear to be "positioned" in some way, with their answers labeled using the letters A, B, C, D, E, F, G, H, I, J, K, and L. The biggest hint as to what is going on is revealed in Problem 160:

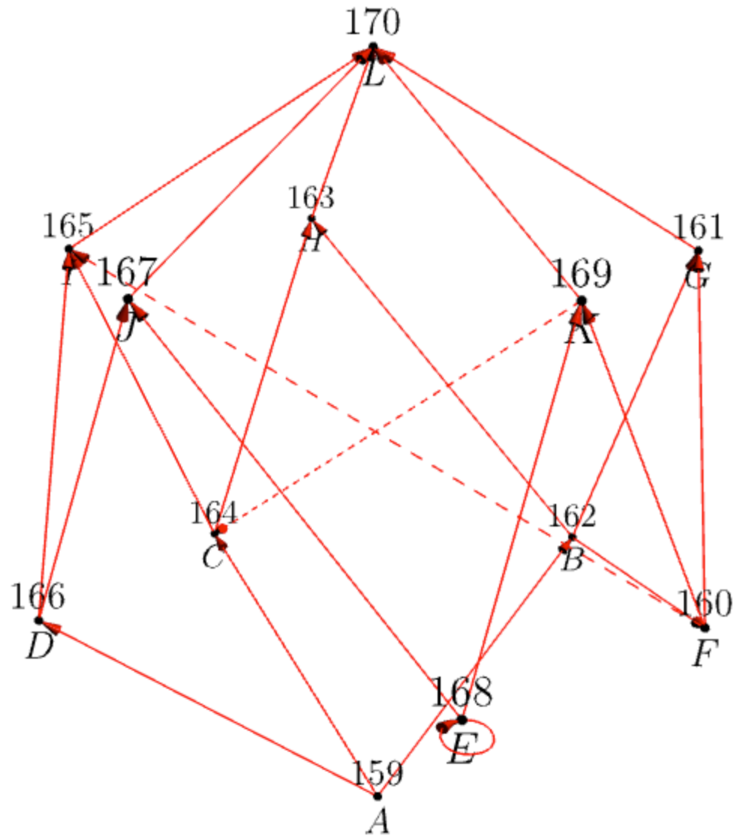
- "Let B be the answer to the problem that is **counter-clockwise adjacent to (and at the same altitude as) this one...**"
- "The 'counter-clockwise' direction is **from the perspective of an observer looking down from above**..."

This suggests that the problems are positioned as points in 3D space, and connected to each other in some way in order to determine which problems are "adjacent".

The exact shape formed by these problems and their connections must be inferred from some other details. These include:

- There are exactly 12 problems.
- One problem is labeled the "abyss", and one problem is labeled the "peak".
- Problems 162, 164, and 166 imply that there is only one problem below them.
- Problems 161, 163, 165, 167, 169 all mention the "two adjacent problems below" them.
- Problem 170, the "peak", says there are exactly **five** problems adjacent to it.

The number five is a particularly damning piece of evidence: This is an icosahedron.



(A is the bottom-most vertex. Above A is the regular pentagon BCDEF, whose vertices are written in counter-clockwise order from when looking from above. Above this is pentagon GHIJK, also in counter-clockwise order. L is the top-most vertex.)

Using the relative positionings implied by the problem statements, we can match up letters with problem numbers. The assignment is confirmed when one notices that the sequences $A, B, \ldots, K, L$ and $159, 160, \ldots, 169, 170$ both correspond to paths connecting adjacent vertices starting from the bottom and ending at the top. Note that the letters G, H, I, J, and L are never mentioned in the problems, but their existence and locations can be inferred from the previous points.

The arrows in the above diagram represent the dependencies required to solve each problem. However, none of the answers are given, and the only problem with no dependencies, Problem 159 (A), has been corrupted and hence cannot be solved. Thus, we will need to get creative.

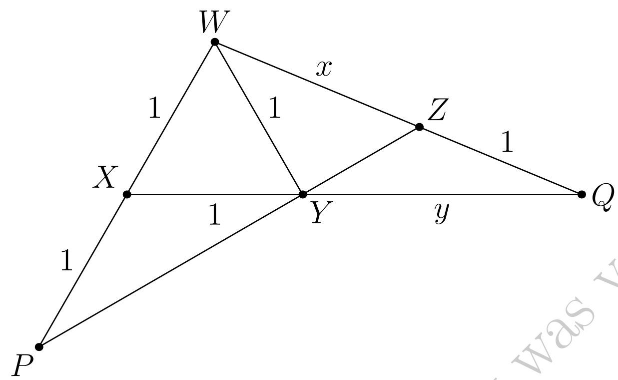
Problem 159 (A)

There is no information here, so the value of A must be inferred from the problems that reference it.

Problem 160 (F)

We claim that $WZ = B\sqrt[3]{2}$.

For the sake of elegance, scale the diagram down by a factor of B so that $WX = 1$.



It is clear that $\triangle WXY$ is equilateral and $\triangle XPY$ is isosceles. Chasing angles, we find that $\angle WYZ = 90^\circ$ and $\angle ZYQ = 30^\circ$. If we let $WZ = x$ and $YQ = y$, then by the sine area formula for triangles,

$$x = \frac{WZ}{ZQ} = \frac{[WYZ]}{[ZYQ]} = \frac{\frac{1}{2}(YW)(YZ)\sin 90^\circ}{\frac{1}{2}(YZ)(YQ)\sin 30^\circ} = \frac{2}{y},$$

so $xy = 2$. By Law of Cosines on $\triangle YQW$,

$$(x+1)^2 = 1^2 + y^2 + y,$$

or $x^2 + 2x = y^2 + y$. Substituting $y = 2/x$ gives

$$x^4 + 2x^3 - 2x - 4 = 0$$

which factors as $(x^3 - 2)(x + 2) = 0$. Thus $x = \sqrt[3]{2}$. After undoing the scaling by B from the beginning of this solution, we get $WZ = B\sqrt[3]{2}$, as claimed.

Now $\log_B(WZ) = 1 + \frac{1}{3}\log_B 2$. Since this is rational, $\log_B 2$ must be rational. So B is a rational power of 2. Since the answer to Problem 162 (B) is a positive integer, B must in fact be 2^k for a positive integer k (note that $k \neq 0$ because the base of a logarithm cannot be 1). Given k, we can then write

$$1 + \frac{1}{3}\log_B 2 = 1 + \frac{1}{k} = \frac{k+1}{k} = \frac{p}{q}.$$

No matter the value of k, it must be the case that $p = k$ and $q = k + 1$. So $|p - q| = 1$, giving $\boxed{F = 1}$.

Problem 161 (G)

This is actually quite subtle, and so we will come back to this later.

Problem 162 (B)

We claim that $A = n!$.

Assign each square a *rank* based on how far north-east they are, with the long diagonal being rank 0.

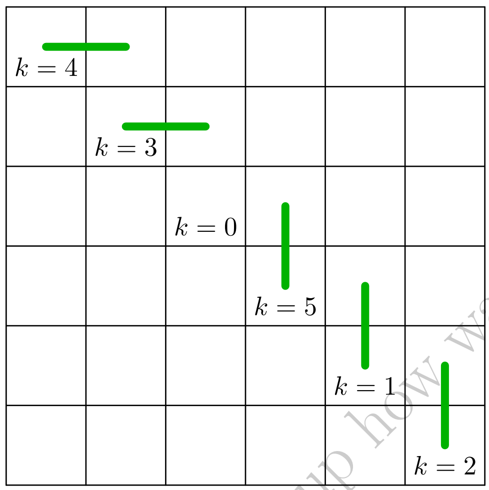
0	1	2	3	4	5
-1	0	1	2	3	4
-2	-1	0	1	2	3
-3	-2	-1	0	1	2
-4	-3	-2	-1	0	1
-5	-4	-3	-2	-1	0

(Ranks for $n = 6$)

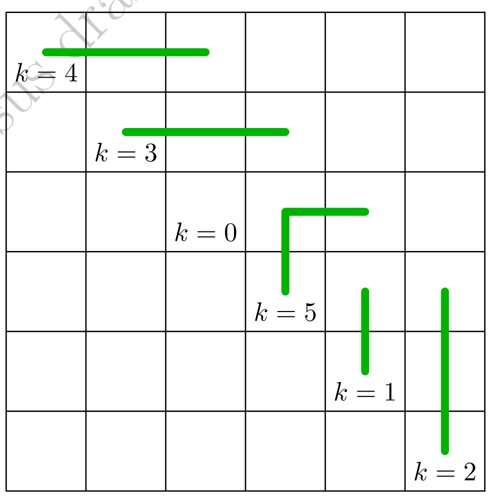
Observe that no snake can occupy more than two rank-0 squares. Now let us focus on the squares of positive rank, from 1 to $n-1$.

- One of the n snakes must reach the top-right-most square, i.e. the sole square of rank $n-1$.
- That snake will occupy one of the squares of rank $n-2$, leaving just one more square of rank $n-2$ unoccupied by them. That square must be occupied by one of the other $n-1$ snakes.
- The two snakes from the previous two bullet points must occupy two squares of rank $n-3$, leaving just one more square of rank $n-3$ unoccupied by them. That square must be occupied by one of the other $n-2$ snakes.
- ...
- The $n-1$ snakes from the previous $n-1$ bullet points must occupied $n-1$ squares of rank 1, leaving just one more square of rank 1 unoccupied by them. That square must be occupied by the last remaining snake.

From this reasoning, we deduce that for each $0 \leq k \leq n-1$, there is exactly one snake that occupies k squares of positive rank. There are $n!$ ways to assign these snakes to the n squares of rank 0, and this uniquely determines how the snake covers the squares! This is because the identity of the snake corresponding to $k = 0$, i.e. the snake that does not reach rank 1, uniquely determines how the other $n-1$ snakes must occupy rank 1.



Then, one of these $n-1$ snakes corresponds to $k = 1$, meaning that this snake does not reach rank 2, and this uniquely determines how the other $n-2$ snakes must occupy rank 2.



Inductively we obtain the uniqueness of the covering for the positive rank squares, and by an entirely symmetrical argument we get the uniqueness of the covering for the negative rank squares as well. This completes the proof that there are exactly $n!$ coverings.

Hence $A = B!$.

Problem 163 (H)

We immediately get the information that B and C are perfect squares greater than 1.

The lengths of the numbers $\{b^n\}_{n\geq 1}$ in base $\sqrt{B}$ are

$$\{\lfloor \log_{\sqrt{B}} b^n \rfloor + 1\}_{n\geq 1} = \{\lfloor n \log_{\sqrt{B}} b \rfloor + 1\}_{n\geq 1},$$

and their lengths in base $\sqrt{C}$ are

$$\{\lfloor \log_{\sqrt{C}} b^n \rfloor + 1\}_{n\geq 1} = \{\lfloor n \log_{\sqrt{C}} b \rfloor + 1\}_{n\geq 1}.$$

So the sequences $\{\lfloor n \log_{\sqrt{B}} b \rfloor\}_{n\geq 1}$ and $\{\lfloor n \log_{\sqrt{C}} b \rfloor\}_{n\geq 1}$ partition $\mathbb{N}$. We can now rely on asymptotics: the former sequence has density $\frac{1}{\log_{\sqrt{B}} b}$ in $\mathbb{N}$, and the latter has density $\frac{1}{\log_{\sqrt{C}} b}$, thus

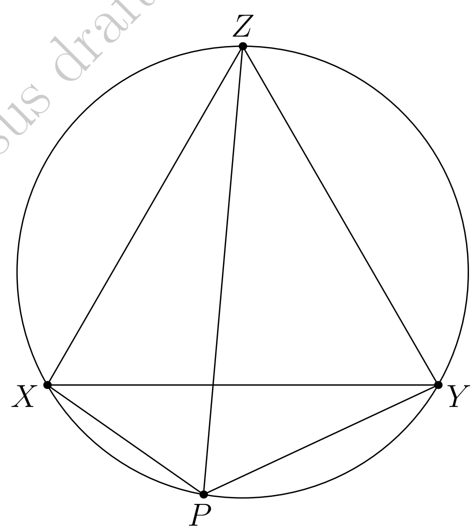
$$\frac{1}{\log_{\sqrt{B}} b} + \frac{1}{\log_{\sqrt{C}} b} = 1.$$

(See also *Rayleigh's Theorem*: https://en.wikipedia.org/wiki/Beatty_sequence). Now,

$$1 = \log_b \sqrt{B} + \log_b \sqrt{C} = \log_b \sqrt{BC},$$

which entails that $b = \sqrt{BC}$. So $H = \sqrt{BC}$.

Problem 164 (C)



By Ptolemy's theorem on quadrilateral $PXZY$,

$$PX \cdot ZY + PY \cdot XZ = PZ \cdot XY.$$

But $XY = YZ = XZ$, so $PX + PY = PZ$. Hence $C = A + K$.

Problem 165 (I)

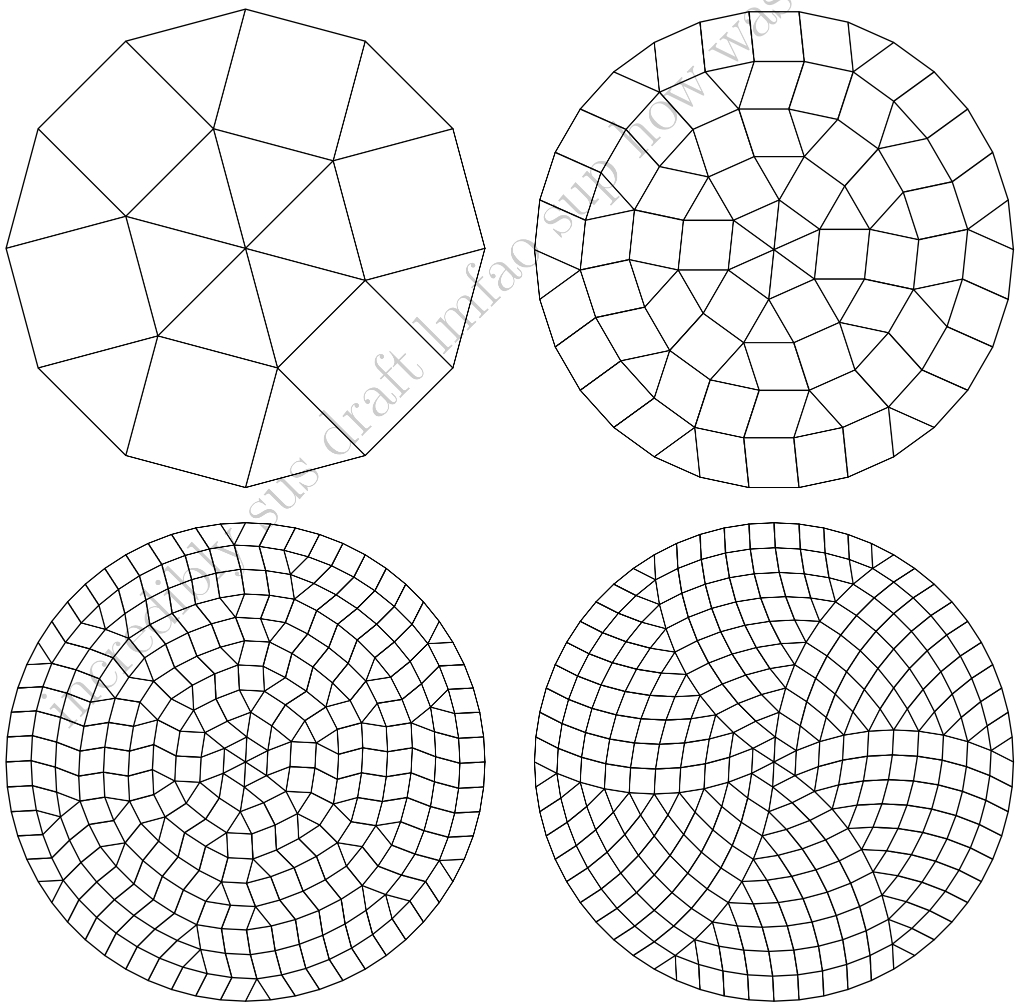
By the factorization $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, we have the implication $a+b+c=0 \implies a^3+b^3+c^3 = 3abc$. Applying this to $x-y$, $y-z$, and $z-x$, we find that $\mu = (x-y)(y-z)(z-x)$. So

$$C + D - F = (x-y)(y-z)(z-x).$$

It is difficult to make progress with this without knowing C, D, and F, so we must move on.

Problem 166 (D)

We claim that A must be a multiple of 6, and that the answer is $(A/6)!$. A few example tilings are shown below.



To argue that A must be a multiple of 6, first note that an angle of the A-gon can only be partitioned by the angles of at most two rhombi, because all angles are strictly less than

$180°$ and $\frac{180°}{60°} = 3$. It follows that, among the triangles and rhombi that share a side with the A-gon (which can be thought of as the "first layer"), the consecutive sequences of rhombi must be connected by a sequences of sides that are all mutually parallel. From this, it is not hard to deduce that there exists at least one equilateral triangle in this "first layer". Then, via a computation or otherwise, it can be shown that the first layer must have exactly six equilateral triangles, positioned symmetrically, which forces A to be divisible by 6.

More generally, if we let $A = 6n$, it can be seen that any such tiling of the $6n$-gon must consist of n layers of rhombi and triangles. Each layer, starting with the outermost one, consists of exactly 6 equilateral triangles placed symmetrically, with the gaps in between them filled rhombi. When this layer is removed, we are left with a (not necessarily regular) polygon with 6 fewer sides.

Intuitively, you should view the first layer as "removing" those 6 sides marked by the equilateral triangles, and all other sides are translated along the sides of these equilateral triangles to form the boundary of the smaller polygon. This process can then be repeated for the smaller polygon, again and again, until we are left with the "0-sided polygon" at the center.

For the first layer, there are n ways to choose the positions of the equilateral triangles. For the next layer, since there are now $6(n-1)$ sides, there will be $(n-1)$ ways to choose the positions of the equilateral triangles. If we keep going, the conclusion is not too hard to infer: There will be $n!$ ways in total to choose the positions of the triangles, and therefore, $n!$ ways to tile the $6n$-gon. Since $n = A/6$, we get $D = (A/6)!$.

Problem 167 (J)

This is a bit tricky without knowing D and E, so we will return later.

Problem 168 (E)

It is clear that E is a positive integer and that $E \geq 2$. Unfortunately, we claim that no more information can be deduced about E from this problem alone.

This is because if the sizes of the fish are $X_1, X_2, \ldots, X_{E-1}$, and Y is their minimum, then for $t \in (0,1)$,

$$\begin{aligned}\mathbb{P}(Y \geq t) = \mathbb{P}(X_1 \geq t, \ldots, X_{E-1} \geq t) &= \prod_{i=1}^{E-1} \mathbb{P}(X_i \geq t) \\ &= \mathbb{P}(X_1 \geq t)^{E-1} = (1-t)^{E-1},\end{aligned}$$

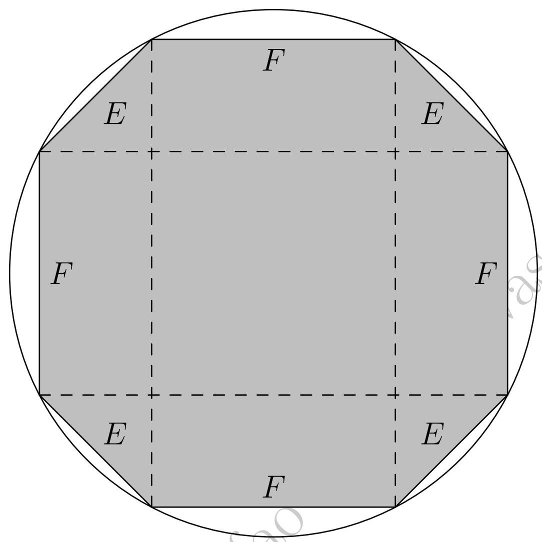
so

$$\mathbb{E}Y = \int_0^1 \mathbb{P}(Y \geq t)\, dt = \int_0^1 (1-t)^{E-1}\, dt = \frac{1}{E}.$$

So the answer, E, is given by $\left(\frac{1}{E}\right)^{-1}$, reducing to $E = E$, telling us nothing.

Problem 169 (K)

The key idea is that rearranging the sides of a cyclic polygon does not change its area! We can therefore rearrange the sides into the following far more pleasing octagon.



Subdividing along the dashed lines, it becomes plain to see that the area is

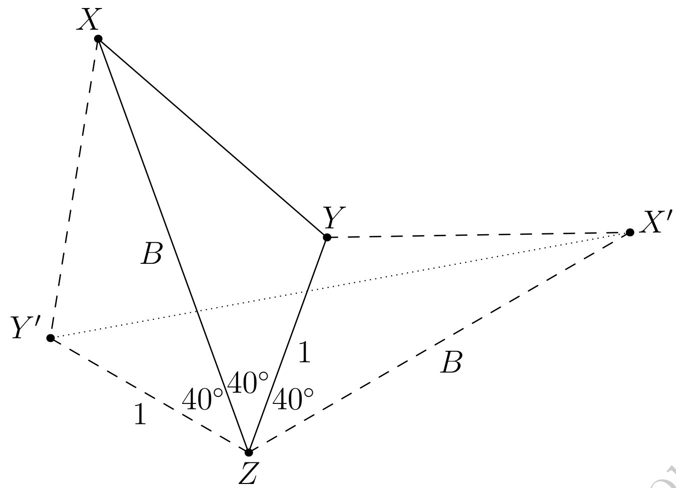
$$E^2 + F^2 + 2\sqrt{2}EF.$$

When this is expressed as $m + n\sqrt{2}$, we have $m + E^2 + F^2$ and $n = 2EF$, so $m + n = E^2 + F^2 = 2EF = (E + F)^2$. That is, $K = (E + F)^2$. Since E and F are integers, this tells us that K is a perfect square! This will be crucial for later.

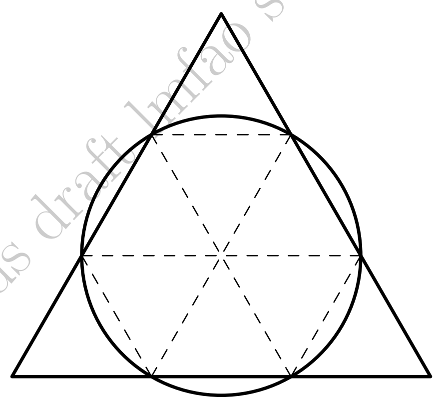
Problem 161 Revisited (G)

We know that $F = 1$, so the only unknown dependency is B, which we know to be a positive integer. From Problem 160, we know that B is a power of 2 that is greater than 1. From Problem 163, we know that B is a perfect square greater than 1. Thus B is a power of 4. Solving this problem will, at last, allow us to deduce the value of B.

By a reflection argument, the answer is essentially the length of the shortest path from the X' to Y', where X' is the reflection of X over $\overline{YZ}$ and Y' is the reflection of Y over $\overline{XZ}$.



At first glance, the answer appears to simply be the length of segment $\overline{X'Y'}$, which can be found by the Law of Cosines. However, if segment $\overline{X'Y'}$ exits the interior of pentagon $ZX'YXY'$, then after reflecting back, this segment will correspond to a path that exits the triangle and fails to visit one of the segments $\overline{ZX}$ or $\overline{ZY}$. This could occur if one segment is too long compare to the other one.



(It's a bit hard to see, but segment $\overline{Y'X'}$ lies slightly above segment $\overline{YX'}$.)

If this is the case, then the shortest path will instead be the union of the segments $\overline{Y'Y}$ and $\overline{YX'}$, which has length $2\sin 40^\circ + \sqrt{1 + B^2 - 2B\cos 40^\circ}$. It is inconceivable that this could ever be written in the form $\sqrt{n}$ for integer n (though I must confess that I have no rigorous proof of this), so we must prevent this case from occurring.

Since $F = 1$ and B is an integer, $\overline{XZ}$ is the longer segment. We can now compute the threshold that B would have to exceed for segment $\overline{X'Y'}$ to exit the pentagon. This threshold is exactly when Y', Y, and X are collinear. In which case, we may compute $\angle ZY'X' = 50^\circ$ and $\angle ZX'Y' = 10^\circ$, and now Law of Sines applied to $\triangle ZX'Y'$ gives

$$\frac{B}{\sin 50^\circ} = \frac{1}{\sin 10^\circ}$$

or $B = \frac{\sin 50^\circ}{\sin 10^\circ} \approx 4.411$. So B cannot exceed 4. But we know that B is a power of 4 that is greater than 1, therefore $\boxed{B = 4}$.

We now obtain an explosion of information:

- The answer to this problem, by the Law of Cosines, is given by

$$\sqrt{G} = \sqrt{1^2 + 4^2 - 2(1)(4)\cos 120^\circ} = \sqrt{21},$$

  so $\boxed{G = 21}$.

- From Problem 162, $A = B! = 4!$ and so $\boxed{A = 24}$.

- From Problem 166, $D = (A/6)! = 4!$ and so $\boxed{D = 24}$.

- From Problem 164, $C = A + K = 24 + K$. From Problem 169, $K = (E+F)^2 = (E+1)^2$. From Problem 163, C is a perfect square. Thus we have the factorization

$$24 = C - K = \sqrt{C}^2 - (E+1)^2 = (\sqrt{C} + E + 1)(\sqrt{C} - E - 1).$$

  These factors sum to $2\sqrt{C}$ which is even, so they must have the same parity. This gives two possible cases:

$$\begin{cases} \sqrt{C} + E + 1 = 12 \\ \sqrt{C} - E - 1 = 2 \end{cases} \quad \text{or} \quad \begin{cases} \sqrt{C} + E + 1 = 6 \\ \sqrt{C} - E - 1 = 4 \end{cases}$$

  These cases solve to $(\sqrt{C}, E) = (7, 4)$ and $(\sqrt{C}, E) = (5, 0)$ respectively. However, from Problem 168, we know that $E \geq 2$, which eliminates the second case and gives us $\boxed{E = 4}$ and $\boxed{C = 49}$.

- Since $K = (E+1)^2 = (4+1)^2$, we get $\boxed{K = 25}$.

- From Problem 163, we know that $H = \sqrt{BC} = \sqrt{4 \cdot 49}$, so $\boxed{H = 14}$.

Problem 165 Revisited (I)

Now that we know that $C = 49$, $D = 24$, and $F = 1$, we have

$$(x - y)(y - z)(z - x) = 72.$$

Up to cyclic symmetry there are two possible orderings for x, y and z: Either $x < y < z$ or $x > y > z$. (Note that no two can be equal.) The latter is impossible since this then $(x - y)(y - z)(z - x)$ would be negative, so $x < y < z$. Now write

$$(y - x)(z - y)(z - x) = 72$$

so that all factors are positive. We see that the first two factors sum to the third. Studying the divisors of $72 = 2^3 \cdot 3^2$, particularly the power of 2, a parity analysis gives two possibilities: Either each of the three factors is even, or one of the factors is divisible by 2^3. The former possibility can be ruled out easily, and so the factors are thus 1, 8, 9 in some order. We are left with two cases:

$$\begin{cases} y - x = 1 \\ z - y = 8 \\ z - x = 9 \end{cases} \quad \text{or} \quad \begin{cases} y - x = 8 \\ z - y = 1 \\ z - x = 9 \end{cases}$$

Solving in terms of x, the first case gives $(x, y, z) = (x, x + 1, x + 9)$ and the second case gives $(x, y, z) = (x, x + 8, x + 9)$. In either case, the maximum among $\{x, y, z\}$ is $x + 9$ and the minimum is x, and the difference between these extremes will always be 9. We conclude that $\boxed{I = 9}$.

Problem 167 Revisited (J)

We now know that $D + E^2 = 24 + 4^2 = 40$. So we wish to determine the factorial $n!$ that should be removed from the product

$$(1!)(2!)\dots(40!)$$

so that what remains is a perfect square. We may rewrite this product into the form

$$\begin{aligned} (1!)(1! \cdot 2) \cdot (3!)(3! \cdot 4) \cdot \dots \cdot (39!)(39! \cdot 40) &= (1! \cdot 3! \cdot \dots \cdot 39!)^2 (2 \cdot 4 \cdot \dots \cdot 40) \\ &= (1! \cdot 3! \cdot \dots \cdot 39!)^2 \cdot 2^{20} \cdot 20! \\ &= (1! \cdot 3! \cdot \dots \cdot 39!)^2 (2^{10})^2 \cdot 20!, \end{aligned}$$

which makes it clear that removing 20! will result in a perfect square, $(1! \cdot 3! \cdot \dots \cdot 39!)^2 (2^{10})^2$. (The problem of investigating if this is the unique solution is left to the reader!) Thus $\boxed{J = 20}$.

Problem 170

We gather up the answers to the 5 adjacent problems:

- $G = 21$
- $H = 14$
- $I = 9$
- $J = 20$
- $K = 25$

Using the *A1Z26* cipher, the numbers $21, 14, 9, 20$, and 25 correspond to the letters U, N, I, T, and Y. The answer to the final problem is $\boxed{\text{UNITY}}$.

■

Indeed, the hidden purpose of the CMUMC POTD was to strengthen the sense of community within the CMU Math Club and, therefore, unify its members. I'm overjoyed to know that the countless fascinating problems that I've accumulated over the years have found such a fulfilling purpose. Dear reader, whether you are from CMU or another place, whether you've experienced the POTD during its lifespan or are solving from another time, I thank you for your participation.

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