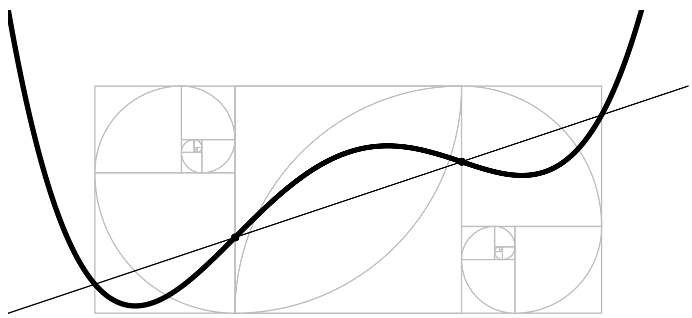
CMUMC

PROBLEM OF THE DAY

Thomas Lam

The Carnegie Mellon University Math Club Problem of the Day



Curated by Thomas Lam

Courant Institute of Mathematical Sciences

Published by me.

First published August 2025.

Printed nowhere because this is intended to be an ebook but you can print it if you really want to.

Written with LaTeX and way too much Asymptote.

Contents

CHAPTER 1

Prelude

This version of the book removes the hints and solutions, leaving only the problems.

Structure

The problems tend to fall into three difficulties. To be precise, it is generally the case that for each integer $n \geq 0$,

- Problem $3n + 1$ is on the easier side,
- Problem $3n + 2$ is of medium difficulty, and
- Problem $3n + 3$ is arbitrarily hard.

Of course, there are some exceptions.

Problems that are unusually or surprisingly difficult are marked with stars. I include these stars out of respect for your time — who would want to spend hours thinking about a problem only to find out that they had virtually no chance of solving it? As a general guideline starred problems (★) are (a subset of) those problems that can be expected to put up a bit of a fight, and doubly-starred problems (★★) are problems whose solutions make me think "Who on earth managed to come up with this???". Ultimately this is a subjective measure, but I hope it is useful.

The final twelve problems — Problems 159 to 170 — were all given on the same day: Friday, April 28th, 2023.

How to Best Use This Book

There is no need to look at the problems in the order given. In fact, I suggest **simply scrolling through the problems** and seeing what catches your eye.

Necessary Background

The vast majority of the problems can be solved with just a standard knowledge of high school mathematics. However, since the problem audience consisted of college undergraduates, you can expect some topics from college mathematics to come up as well. These include real analysis, topology, linear algebra, and more. (Unfortunately, lovers of abstract algebra may find themselves disappointed with this collection. This is because I am allergic to abstract algebra.)

Problem Sources

A source of each problem is given in their solution if I remember where I got it from. The source I list may not be very informative. (Some problems may have multiple plausible sources that they can be ascribed to, but I'll list just one.) Only a few of the problems are my own creation. If a problem is marked as being "proposed" by someone, then that someone is the original source of that problem.

Many problems have been reworded greatly from their sources. I've taken this liberty in an effort to make problem statements a little more natural or less contrived. For example, Problem 15 is normally phrased in terms of a line of coins, which I personally find very difficult to wrap my head around.

Acknowledgments

Special thanks to:

- Edward Hou for 11 problem suggestions and proposals
- "tenth" for 11 problem suggestions and proposals
- "tanoshii" for 4 problem suggestions
- David Altizio for 3 problem suggestions
- "Linus" for 2 problem suggestions
- "Kaz" for 1 problem suggestion
- "Nishant" for 1 problem suggestion
- "asbodke" for 1 problem suggestion

The POTD would not have been possible without all of your contributions.

Major thanks to those that participated in solving the POTD. I very much enjoyed your insightful solutions, many of which introduced novel ideas that I have included in this book. There would have been no point in continuing the POTD without your support.

Thanks to the following individuals for test-solving the grand finale:

- Yannick Yao
- “Deusovi”
- “axcaea”

Lastly, I thank the following references for being great sources of nice problems:

- Tanya Khovanova’s blog
- Francis Su’s *Math Fun Facts*
- The Leningrad Math Olympiad

Without further ado, happy solving!

CHAPTER 2

Problems

Problem 1

Good Morning. I have a cup of tea and a cup of milk, in equal quantities.

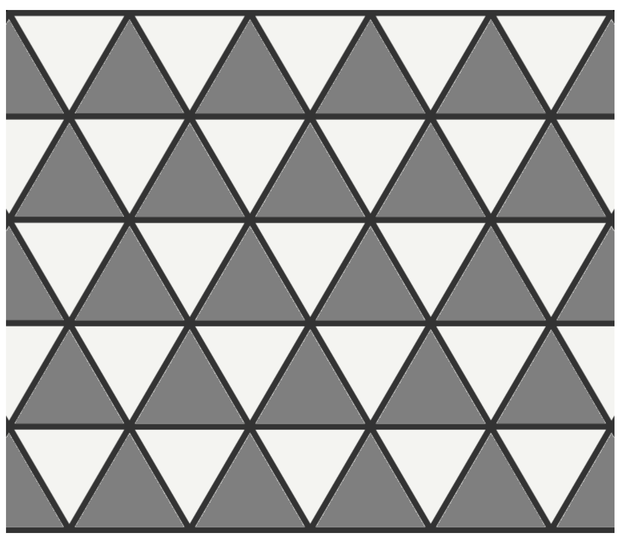
I take a spoonful of the tea and stir it into the milk. I then take a spoonful of the milk/tea mixture and stir it into the tea.

Which cup is more contaminated?

Problem 2 (Proposed by Edward Hou)

What is the greatest "fraction" of a plane that can be covered by non-intersecting equilateral triangles of same size and orientation?

For example, this arrangement of triangles covers "half" of the plane:



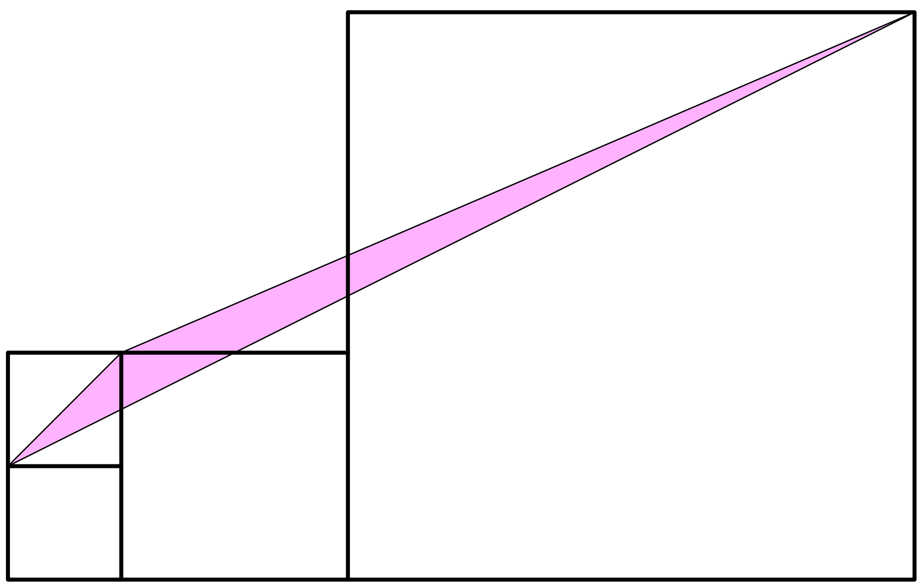
Problem 3

There is a very weird solar system that cannot be seen because the planets block out all light from the sun. What is the minimum possible number of planets in the solar system?

(You may assume that the sun is a single point which emits rays of light.)

Problem 4

The diagram below consists of four squares. The two smallest squares have side length 1. Find the area of the shaded triangle.



Problem 5

Back in September, Kaz needed quarters so that he could use the washing machine. I checked my wallet, and found that I had 16 quarters! Unfortunately for Kaz, I saw this as the perfect opportunity to be a stereotypical logic puzzle antagonist.

I placed them on the table in a 4×4 grid, and observed that they made 10 lines of four: 4 horizontal, 4 vertical, and 2 diagonal. I told Kaz that he could take my quarters if he can rearrange the 16 quarters to form 15 lines of four.

Can you help Kaz take my quarters?

(i.e. Find 16 distinct points on a plane such that there exist 15 distinct lines that each contain at least four of the points.)

Problem 6 (★★)

Alice and Bob are taking a geometry test, which consists of a single question: Construct (using a compass and straightedge) the midpoint of a segment AB, given only the points A and B.

Alice only brought her compass, and Bob only brought his straightedge. Which of them, if either, can pass the test?

Problem 7

A (base ten) number N is formed by a strictly increasing sequence of digits (e.g. 2357, but not 3558). Compute the sum of the digits of $9N$.

Problem 8

Solve for x:

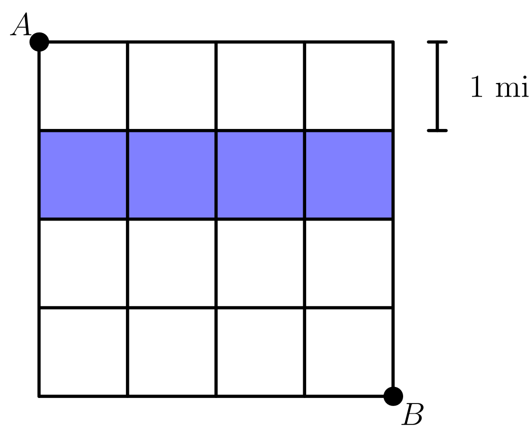
$$\sqrt{5-x} = 5 - x^2$$

Problem 9

Let $n \in \mathbb{N}$. Prove that $\left\lfloor \frac{n!}{e} \right\rfloor$ is even.

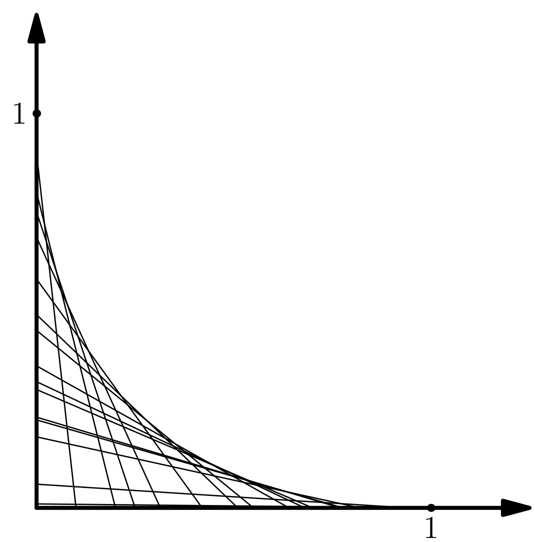
Problem 10

Towns A and B are tired of swimming to each other. Describe where to place a vertical bridge across the river such that the walking distance between the towns is minimized, and determine this minimum distance.



Problem 11

Bob is in detention for failing his exam. His detention assignment is to repeatedly pick a real number t between 0 and 1, and then draw the line segment from $(t, 0)$ to $(0, 1-t)$ using his straightedge. As Bob draws more segments of this form, they begin to bound a curve. What is the equation of this curve?



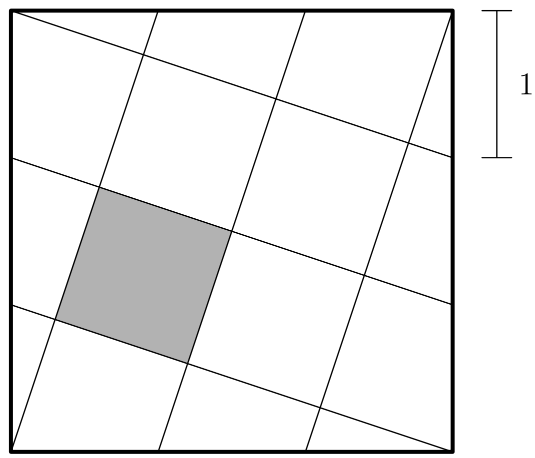
Problem 12 (★)

Fill in the blank in the following conversation between Alice and Beth.

> Alice: "Did you know that your favorite number is the sum of the ages of my stuffed animal turtles, and that my favorite number is their product?"
>
> Beth: "I wouldn't know because I don't know your favorite number. If you tell me your favorite number and how many stuffed animal turtles you have, would I know the ages of your stuffed animal turtles?"
>
> Alice: "No."
>
> Beth: "Oh, so your favorite number is ___________!"

Problem 13

Lines are drawn through the trisection points of the sides of a square, as shown. What is the area of the shaded region?



Problem 14 (Suggested by "tenth")

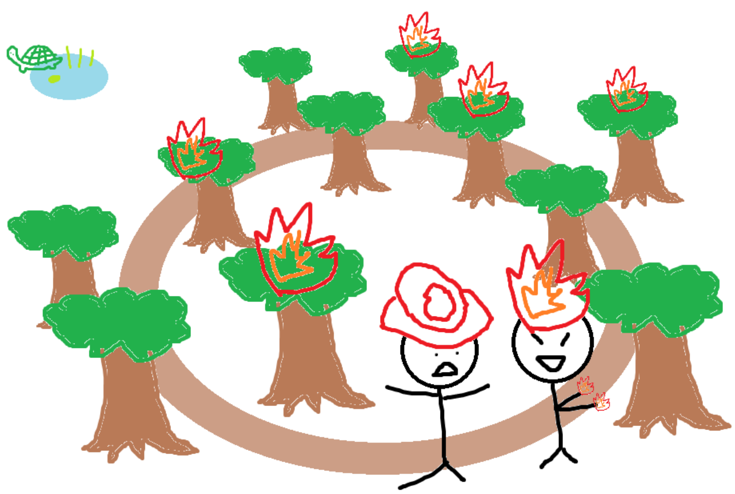
There are two positive reals x satisfying $2^x + 3^{1/x} = 5$. One of them is 1. What's the other one?

Problem 15 (Suggested by Kaz)

A firefighter and a pyromaniac are running together along a circular hiking trail, lined with 2022 trees. Some of the trees are on fire.

- For each burning tree they encounter, the firefighter can choose whether or not to extinguish the tree before they move on.
- For each not-on-fire tree they encounter, the pyromaniac can choose whether or not to set it ablaze before they move on.

Does the firefighter have a strategy that guarantees that all flames will be vanquished at some point?



Problem 16

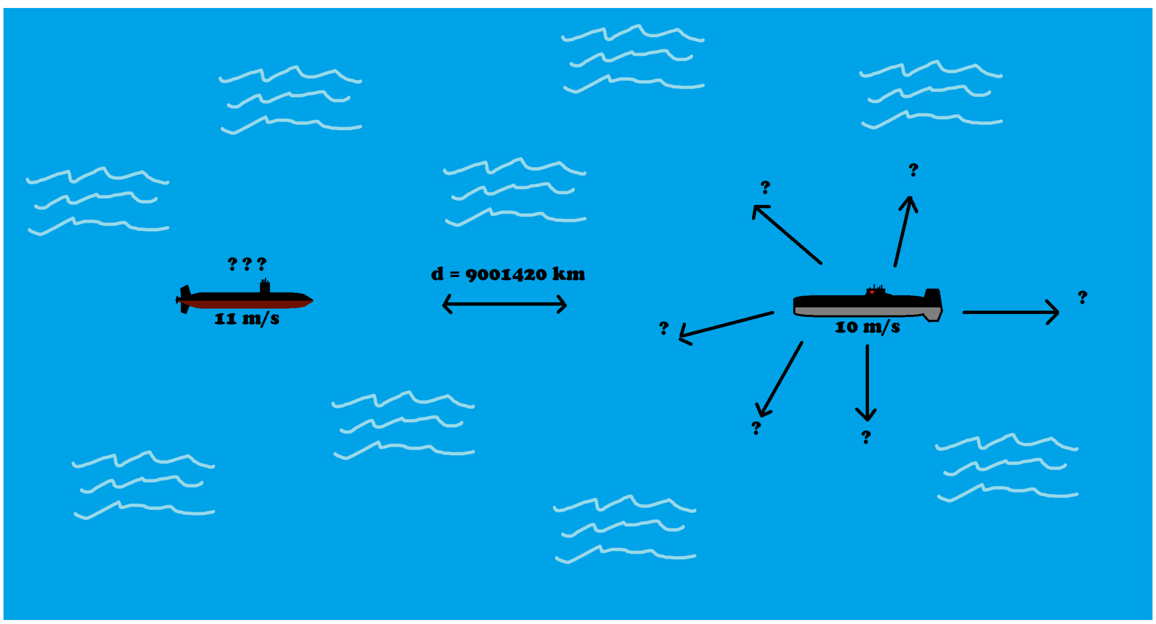
Prove that any function $f : \mathbb{R} \to \mathbb{R}$ may be written as the sum of an odd function and an even function.

Problem 17

You are captain of the USS Dallas, a submarine in a dark ocean searching for the Red October, which is traveling in a straight line. Suddenly, your sonar detects the exact location of the Red October, but not its direction of travel. Your sonar then breaks.

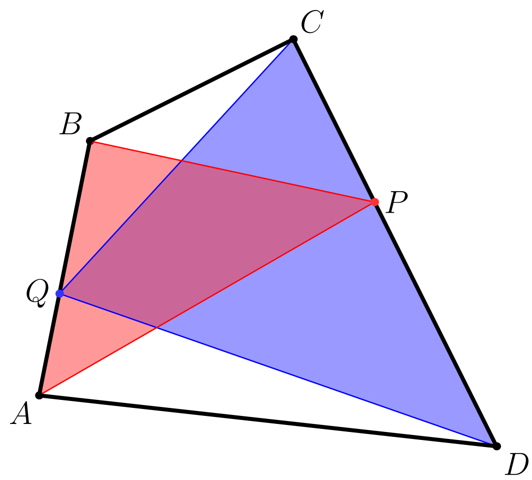
Since you're an amazing captain, you happen to know the exact speed of both the USS Dallas and the Red October (of course, all submarines travel at constant speeds at all times). In particular, your ship can travel slightly faster than the Red October.

Can you track down the Red October?



Problem 18

A convex quadrilateral $ABCD$ is given. Point P is on $\overline{CD}$ and point Q is on $\overline{AB}$ such that $AQ : QB = CP : PD$.



Prove that $[APB] + [CQD] = [ABCD]$.

(Square brackets denote area.)

Problem 19

I have two ropes. The first burns up in one minute when lit from one end, and the second similarly burns up in two minutes. Burning rate is not necessarily uniform over the lengths of these ropes. Can you measure 75 seconds?

Problem 20

Sydney the squirrel is at $(0, 0)$ and is trying to get to $(1, 0)$. She can move only by reflecting her position over any line that can be formed by connecting two lattice points, provided that the reflection puts her on another lattice point.

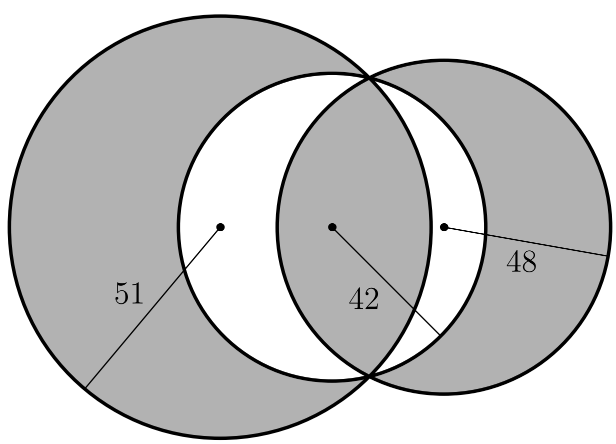
Can Sydney make it to $(1, 0)$?

Problem 21 (★)

What if Sydney need not step on lattice points?

Problem 22

Find the shaded area.



Problem 23

Given that $(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}) = 1$, what is the value of $x + y$?

Problem 24

10 crewmates fell off the USS Dallas, and now we have to save them. We have 10 giant circular life-buoys, each of radius 10 meters. For the rescue to be a success, we need to throw them into the ocean such that each person is inside a lifebuoy, but no two lifebuoys intersect. Can this be done?

Problem 25

Minsung is acting highly sus, which is why we need to try and prevent him from walking too far. Once he gets 10 feet away from his spawnpoint, we'll lose sight of him and he might be able to murder all of us.

But for now, we can impede his movement with the following procedure: Every second, Minsung chooses a direction to face. Then we can either let him take a 1-foot step forward, or fire a laser gun to scare him so that he steps 1 foot backward.

Assuming ideal strategies, will this plan keep Minsung at bay indefinitely? If not, how long will it be until Minsung gets 10 feet away from spawn and possibly murders us all?

Problem 26 (Suggested by David Altizio)

In a certain country, a dollar is 100 cents and coins have denominations 1, 2, 5, 10, 20, 50, and 100 cents.

Suppose that one can make A cents using exactly B coins. Prove that it is possible to make B dollars using exactly A coins.

Problem 27 (Suggested by "Nishant")

Does there exist a ring endomorphism of the real numbers that is not the identity?

(A ring endomorphism of the real numbers is a function f satisfying $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, and $f(1) = 1$.

Problem 28

Call a triangle *lit* if it is non-degenerate and has integer side lengths. Which quantity is greater, if either?

- The number of lit triangles with perimeter 2019
- The number of lit triangles with perimeter 2022

Problem 29

For a positive integer n, what is the dimension of the (real) vector space formed by all $n \times n$ magic squares?

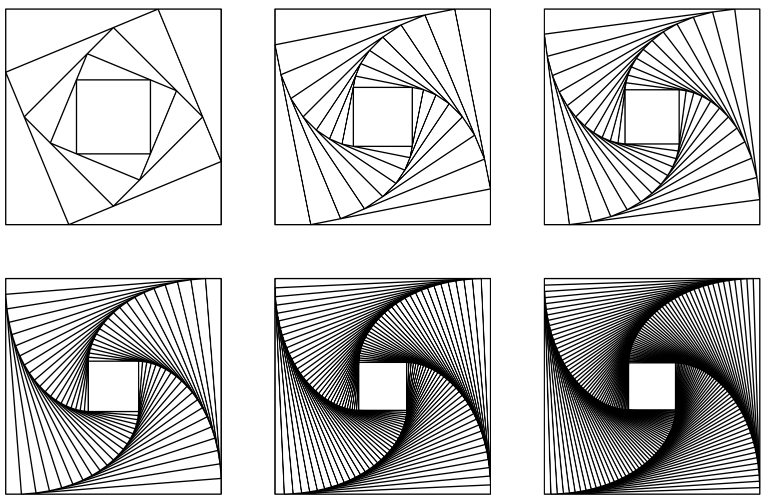
(Recall that a magic square is a grid of real numbers, each of whose rows, columns, and main diagonals sum to a common number. We define this vector space in the obvious way, with entry-wise scaling and addition.)

Problem 30 (Proposed by "tenth")

The side length of the largest square is 1.

As the number of squares increases, to what limit does the area of the smallest square approach, if any?

(The angle between each two successive squares is the same, and the "total angle of rotation" is 90 degrees.)



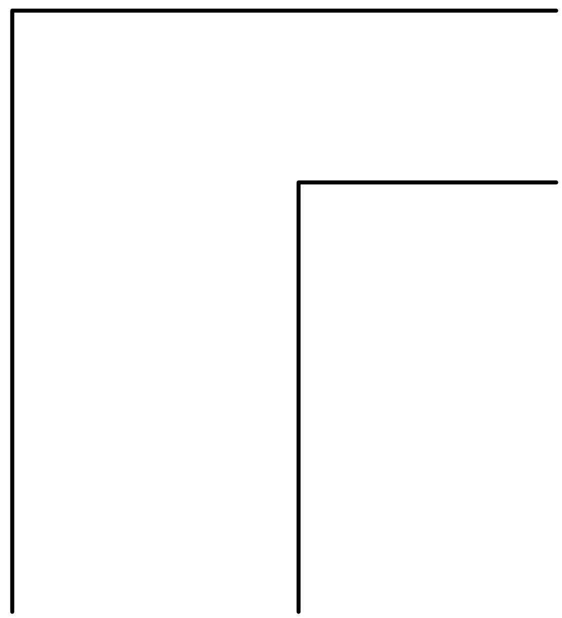
Problem 31

Good morning! At department tea, there are several cups of milk-contaminated tea, each filled to the same amount. The cup I'm holding has a sixth of all the milk and a quarter of all the tea.

How many cups are there?

Problem 32

A hallway of width a turns into another hallway of width b at a right angle, as shown. What is the length of the longest stick I can carry through this hallway?



Problem 33

P is a polynomial with real coefficients such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that we can write

$$P(x) = A(x)^2 + B(x)^2$$

for some polynomials A, B with real coefficients.

Problem 34

Alex, Blaire, and Clara have decided to settle their differences with a three-way duel. In some randomly-chosen firing order, they will take turns shooting at another duelist until there is one person left standing.

It is well-known that Alex and Blaire are expert markswomen, hitting all their shots with 100% accuracy, whereas Clara can only hit her target with 50% probability.

Who is most likely to win the duel? (Assume that nobody misses on purpose.)

Problem 35

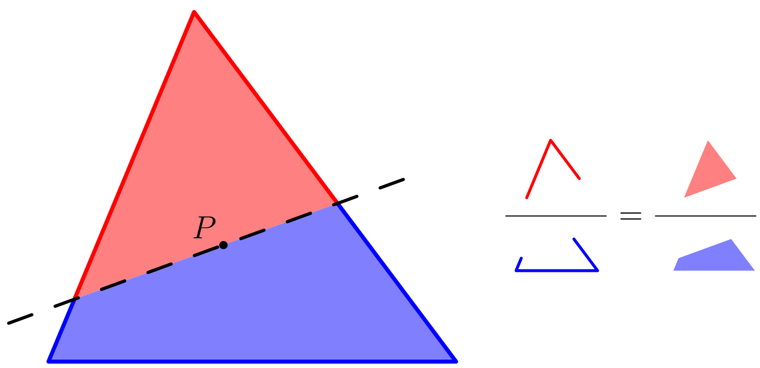
31415 people got into a nasty argument, and have decided to settle their differences with a duel battle royale. They stand on a plane in such a way that their pairwise distances are all distinct, and then the moment it's high noon, everyone immediately takes out a gun and fatally shoots the duelist closest to them.

Prove that at least one duelist is still alive.

Problem 36

For the summer weekly pizza seminar, Ricky and Owen brought a triangular pizza. They decide to share this pizza so that nobody else is subjected to the horror of having a triangular pizza.

They want to cut the pizza along a straight line such that the pizza's area and its crust are divided into the same ratio. Prove that any such cut must pass through a common point, and identify this point.



(Preemptive Remark: The converse also holds! Any cut through this point will divide the pizza's area and crust into the same ratio.)

Problem 37

The base ten representation of the integer 2^{29} has 9 digits, all of them distinct. Which digit (from 0 through 9 inclusive) is missing?

Problem 38

A ship of seven Hearthians lands on a small spherical planet. One Hearthian, Arkose, explores the planet (starting from the ship) by

1. walking 30 km in a straight line in some direction,
2. turning 90° counter-clockwise and walking another 30 km, and then
3. turning 90° degrees counter-clockwise (again) and walking another 30 km.

The Hearthians Bastite, Chert, Desmine, Esker, Feldspar, and Gabbro do the same procedure, but using the numbers 40, 50, 60, 70, 80, and 90 respectively (instead of 30).

Once everyone finished their journeys, they found themselves together again, at one point! ...Well, except for one Hearthian. The Hearthians found this bizarre: Could this situation be possible, or did they mess up?

Problem 39

Prove or disprove: Every totally-ordered family of subsets of natural numbers is at most countable.

(A family $\mathcal{F}$ of subsets of $\mathbb{N}$ is *totally-ordered* if for any two sets S, T in $\mathcal{F}$, we either have $S \subseteq T$ or $S \supseteq T$. For example, the family of subsets

$$\mathcal{F} := \big\{\{\}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,5\}, \{1,2,3,5,8\}, \cdots\big\}$$

is totally-ordered. Any two sets in $\mathcal{F}$ are comparable by set inclusion. In contrast, the family of subsets

$$\mathcal{F} := \big\{\{2\}, \{1,2\}, \{2,3\}\big\}$$

is *not* totally-ordered, because the sets $\{1,2\}$ and $\{2,3\}$ cannot be compared.)

Themed Week: Prisoners and Hats

Problem 40

A bored warden decides to play a game. He puts 10 prisoners in a single-file line, and then places either a white or black hat on every prisoner's head. The number of white and black hats is not known by the prisoners. Each prisoner is only able to see the colors of the hats in front of them.

Starting from the back of the line, the warden asks each prisoner to guess the color of their hat. If a prisoner guesses incorrectly, they are shot. The prisoners can hear the guesses and the shots.

What is the maximum number of prisoners that can be guaranteed to survive the game? The prisoners may formulate a plan beforehand.

Problem 41 (Suggested by "Linus")

A bored warden decides to play a game. He puts **100 prisoners** in a single-file line, and then on each of their heads he places one of his **101 different-colored hats**. The prisoners know what the 101 different colors are, and they know that **every prisoner gets a different hat color**. Each prisoner is only able to see the colors of the hats in front of them.

Starting from the back of the line, the warden asks each prisoner to guess the color of their hat. **They may not guess a hat color that has been previously guessed**. If a prisoner guesses incorrectly, they are shot. The prisoners can hear the guesses and the shots.

What is the maximum number of prisoners that can be guaranteed to survive the game? The prisoners may formulate a plan beforehand.

Problem 42 (★)

A bored warden decides to play a game. He puts a **countably infinite** number of prisoners in a single-file line, in such a way that there exists a back-most prisoner. *(E.g. imagine putting the prisoners on the natural numbers of the real line, with all of them facing in the positive direction)* The warden then places a hat on each prisoner's head. Each hat has a **real number** written on it. Each prisoner is only able to see the numbers on the hats in front of them. Each prisoner knows where they stand in line.

Starting from the back of the line, the warden asks each prisoner to guess the real number written on their hat. If a prisoner guesses incorrectly, they are shot.

By formulating a plan beforehand, can the prisoners ensure that **only finitely many of them die**?

Oh, and one last thing: **The prisoners are deaf.**

Problem 43

Is there a differentiable function $f : \mathbb{R} \to \mathbb{R}$ for which

$$\lim_{x\to\infty} f(x) = 2$$

and

$$\lim_{x\to\infty} f'(x) = 1?$$

Problem 44

Ana and Beth decide to play a math game. Starting with Ana, they take turns saying an integer between 1 and 9 inclusive that has not yet been said by either of them. The first player to have said three numbers that sum to 15 is the winner. If all numbers have been said and neither player has won, then the game ends in a draw.

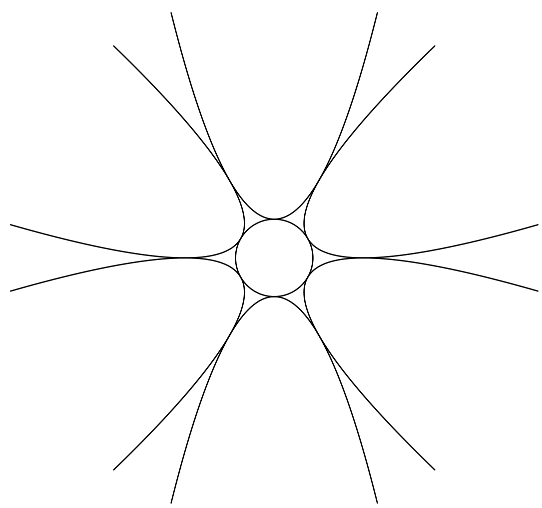
Which player has the winning strategy, if either?

Problem 45

A unit cube is positioned somewhere in space, with some orientation. Prove that the area of its projection unto the xy-plane is equal to the length of its projection unto the z-axis.

Problem 46 (Suggested by David Altizio)

Six copies of the parabola $y = x^2$ are arranged in the following way, tangent to some circle:



What is the radius of this cirle?

Problem 47

Let m and n be positive integers. Prove that $(1-x^m)^n + (1-(1-x)^n)^m \geq 1$ for all $x \in [0, 1]$.

Problem 48

There is a red block and a blue block of equal size. The red block is $100^\circ\ C$ and the blue block is $0^\circ\ C$. You can cut the blocks into pieces and press pieces together. When pieces are pressed together, they will reach thermal equilibrium.

In the end, you reassemble the red and blue blocks. Once it is reassembled, how hot could the blue block be? What is the maximum temperature that the blue block can attain, if any?

Problem 49

In 1734, Euler solved the Basel Problem, which involved proving that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Now it's your turn. What is

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots ?$$

Problem 50

Ari and Beth are playing a game. Beth writes down two distinct real numbers on two cards and gives them to Ari face down. Ari must then choose a card, look at it, and guess whether it was the higher card or lower card. Ari wins the game if and only if her guess is right.

Can Ari follow a strategy that guarantees that her probability of winning the game is strictly greater than 50%?

(We know nothing about how Beth may choose the real numbers. For example, she could be choosing them from a Gaussian distribution, or she may be dead set on choosing 0 and 1.)

Problem 51 (Suggested by Edward Hou)

A hydra happened to harness the hospitality of Hilbert's hotel, an infinite hotel whose rooms are indexed by the integers. Hilbert didn't think this would be an issue until the hydra started antagonizing the guests in room 0. Hilbert has hired you, an expert on hydras, to try and move the hydra to another room.

If you lop a hydra head in some room n, then the hydra will grow two new heads, in rooms $n-1$ and $n+1$. You can also reverse this process: If there exist a hydra head in rooms $n-1$ and and another in room $n+1$ for some integer n, and you lop them off at the same time, then a new head grows in room n. Rooms can have multiple hydra heads.

For which integers n, if any, can we move the hydra to room n, in the sense that after some finite sequence of hydra head-huntings, there could exist hydra heads only in room n?

Problem 52 (Suggested by "tenth")

A triangle T is given. Prove that you can dissect T into three pieces and rearrange them (without any flipping) to obtain a reflected image of T.

Problem 53

A standard 6-sided die's sides are numbered from 1 to 6. A pair of such dice can be rolled to obtain various sums, each having some probability of occurring. Does there exist a different pair of 6-sided dice, with each side being a positive integer, such that the probabilities for each sum that can be obtained by rolling them are the same as that of the pair of standard dice?

This different pair of dice need not be identical to each other.

Problem 54 (★)

Prove that, up to similarity, there is a unique convex equilateral tridecagon whose angles are multiples of 20 degrees.

("Trideca-" means 13.)

Problem 55

I have a rectangular piece of paper that "fits" on my circular plate. That is, it can be placed flatly on the plate without hanging off the edge. Prove that if I fold the paper along any straight line, then it will still fit on the plate.

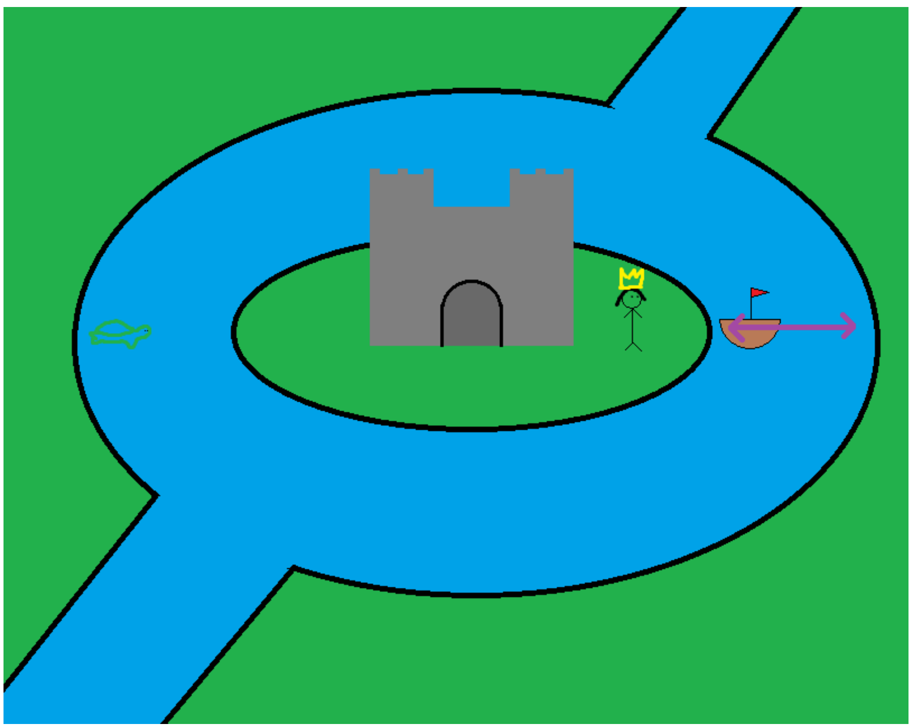
Problem 56

Baka the Bunny has a carrot. In one bite, Baka eats some random amount of it (uniformly at random from no carrot to entire carrot). Baka continues taking bites, with each bite eating the same amount of carrot as the first bite, until there is no longer enough carrot to take a full bite. What is the expected fraction of the carrot eaten by Baka?

Problem 57

Revolution has befallen the Kingdom of Mondstadt! The King is dead and the Queen has gone into hiding.

The Queen's only method of escape is by secretly hiding on a cargo ship that sails to Monstadt and back once a day (see diagram). Knowing this, the Insurrection may choose to bomb any cargo ship in an attempt to assassinate the Queen. Fortunately, you happen to know that the Insurrection has **only finitely many bombs**, but you don't know the exact number.



You may send an escape plan to the Queen for her to follow, but due to the presence of spies, **your message will be intercepted by the Insurrection**.

Prove that you can save the Queen with at least 99% probability.

Problem 58

Cherie is 5 ft tall. She looked in the mirror, and found that she could see her entire body. What is the minimum possible height of the mirror?

Problem 59

Can you cover the plane with the interiors of finitely many non-degenerate parabolas?

Problem 60 (Suggested by "tenth")

Let $0 < x < \pi$. Initially, you have a cake with pink top and purple bottom. You now perform a sequence of steps as follows: On the kth step, you take the x-radian slice of the cake between $(k-1)x$ and kx radians, **flip it over**, and slot it back into the cake.

Prove that eventually (i.e. in a finite, positive number of steps), the cake returns to its original state, with a completely pink top and completely purple bottom.

Note: x/π is not necessarily rational.

Themed Week: Lights Out

The Lights Out Puzzle is played on an $n \times n$ grid of lights. At the start, some subset of the lights are on. A move consists of pressing any of the lights, which flips the on/off state of that light as well as the orthogonally adjacent lights.

Problem 61

Prove that there exists a 4×4 Lights Out puzzle that cannot be solved.

Problem 62

Let us play with a variant now.

The *Rook-Toggle Lights Out puzzle* is played on an $n \times n$ board. It is identical to the standard Lights Out puzzle, except the *toggle rule* is changed. In this variant, if you press a light, then that light as well as all lights in the same row and column are toggled from on to off and vice versa. (That is, all squares “attacked by a rook” are toggled.)

Given a board of lights, an *easy run* consists of pressing all lights that are on simultaneously.

Prove that any solvable $n \times n$ Rook-Toggle Lights Out puzzle will be solved in two easy runs.

Problem 63

Finally, let us discuss the Lights Out game in full generality.

The *Lights Out puzzle* is played on a graph G with n vertices. Each vertex is a light that may start on or off. In a move, you may press a light, which toggles the state of that light and all of its neighbors.

Klaus Sutner (a CMU professor!) proved the following theorem: *Every Lights Out puzzle in which all lights start in the on state can be solved.* His proof uses linear algebra, but be rest assured that this is not necessary.

Prove the above theorem.

Problem 64

There are 20 ants on the left half of a stick, facing right, and 22 ants on the right half of the stick, facing left. They all start moving forward at the same speed. When two ants collide, they both switch directions. When an ant reaches either end of the stick, it falls off.

a) How many ants fall off of each side?

b) How many collisions will there be?

Problem 65

You are given the graph of $y = x^3$. Given a compass and straightedge, construct the coordinate axes.

Problem 66

A wounded king strolls the battlefield, encountering an 8×8 chessboard that he must cross. The chessboard is then set on fire, with each square having a 50% chance of burning up and becoming impassable. The king, being wounded, can only move in the four cardinal directions, as well as up-right or bottom-left. Compute the probability that the king can start on the bottom row of the board and reach the top.

Problem 67

14	16	12
18	14	10
16	18	14

Inside of each of the nine rectangles above is the perimeter of said rectangle. ...Er, actually one of the numbers is lying. Which one?

Problem 68 (Suggested by Edward)

Booster is trying to find Mario, who is hiding behind one of countably many curtains indexed by the naturals. Booster can open and close any curtain to check if Mario is there, and then Mario will move to an adjacent curtain (i.e. a curtain whose index differs from Mario's current curtain's index by exactly 1). Can Booster guarantee that he will eventually find Mario?

Problem 69 (★)

Points A and B are 10 miles apart. You have nothing but a 1-inch straightedge and a pencil. Construct the line segment between A and B.

Problem 70

Emily and Sydney take turns placing quarters on a rectangular table, such that no quarter hangs off the edge or lies on top of another quarter. The player that cannot place a quarter loses. Who has the winning strategy?

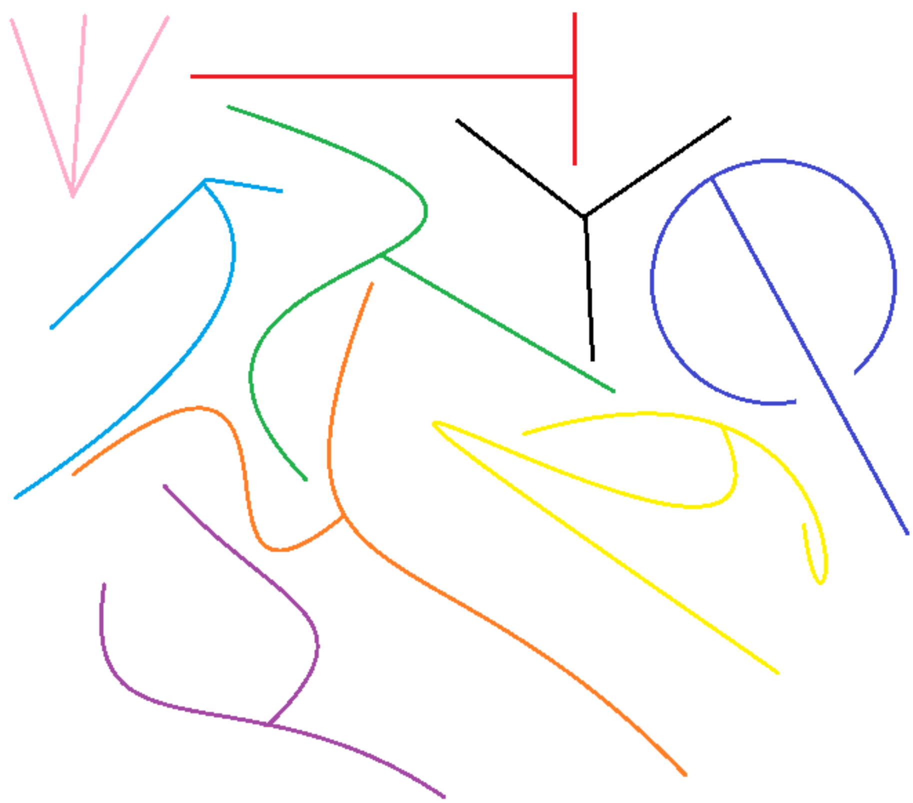
Problem 71

Evaluate the integral

$$\int_0^\infty e^{-x^2-1/x^2}\,dx.$$

Problem 72 (★)

Do there exist uncountably many pairwise-disjoint subsets of a plane, each homeomorphic to the letter Y?



Problem 73

An ant lies on a vertex of a unit cube. What is the length of its shortest path along the surface to the opposite vertex?

Problem 74 (Suggested by Edward)

An ant lies on a vertex of a $1 \times 1 \times 2$ box. What point on the box is farthest (along the surface) from the ant? Exactly how far is it from the ant?

Problem 75

Let $P(x)$ be a non-constant polynomial with complex coefficients. Prove that the roots of $P'(x)$ lie in the convex hull of the roots of $P(x)$.

Problem 76

Let P be a convex polygon with 180-degree rotational symmetry. Prove that P may be subdivided into parallelograms.

Problem 77

a) Fix $0 < p < 1$. Given a fair coin, simulate a biased coin that flips head with probability p.

b) Given a biased coin that flips heads with some probability $0 < p < 1$, simulate a fair coin. You do not know the value of p.

Problem 78 (Suggested by Edward)

a) Show that if $A > 0$ is large enough, then any collection of squares whose areas sum to A can cover a unit square.

b) Show that if $a > 0$ is small enough, then any collection of squares whose areas sum to a can fit inside a unit square.

Problem 79

Magellan is trying to circumnavigate the ocean world Planet 4546B by sailing one full revolution around the equator.

Magellan has three ships docked at some port, where he can supply his ships with supplies. But, each of his ships can only carry enough supplies to sail halfway around the equator.

Can Magellan fulfill his dream?

Problem 80

The derivatives of $\sin(x)$ repeat after every 4 derivatives. Can you find a smooth function $f : \mathbb{R} \to \mathbb{R}$, expressed in closed form using elementary functions, such that its sequence of derivatives repeats after every 3 derivatives?

Problem 81 (★★) (Proposed by Edward Hou)

The circle shape can generate Venn diagrams of size at most 3.

Prove that there is a convex shape that can generate Venn diagrams of any size.

(Rigorously, show that there is a convex open $K \subseteq \mathbb{R}^2$ such that for any $n \in \mathbb{N}$, you can arrange n copies $K_1, K_2, \cdots, K_n$ of K, with translation and rotation allowed, such that for any $A \subseteq \{1, 2, \cdots, n\}$ we have that $\bigcap_{i \in A} K_i \cap \bigcap_{i \notin A} K_i^c$ is non-empty.)

Themed Week: Pizza

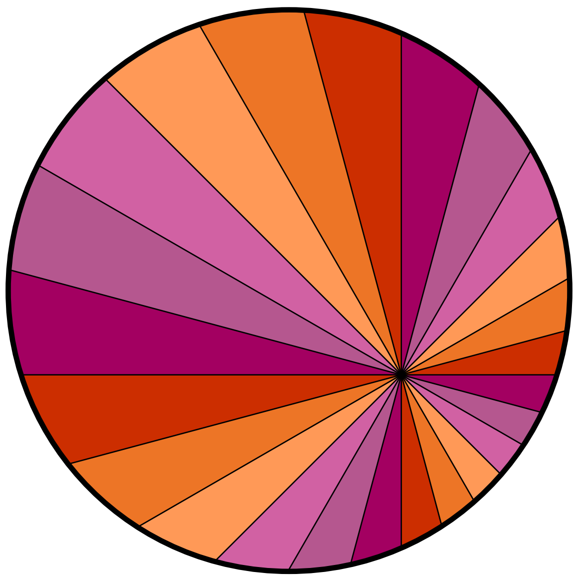
Problem 82

It's a Halloween pizza party! But little Timmie doesn't like eating the crust. Ugh. Can you divide a circular pizza into finitely many congruent pieces such that there exists a piece with no (positive-length) crust?

(So, a piece that intersects the crust at exactly one point is okay!)

Problem 83

N people want to split a pizza equally. But I'm bad at finding the center of a pizza. So I just picked some point in the pizza and made $2N$ cuts through it at equal angles, like so:

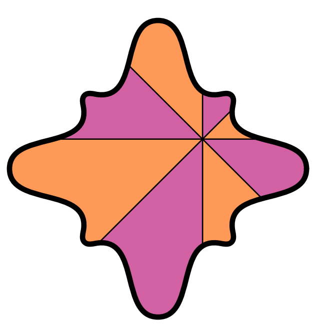


This divides the pizza into $4N$ slices, and now each of the N people takes every Nth slice (so, 4 slices each).

Prove that everyone gets the same amount of pizza.

Further Ventures:

- Can you prove also that if $N \geq 2$, then alternating the slices will evenly divide the pizza between two people?
- Can you prove that the following pizza is evenly divided?



Problem 84 (★★)

Allison and Beth are sharing a pizza. Beth cuts the pizza into slices (all sectors), not necessarily all the same size. Then, starting with Allison, they alternate taking slices such that the remaining pizza is always one contiguous piece (you can think of this as "each slice taken is adjacent to the existing empty space").

Does Beth have a strategy that ensures that she gets to eat strictly more than half the pizza?

Problem 85

It is given that the minimum value of x^x over $(0, \infty)$ is M. Prove that the maximum value of $x^{1/x}$ over $(0, \infty)$ is $1/M$.

Problem 86

Find a polygon with the following properties, or prove that it is impossible:

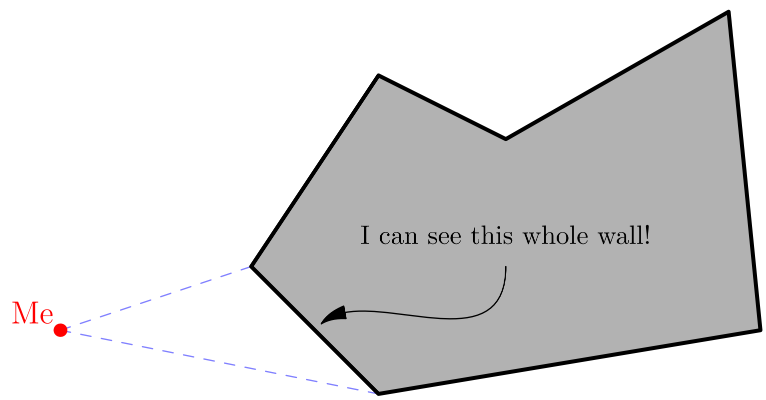
- The polygon has perimeter 314.
- All of the polygon's side lengths have length 1.
- All of the polygon's internal angles are either 90 or 270 degrees.

Problem 87

I'm somewhere in the interior of a square room whose walls are mirrors. Can my friends arrange themselves inside (or on the border of) this square room in a particular way so that I cannot see my own reflection?

Problem 88 (Courtesy of Lance Lampert)

I'm standing outside a polygonal building. Prove or disprove: I must be able to see one of its walls in entirety.



Problem 89

The floor of my house is uneven. Show that if I buy a perfectly-shaped square table of sufficient height, I can position it in such a way that all of the legs touch the floor.

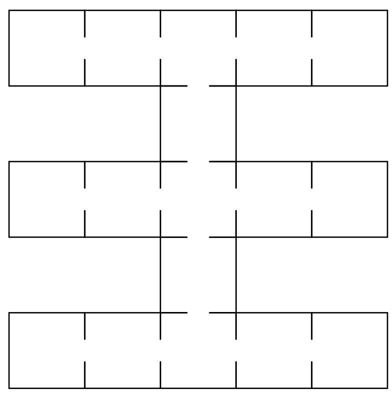
(For rigor's sake, you may make the following realistic assumptions: The floor is quite large compared to the table and is Lipschitz continuous with Lipschitz constant 1 — that is, the slope of the line between any two points on the floor will never exceed 1.)

Problem 90

At last, a nefarious crime lord's hiding place has been tracked down: a 1000-room palace. The floor plan is a secret, but it is known that the palace's adjacency graph is a tree. That is, the rooms are connected in such a way that there is no loop of rooms.

What is the minimum number of soldiers that should be sent to infiltrate the palace in order to ensure the crime lord's capture? The crime lord is considered to be caught if they are ever in the same room as a soldier.

As an example, a criminal hiding in the palace below could evade a search from one soldier, but two soldiers can guarantee the capture.



Clarifications:

- *The crime lord cannot "swap places" with a soldier while evading said soldier, like in real life.*
- *Time can be interpreted in either a continuous or discrete sense as long as the above point is obeyed.*
- *The crime lord's speed can be arbitrarily large.*
- *The soldiers do not know anything about the crime lord's whereabouts other than the fact that they are present in the palace.*
- *There are no "hallways" between rooms. They are connected by virtue of being adjacent to each other.*
- *The crime lord's capture must be guaranteed in finite time, not just almost surely in the limit.*

Problem 91

a) The shape of Gloria's house is a circle. She builds a fence around her house such that all points on the fence are exactly 1 foot away from her house.

   Prove that the fence is exactly 2π feet longer than the perimeter of her house.

b) What if the shape of Gloria's house was a convex polygon instead?

c) (Bonus) What if the shape of Gloria's house was an arbitrary bounded convex set?

Problem 92 (Suggested by "asbodke")

Consider a 4×3 rectangle.

a) Prove that among any 7 points in the rectangle, there exist two of those points that are at most $\sqrt{5}$ apart.

b) Can we do better than 7?

Problem 93 (Suggested by "tanoshii")

There are $\frac{n(n+1)}{2}$ stones in piles. Every minute, we remove a stone from each pile and gather them to form a new pile.

Prove that at some point, the piles' sizes will forever be $1, 2, \cdots, n$.

Problem 94

Choose a point uniformly at random from a 9001×420 rectangle. What is the probability that this point is closer to the center of the rectangle than any of the four vertices of the rectangle?

Problem 95

a) Show that any increasing $f : [0,1] \to [0,1]$ has a fixed point.

b) Show that any increasing $f : P(X) \to P(X)$ has a fixed point, where X is any set.

In Part (b), $P(X)$ denotes the power set of X, and f is increasing in the sense that if $A \subseteq B$ then $f(A) \subseteq f(B)$.

Problem 96 (Suggested by “Linus”)

For a positive integer n, let $o(n)$ be the number of odd digits of n. Is

$$\sum_{n=0}^{\infty} \frac{o(2^n)}{2^n}$$

rational?

Problem 97

A group of 25 people stand in a ring on a field. All the people have different heights. Each person is asked whether they are taller than their neighbors, shorter than their neighbors, or in-between.

5 people said "taller". How many said "in-between"?

Problem 98

A turtle is on a field. It crawled forward for 6 hours (the turtle's speed can vary, and the turtle need not always be moving). The turtle's crawling was always watched by at least one turtle enthusiast at all times. Each enthusiast watched the turtle crawl for an hour before leaving, reporting that they saw the turtle crawl exactly 1 inch in that hour.

What's the farthest distance that the turtle could have crawled during the 6 hours?

Clarification: You may assume that all relevant time intervals are closed.

Problem 99

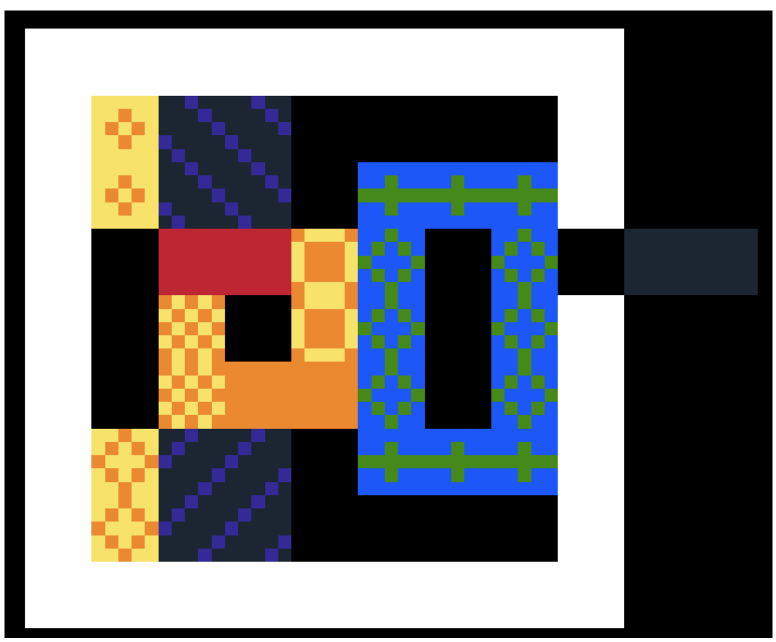
The turtle from Problem 98 is on a field again. This time it plans to crawl 1 foot in 1 hour. The people in Problem 97 found a real number t such that no matter how the turtle moves, there will exist an interval of length t (contained within the hour) during which the turtle crawls t feet. Find all possible values of t.

Themed Week: Rush Hour

Problem 100

Solve the following Rush Hour variant puzzle. The rules are as follows.

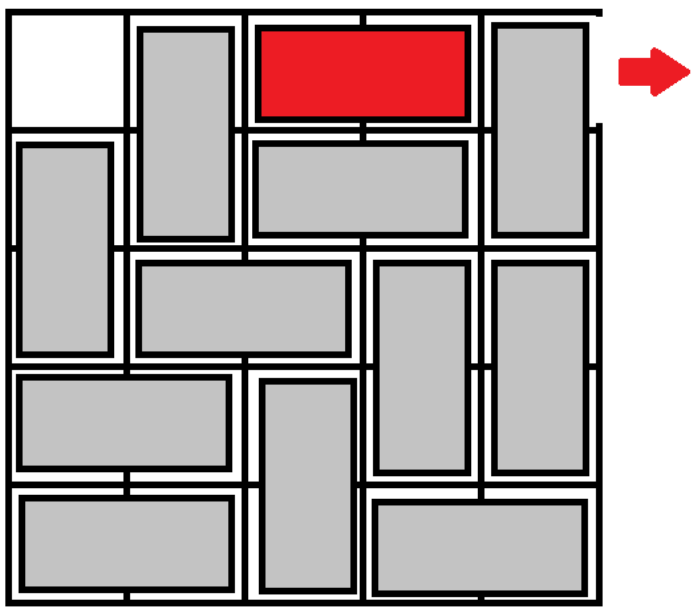
- Each piece in the puzzle is a "train".
- The goal is to move the 1×2 **red** train out of the enclosure.
- Each horizontal 1×2 or 1×3 train can only move horizontally. Ditto for the vertical trains.
- The square 2×2 trains can move in any orthogonal direction.



(Click here to play)

Problem 101 (Suggested by "tenth")

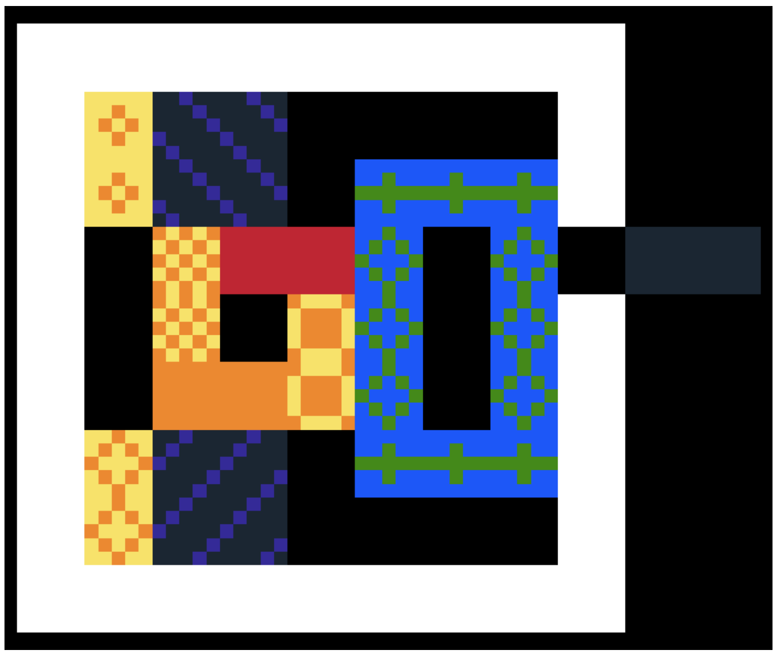
A rush-hour puzzle on a 2021×2021 grid is designed as follows: A corner of the grid is chosen, and an edge adjacent to that corner is removed, forming the exit point. The 1×2 red train is then placed such that it is one square away from the exit. Lastly, the entire grid is filled with 1×2 obstacle cars, except for some corner square. (The diagram below shows an example for a smaller-sized grid.)



Prove that the rush-hour puzzle that we have constructed is solvable.

Problem 102 (★)

My sincerest apologies for Problem 100. Since this is the CMU *Math* Club, you were probably expecting a really interesting math problem, but I gave a trivial puzzle instead. This was undoubtedly quite disappointing. To make up for this error in judgment, I'm making today's problem by taking Problem 100 and pushing the red train forward by one square. This should make the puzzle slightly easier and make for a relaxing Friday problem. Thank you for your understanding.



(Click here to play)

Problem 103 (Suggested by Edward Hou)

There are 100 bottles of beer on the wall. 100 bottles of beer! You can take some down, pass 'em around, and whenever you have three empty bottles you can trade them in for one full bottle of beer.

Your friend is willing to give you as many empty bottles as you want, as long as you return them. How many bottles of beer can be drunk?

Problem 104

You enter your kitchen to see a disaster! Each of your n chefs started cooking a personalized pancake on their own pan before falling asleep... and it's about time to flip the pancakes!

To help with the flipping, you brought your own pan (for a total of $n+1$ pans, one empty). You can take any pancake-filled pan and transfer its pancake unto the empty pan by putting the pans on top of each other and turning them around, flipping the pancake in the process.

Is it possible to execute a sequence of such moves such that each of the n pancakes ends up in the pan it was originally in, except flipped?

Problem 105

I have a 2×200 box.

a) Show that 400 coins of diameter 1 can fit inside the box.

b) Can we fit any more than that?

Problem 106

Alyssa and Beth bought a 101×101 chocolate bar for some reason. They take turns taking a piece of chocolate and breaking it up along a gridline. Whoever can't move is the loser (i.e. when all pieces are 1×1). Who has the winning strategy?

Problem 107

5^{2023} is a 1415-digit number that starts with 1. How many smaller powers of 5 (from 5^0 to 5^{2022} inclusive) also start with 1?

Problem 108

Let $f : [0,1] \to [0,1]$ be continuous and strictly increasing with $f(0) = 0$ and $f(1) = 1$. Prove that

$$\sum_{n=1}^{9} f(n/10) + f^{-1}(n/10) \leq \frac{99}{10}.$$

Problem 109

Let n be a positive integer. Prove that the penultimate digit of 3^n is even.

Problem 110 (★★)

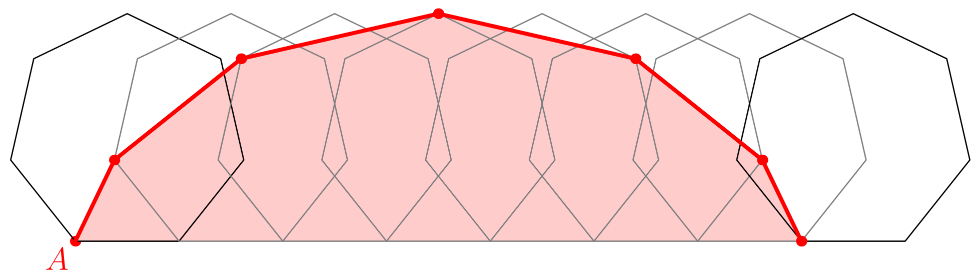
Let I be an interval and $f : I \to I$ be continuous. We say that a point $x \in I$ has period k if the sequence $x, f(x), f(f(x)), \cdots$ has period k.

Suppose that some point has period 3. Prove that there is a point with period n for all natural n.

(Apology: Due to an error of my own, this problem is much harder than intended. It's still true though, and there's an elementary proof.)

Problem 111 (★)

P is a regular n-gon with area 1 such that one of its sides, $\overline{AB}$, is resting on a line. We now roll P forwards n times. The images of point A under each of these rolls form a new n-gon Q.



Prove that the area of Q is 3.

Problem 112

There is a 5×5 grid of rooms, and between every two orthogonally adjacent rooms is a Magic Door. If you pass through a Magic Door in one direction, you will gain a dollar. But if you pass through that same Magic Door in the opposite direction, you will lose a dollar.

Starting with the room in the northwest corner, I wandered around the rooms, and at one point I discovered that I had gained \$8. Exploring more, I further discovered that if I walk through any loop of doors, the net monetary gain will always be \$0.

Can you figure out whether I will gain or lose a dollar if I exit the central room through its north door?

Problem 113 (Suggested by "tanoshii")

Let $1 \leq k \leq n$. I'm thinking of a secret number. Can I communicate with my n friends in such a way that no $k-1$ of them can figure it out by combining their knowledge, but any k of them can?

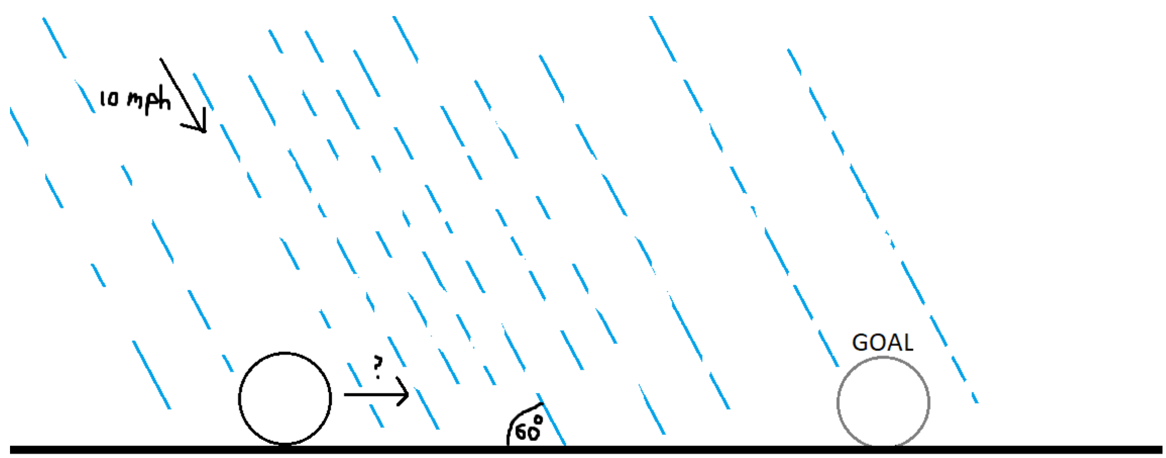
Problem 114 (Suggested by Edward Hou)

a) You are in a circular cage with a hungry lion in the center, who can move at the same speed as you. For many years, it was thought that you would surely be eaten if the lion follows the simple strategy of moving towards you whilst always staying on the line segment connecting your position with the center. It turns out that this doesn't work. Show that you can survive indefinitely if the lion follows this strategy.

b) Suppose now that there are two lions somewhere in the cage. Show that the lions can coordinate in some way to eat you.

Problem 115

Rain is falling at 10 mph at an angle of 60° (directed away from you). How fast should you run to a certain destination in front of you to minimize how wet you get?

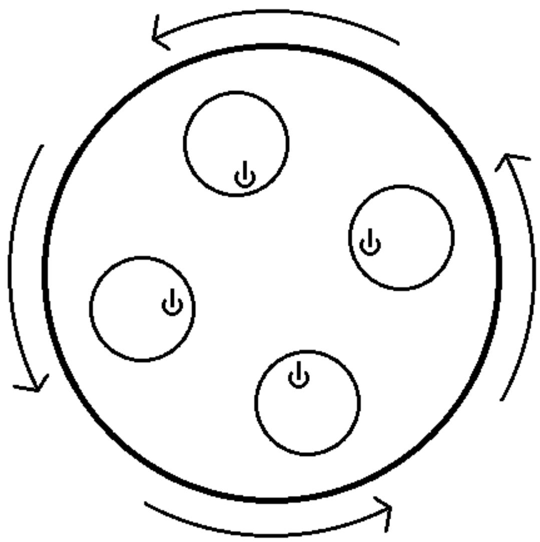


Assume that you are shaped like a ball.

Problem 116

You have reached the enemy's server room, which is a rapidly rotating circular room consisting of four indistinguishable cylindrical servers arranged symmetrically in a square. At least one server is on, but some could be off.

Each server has a power button that can be pressed to switch its state from on to off or vice versa, but this change will not take effect until you exit the room and close the door for a few seconds. You won't be able to tell which server was which upon re-entering. Moreover there is no visual indicator for whether a server is on or off.



Your goal is to turn off all the servers so that the hackers can access the mainframe and like uh win everything. The hackers will inform you the moment all servers are off. Can you guarantee success in some finite number of steps?

Problem 117

You and your 2023 friends are sitting around a table. The bottle of grape juice you have is really good, so you randomly choose one of your two neighbors to give it to. They in turn think it's really good, and they too pass the bottle to a random neighbor. This keeps going until everyone tastes the grape juice.

Who is the most likely to try the grape juice last?

Problem 118

A ladder slides down a wall. What shape is traced out by its midpoint?

Problem 119 (Suggested by "tanoshii")

For which $n \geq 3$ does there exist a regular n-gon in the xy-plane whose vertices are lattice points?

Problem 120 (★) (Suggested by Edward Hou)

In 3D space, what is the largest number of lines that can be selected such that all the lines intersect at one point and each pair of distinct lines intersect at the same angle?

For example, we can obtain 3 by choosing the x, y, and z axes, which all intersect at the origin with every pair of axes intersecting at an angle of $90°$.

Problem 121

There are 5 apples for sale with 5 different sizes and 5 different positive integer prices from \$1 to \$5. In dollars, what is the price of the apple that's bigger than the apple that costs more than the apple that's smaller than the apple that's cheaper than the apple that's green, given that it is red?

Problem 122

Can you drill a hole through a wooden cube such that a larger cube can be passed through the hole?

Problem 123 (Suggested by "tenth")

The digits of a positive integer are all greater than 5. Could the digits of its square be all less than 5?

Problem 124

How full is the wine bottle?

Problem 125

In an effort to commit as many FERPA violations as possible, I lazily handed back midterm exams to the students in my recitation randomly. Each student got back an exam, but it might not be theirs.

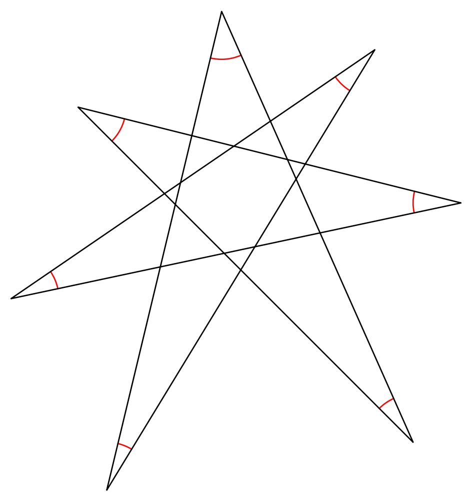
In one round, each student may choose a partner and switch exams with them. Prove that everyone can get back their correct midterm exam in at most two rounds.

Problem 126 (★) (Suggested by "tenth")

There is a triangle with area T and an ellipse with area E. Their intersection has area A. Show that $\frac{T}{3} + \frac{E}{2} \geq A$.

Problem 127

Find the sum of the seven marked angles.



Problem 128

Is it possible to partition the positive integers into multiple arithmetic progressions, each with a distinct common difference?

Problem 129

Detective Yohane's area of vision is an open disk of positive radius centered at her location. She has found a crucial clue in her investigation: A finite set of footprints in the plane! Yohane decides to track down the murderer with the following search strategy: Every minute, she will travel to the centroid of the set of all footprints in her vision.

Show that Yohane will eventually stop moving.

Problem 130 (Suggested by David Altizio)

A triangle's vertices are lattice points. There are no other lattice points on the triangle's boundary. Suppose that there is exactly one lattice point in the triangle's interior. Prove that this point is the triangle's centroid.

Problem 131 (Suggested by Edward Hou)

I have a large supply of unit-length rods that can be linked only at their endpoints. I can join three of them to form an equilateral triangle, which will be a rigid shape. The same is not true if I make a unit square, since the rods can rotate about their ends to deform the square into a parallelogram.

Is it possible to brace a unit square by adding more unit-length rods so that it becomes rigid? The rods may intersect.

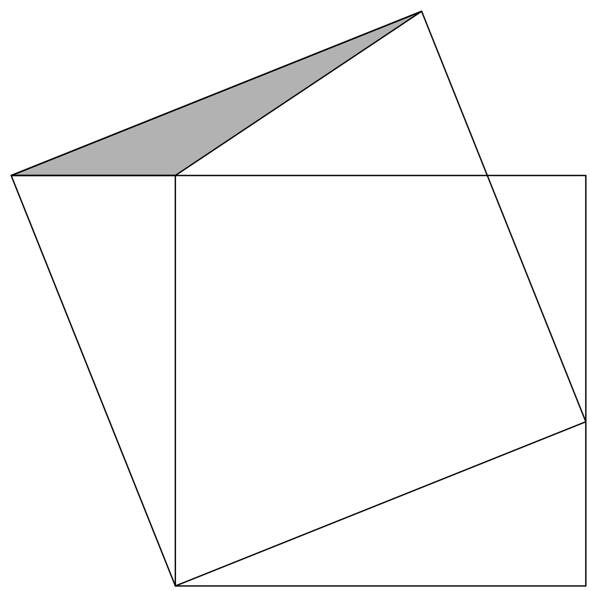
Problem 132

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n} \qquad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n}$$

One of these series converges for all θ. The other one diverges for some θ. Which is which?

Problem 133 (Suggested by several people)

Two squares are overlapped as shown. The smaller square has area 16, and the gray triangle has area 1. What is the area of the larger square?



Problem 134

I've struck it rich! At the bottom of a circular lake, I found a treasure chest full of gold. Unfortunately, the evil pirate king has taken notice, and has sent his army of pirates to patrol the boundary of the lake to prevent my escape.

The pirates' running speed is the same as the speed of my boat. If I can reach any part of the shore that isn't being occupied by a pirate, I'll be safe because I can outrun the pirates.

How many pirates does the pirate king need to send in order to prevent my escape?

(Open the first hint for an extension to this problem!)

Problem 135 (★★) (Suggested by "tanoshii")

Let S be a finite set of positive integers such that $\sum_{k \in S} \frac{1}{k} \geq 2$. Prove that two distinct subsets of S have the same sum.

Problem 136 (Suggested by "tenth")

Can a closed disk be partitioned into two congruent sets?

Problem 137

I'm a disorganized and overexcited birthday party host. I have no idea how many people I invited, and every time another guest arrives, I make a cut into the cake, thereby incrementing the number of cake pieces by one. I will only serve the pieces of cake once I am informed that all guests have arrived.

Can I ensure that the largest cake piece will be strictly less than double the size of the smallest cake piece?

Problem 138

My two friends and I found a treasure chest with 101 pieces of gold! I'm feeling awfully generous, so I agree to take just one piece. We notice that no matter which piece I take, my friends can distribute the remaining 100 pieces into two shares of 50 pieces, with each share weighing the same.

Prove that all 101 gold pieces have the same weight.

Problem 139

I have a shuffled standard deck of 52 cards. I start dealing them face-up, and you can stop me at any point before I run out of cards. You win if, at this point, the card on top of the deck is an ace.

Under an optimal strategy, what are your odds of winning?

Problem 140 (Suggested by Edward Hou)

Four points A, B, C, and D line in a plane. It is given that there does not exist a square such that each of its sides (when extended) passes through one of these four points, with different sides passing through different points.

Prove that D is the orthocenter of $\triangle ABC$.

Problem 141

Prove *Smith's Determinant Identity*:

$$\begin{vmatrix} \gcd(1,1) & \gcd(1,2) & \gcd(1,3) & \cdots & \gcd(1,n) \\ \gcd(2,1) & \gcd(2,2) & \gcd(2,3) & \cdots & \gcd(2,n) \\ \gcd(3,1) & \gcd(3,2) & \gcd(3,3) & \cdots & \gcd(3,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gcd(n,1) & \gcd(n,2) & \gcd(n,3) & \cdots & \gcd(n,n) \end{vmatrix} = \varphi(1)\varphi(2)\varphi(3)\cdots\varphi(n),$$

where φ is the Euler totient function.

Problem 142

I'm making dumplings! I have three unmixed bowls of filling for different types of dumplings. I need to mix each bowl by hand.

My hand gets sweaty really easily, so to prevent this from contaminating the filling, I need to wear a latex glove. Moreover, we don't want cross-contamination between the fillings.

Can I mix all three bowls without dirtying my hand and without any contamination, if I only have two latex gloves at my disposal?

Problem 143

A number of guards were hired for the night shift at a museum. Each guard was assigned to watch a different room. Due to boredom, each guard at some point decides to take a walk around the museum before coming back to their assigned room, having visited every room of the museum exactly once.

Miraculously, no two guards ever saw each other during the night. Prove that, at some point during the night, no guard was watching their assigned room.

Problem 144 (Suggested by "tenth")

Let T be a triangle. Amber and Beth are playing a game. In a move, Amber chooses a point in the plane, and then Beth colors this point either red or blue. Moves of this form are repeated until there are three points of the same color that form a triangle congruent to T, at which point Amber wins.

a) Is there a triangle T for which Amber can force a win?

b) Is there a triangle T for which Beth can forever prevent Amber from winning?

Problem 145

$$\sin(\cos(x)) \qquad \cos(\sin(x))$$

Show that one of the above functions is strictly greater than the other function for all $x \in \mathbb{R}$.

Problem 146

Four distinct circles of radius r are on the surface of a unit sphere such that they are pairwise tangent. What is r?

Problem 147 (★) (Suggested by "tenth")

There are n players that want to play a board game. They need to decide their playing order uniformly at random, and they wish to do this by rolling n fair dice as follows: Each player will roll the die assigned to them, and their playing order is given by the order of the n numbers rolled.

Does there exist n dice that accomplish this task, for every positive integer n?

(To be precise: Find n finite sets $A_1, A_2, \ldots, A_n$ of real numbers such that if we pick elements $x(1), x(2), \ldots, x(n)$ from these sets, each uniformly at random, then the probability that

$$x(\pi(1)) < x(\pi(2)) < \cdots < x(\pi(n))$$

is exactly $\frac{1}{n!}$ for any permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.)

Problem 148

On a 2023×2023 grid, Ashley and Beth take turns claiming (unclaimed) squares, with Ashley going first. The game ends when all squares are claimed, and the winner is the player whose claimed region has the greater perimeter. Which player has the winning strategy, if either?

Problem 149 (Suggested by "tenth")

We call a subset S of $\mathbb{R}^n$ a *two-distance set* if for some $a, b > 0$, we have that the distance between any two distinct points of S is either a or b.

a) Prove that there is a two-distance set of size $\binom{n}{2}$.

b) Prove that there is a two-distance set of size $\binom{n+1}{2}$.

Problem 150

Ai and Beth each have 2023 dollars. They also each have a biased coin that flips heads with probability 51%. Every second, they flip their coins. On every flip, Ai bets a dollar that her coin comes up heads, whereas Beth bets a dollar that her coin comes up tails.

a) Ai and Beth eventually both go broke. Who was more likely to have gone broke first?

b) Suppose now that instead of flipping two different biased coins, they were flipping the *same* biased coin. If Ai and Beth eventually both go broke, who was more likely to have gone broke first?

Problem 151

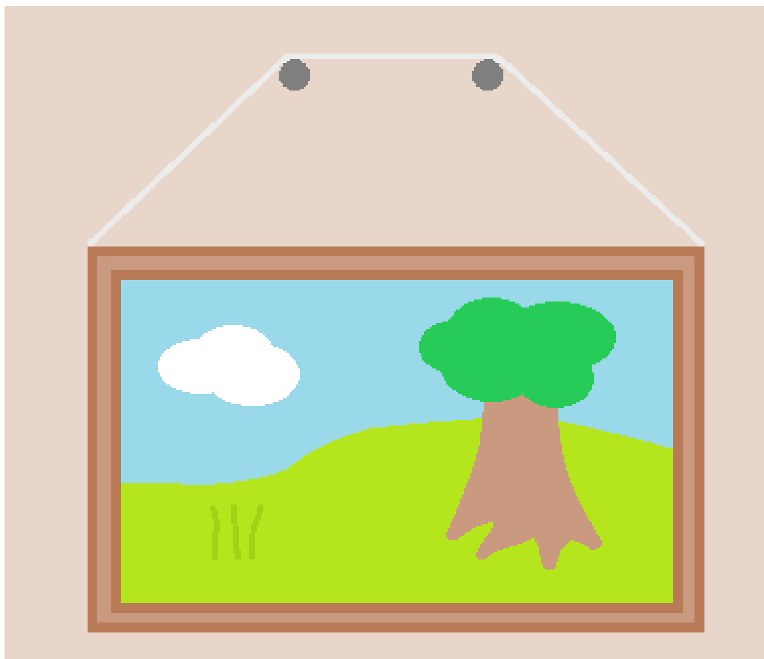
Angela is thinking of a polynomial P with positive integer coefficients, and Beth is trying to guess what it is. Beth can pay Angela a dollar to know the value of $P(n)$ for a positive integer n of Beth's choice. Beth can do this as many times as she wants.

Beth's money is a bit tight, though. At most how much money does Beth need to spend to deduce Angela's polynomial?

Problem 152

A painting is hanging on the wall by a string over two nails. For the painting to fall, both nails must be removed.

Wrap the string around the two nails in such a way that the painting hangs, but will fall if either nail is removed.



Problem 153 (★)

Please complete the below wordsearch. Recall that words can only appear in any of 8 compass directions and that each word you need to find will appear exactly once. The special properties for this wordsearch are as follows:

1. Only the letters 'B' and 'E' appear.
2. I spilled my drink over the wordsearch, so most letters are not visible. Apologies.

									E							
										E						
											E					
			E													
															E	
									E							

WORDS TO FIND

BEE

Problem 154

Let G be the graph of a quartic with two inflection points at A and B. Ray $\overrightarrow{AB}$ intersects G again at a point C. Prove that $\frac{|AC|}{|AB|} = \varphi$ where φ is the golden ratio.

Problem 155

(a) Find a solution to the "differential equation"

$$f'(x) = f^{-1}(x), \qquad x \in (0, \infty)$$

for $f : (0, \infty) \to (0, \infty)$ differentiable and invertible.

(b) (Bonus) Show that the solution is unique.

Problem 156

$z \neq 0$ is a complex number. Given that z is a root of a polynomial whose coefficients are all either 0 or 1, compute the greatest possible lower bound on $|z|$.

Problem 157

Show that

$$\int_0^\infty \frac{1}{(1 + x^\varphi)^\varphi}\, dx = 1$$

where φ is the golden ratio.

Problem 158 (★★)

P is a monic polynomial with integer coefficients. It is given that all of its roots are real, are non-integers, and lie between 0 and 3.

Prove that $P(\varphi^2) = 0$ where φ is the golden ratio.

Grand Finale

For over a year, the POTD has been secretly creating something behind the scenes...

Problem 159: The Abyss

ERROR (problem_data.txt): The file or directory is corrupted and unreadable.

Problem 160

Let B be the answer to the problem that is counter-clockwise adjacent to (and at the same altitude as) this one. $WXYZ$ is a convex quadrilateral. Rays $\overrightarrow{WX}$ and $\overrightarrow{ZY}$ intersect at P, and rays $\overrightarrow{WZ}$ and $\overrightarrow{XY}$ intersect at Q. Suppose that

$$WX = XY = YW = XP = ZQ = B.$$

Then $\log_B(WZ)$ may be written as p/q where p and q are positive integers and $\gcd(p, q) = 1$. Compute $|p - q|$.

Clarification: The "counter-clockwise" direction is from the perspective of an observer looking down from above.

Problem 161

Let B and F be the answers to the two adjacent problems below this one. $\triangle XYZ$ satisfies $XZ = B$, $YZ = F$, and $\angle XZY = 40°$. The shortest path that starts from X, visits segment YZ, visits segment XZ, and then ends at Y has length $\sqrt{n}$ for an integer n. What is n?

Problem 162

Let A be the answer to the problem below this one. Let n be a positive integer. A sequence of n squares of an $n \times n$ grid is called a *snake* if each square in the sequence (after the first) is either the rightward neighbor or the upward neighbor of the previous one. It turns out that there are A ways to partition an $n \times n$ grid into n snakes. Compute n.

Problem 163

Let B and C be the answers to the two adjacent problems below this one. $b \geq 2$ is a positive integer. Consider the sequence

$$b, b^2, b^3, b^4, b^5, \cdots$$

formed by the powers of b, starting with b. Now make one list of positive integers consisting of the lengths of these powers in base $\sqrt{B}$, and make another list of positive integers consisting of the lengths of these powers in base $\sqrt{C}$.

It just so happens that these two lists of positive integers partition the set $\{2, 3, 4, 5, \cdots\}$. What is b?

Problem 164

Let A be the answer to the problem below this one. Let K be the answer to the problem opposite this one. P is a point on the circumcircle of equilateral triangle $\triangle XYZ$ which lies on the minor arc between X and Y. Given that $PX = A$ and $PY = K$, what is PZ?

Problem 165

Let C and D be the answers to the two adjacent problems below this one. Let F be the answer to the problem opposite this one. Let $\mu = C + D - F$. For some integers x, y, z, the average of $(x - y)^3$, $(y - z)^3$, and $(z - x)^3$ is μ. Compute

$$\max(x, y, z) - \min(x, y, z).$$

Problem 166 (Suggested by "tenth")

Let A be the answer to the problem below this one. In how many ways can you tile a regular A-gon with side length 1, such that each tile is either

- an equilateral triangle of side length 1, or
- a rhombus with side length 1, all of whose angles are greater than $60°$?

Problem 167

Let D and E be the answers to the two adjacent problems below this one, with E being counter-clockwise adjacent to D.

Consider the product

$$(1!)(2!)(3!)(4!)(5!)\cdots((D+E^2-1)!)((D+E^2)!).$$

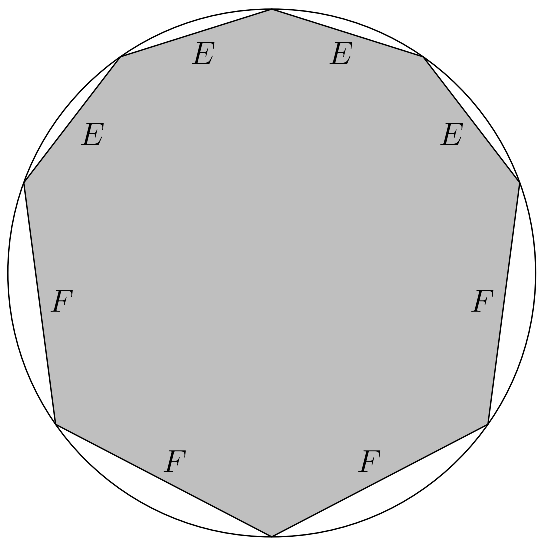
What is the value of n for which the above product, when the term $(n!)$ is removed, will be a perfect square?

Problem 168

Let E be the answer to this problem. Beth has finally found her life's calling: Fishing. Every time she catches a fish, its size is independently and uniformly selected at random from $[0,1]$. Yesterday she caught $E-1$ red fish. What is $(\mathbb{E}[\text{Size of the smallest red fish}])^{-1}$?

Problem 169

Let E and F be the answers to the two adjacent problems below this one. Then the area of the cyclic octagon below is $m + n\sqrt{2}$ for some positive integers m and n. What is $m + n$?



Problem 170: The Peak

Use the answers to the five problems adjacent to this one.

What is the POTD's secret agenda?