NYU Advanced Calculus Workshop

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Contents

0 Notation and Conventions

0.1 Multvariable Calculus

- **Vector Fields and Scalar Fields:** Vector fields are always denoted by capital letters. Scalar fields are always denoted by lowercase letters.
- **Gradients and Jacobians:** For a function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f is the gradient of f. For a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$
- **Integration:**
  - I will not be writing $\iint$, $\iiint$, etc. and will instead represent all integrals as $\int$.
  - $\int_C F \cdot ds$ is the line integral of a vector field F over a path C.
  - dx indicates integration over $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, depending on context. You may be familiar with seeing dA or dV instead. I may instead use $d(x, y)(\equiv dA)$ to clarify integration over $\mathbb{R}^2$, and similarly write $d(x, y, z)(\equiv dV)$ to clarify integration over $\mathbb{R}^3$.
  - dS indicates a surface integral.
  - In writing an integral we will often omit the independent variable for brevity/clarity, e.g. $\int_\Omega f\,dx := \int_\Omega f(x)\,dx$.
  - If, however, the variable is important (particularly when multiple integrals are involved), I may emphasize the name of the variable under the integral, e.g.

$$\int_{x\in E} f(x)\,dx := \int_E f(x)\,dx.$$

- ν is the unit outward normal to the implied surface. (I am allergic to using n or $\hat{n}$.)
- We write $\operatorname{div} F$ for divergence and $\operatorname{curl} F$ for curl (instead of $\nabla \cdot F$ and $\nabla \times F$, respectively).
- $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the basis vectors of $\mathbb{R}^3$. For example $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k} = (3, 4, 5)$.

0.2 Analysis

- $B_n(x, r)$ is the n-dimensional ball centered at x with radius r. That is, $B_n(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$.
- For a set U, ∂U denotes the boundary of U.
- log is the natural log.

1 Day 1: Integral Spam

1.1 Area and Volume

- Usually you want to find area/volume of a region by slicing the region into lower-dimensional cross sections and integrating over the length/area of these sections.
- Consider using polar coordinates or other coordinate systems if such methods seem relevant.

Example 1.1 (Stolen from the GRE): Find the volume of the region bounded by $y = x^2$, $y = 2 - x^2$, $z = 0$, and $z = y + 3$.

Solution. This is

$$\begin{aligned}\int_{x=-1}^{1}\int_{y=x^2}^{2-x^2}\int_{z=0}^{y+3} 1\,dz\,dy\,dx &= \int_{x=-1}^{1}\int_{y=x^2}^{2-x^2} y+3\,dy\,dx \\ &= \int_{x=-1}^{1} \frac{1}{2}(2-x^2)^2 - \frac{1}{2}(x^2)^2 + 3(2-2x^2)\,dx \\ &= \int_{x=-1}^{1} 8 - 8x^2\,dx \\ &= 16 - \frac{16}{3} = \boxed{\frac{32}{3}}\end{aligned}$$

■

Example 1.2: Find the volume of the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, y^2 + z^2 < 1, x^2 + z^2 < 1\}.$$

Solution. By symmetry we can restrict to the octant $x, y, z > 0$ and then multiply the answer we get by 8. A dumb expression for the volume is given by

$$\frac{V}{8} = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 1_{x^2+y^2<1} 1_{y^2+z^2<1} 1_{x^2+z^2<1}\,dz\,dy\,dx,$$

where we use the "indicator function"

$$1_P := \begin{cases} 1, & P \text{ is true} \\ 0, & P \text{ is false} \end{cases}.$$

This sets up a seemingly bare-bones integral, but this ends up working pretty well if you're having trouble imagining what the set looks like.

$$\begin{aligned}\frac{V}{8} &= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\int_{z=0}^{1} 1_{y^2+z^2<1}1_{x^2+z^2<1}\,dz\,dy\,dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\int_{z=0}^{1} 1_{z<\sqrt{1-x^2}}1_{z<\sqrt{1-y^2}}\,dz\,dy\,dx \\ &= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\min(\sqrt{1-x^2},\sqrt{1-y^2})\,dy\,dx \\ &= \int_{x,y>0,x^2+y^2<1}\min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y).\end{aligned}$$

The min is ugly so we now use symmetry again by restricting to the region where $x < y$. This gives

$$\begin{aligned}\frac{V}{16} &= \int_{x,y>0,x^2+y^2<1}\min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y) \\ &= \int_{x,y>0,x<y,x^2+y^2<1}\min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y) \\ &= \int_{x,y>0,x<y,x^2+y^2<1}\sqrt{1-y^2}\,d(x,y) \\ &= \int_{y=0}^{\sqrt{2}/2}\int_{x=0}^{y}\sqrt{1-y^2}\,dx\,dy + \int_{y=\sqrt{2}/2}^{1}\int_{x=0}^{\sqrt{1-y^2}}\sqrt{1-y^2}\,dx\,dy \\ &= \int_{y=0}^{\sqrt{2}/2} y\sqrt{1-y^2}\,dy + \int_{y=\sqrt{2}/2}^{1} 1-y^2\,dy \\ &= \int_{u=1}^{1/2} -\tfrac{1}{2}\sqrt{u}\,du + \left(1-\frac{\sqrt{2}}{2}\right) - \frac{1}{3} + \frac{\sqrt{2}}{12} \\ &= \frac{1}{3} - \frac{(1/2)^{3/2}}{3} + \left(1-\frac{\sqrt{2}}{2}\right) - \frac{1}{3} + \frac{\sqrt{2}}{12} \\ &= \frac{1}{3} - \frac{\sqrt{2}}{12} + 1 - \frac{\sqrt{2}}{2} - \frac{1}{3} + \frac{\sqrt{2}}{12} \\ &= 1 - \frac{\sqrt{2}}{2}.\end{aligned}$$

So $\boxed{V = 8(2-\sqrt{2})}$. ■

1.2 Change of Variables

Let g be a smooth bijection, and let f be sufficiently nice. Then:

$$\begin{aligned}\int_{x\in E} f(x)\,dx \text{ ``} &= \int_{g(y)\in E} f(g(y))\,d(g(y))\,dy \text{ ''}\\ \text{``} &= \int_{g(y)\in E} f(g(y))\frac{d(g(y))}{dy}\,dy \text{ ''}\\ &= \int_{y\in g^{-1}(E)} f(g(y))|\det Dg(y)|\,dy\end{aligned}$$

1.3 Polar and Spherical Coordinates

1.3.1 Polar

The *polar coordinates* are given by the change of variables

$$g(r,\theta) = (r\cos\theta, r\sin\theta),$$

which we can think of as the substitutions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The determinant of the Jacobian is

$$\det Dg(r,\theta) = \begin{vmatrix} \frac{\partial g_1}{\partial r}(r,\theta) & \frac{\partial g_1}{\partial \theta}(r,\theta) \\ \frac{\partial g_2}{\partial r}(r,\theta) & \frac{\partial g_2}{\partial \theta}(r,\theta)\end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta\end{vmatrix}$$

$$= \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r,$$

and since $r > 0$ we simply have $|\det J_g(r,\theta)| = r$. Hence r is the "price" to pay in order to change to polar coordinates. Thus, for example, if $B_2(0,R)$ is the ball with radius R centered at $(0,0)$, then

$$\int_{B_r(0,R)} f(x,y)\,d(x,y) = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} f(r\cos\theta, r\sin\theta)\cdot r\,d\theta\,dr.$$

1.3.2 Cylindrical

The *cylindrical coordinates* are given by the change of variables

$$g(r,\theta,z) = (r\cos\theta, r\sin\theta, z),$$

which we can think of as the substitutions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

So cylindrical is actually pretty lame because it just tacks on a third dimension to polar coordinates. You can find that $|\det Dg(r,\theta,z)| = r$.

1.3.3 Spherical

The *spherical coordinates* are given by the change of variables

$$g(r,\theta,\phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \qquad 0 < \theta < \pi, 0 < \phi < 2\pi$$

which we can think of as the substitutions:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

You can show that $|\det Dg(r,\theta,\phi)| = r^2\sin\theta$. Note that $\sin\theta > 0$ for $0 < \theta < \pi$, so this indeed always non-negative.

I personally find this hard to remember. I highly remember **understanding where spherical coordinates come from** instead of trying to memorize them. That way, you can always rederive them when you need them.

1.4 Path Integrals

Summary:

- A (parametrization of a) path γ is given by $\varphi : [a,b] \to \mathbb{R}^n$.
- Line integral of a *function* $f : \mathbb{R}^n \to \mathbb{R}$ over γ is given by

$$\int_\gamma f\,ds := \int_a^b f(\varphi(t))\|\varphi'(t)\|\,dt.$$

  - Does not depend on the parametrization, φ.

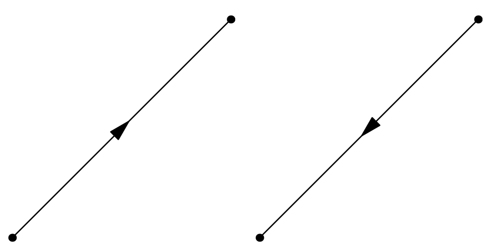
  - $\int_\gamma f\,ds$ does *not* depend on the orientation of γ. It's the same whether we go forward of backward.
  - Can think of $\int_\gamma f\,ds$ as "the area under f along the path γ", a picture best imagined for $f : \mathbb{R}^2 \to \mathbb{R}$.

- Line integral of a *vector field* $F : \mathbb{R}^n \to \mathbb{R}^n$ over γ is given by

$$\int_\gamma F \cdot ds := \int_a^b F(\varphi(t)) \cdot \varphi'(t)\,dt.$$

  - Does not depend on the parametrization either.
  - *Does* depend on the orientatiion of γ. In particular if $-\gamma$ is the path obtained by reversing the direction/orientation of γ, then $\int_{-\gamma} F \cdot ds = -\int_\gamma F \cdot ds$.
  - Can think of $\int_\gamma F \cdot ds$ as "how much F agrees with the velocity of γ as we travel along γ".
  - Other notations include:
    * $\int_\gamma F \cdot dr$
    * $\int_\gamma F$
    * In $\mathbb{R}^2$, $\int_\gamma M\,dx + N\,dy$, where $F = (M, N)$, i.e. M and N are the components of F.

- (For simplicity, we should assume that all functions and paths are smooth, but this assumption can be weakened to "Lipschitz".)



Two paths of the opposite orientation

Generally I'd say that line integrals of *vector fields* are more common and more useful, so I'd focus your energy on those.

Example 1.3: Let $f(x, y) = x^2 + y$. Let $F(x, y) = (xy, x - y)$. Let γ be the unit circle centered at $(0, 0)$ oriented counter-clockwise.

(a) Compute $\int_\gamma f\,ds$.

(b) Compute $\int_\gamma F \cdot ds$.

(c) Compute $\int_\gamma x\,dy$.

Solution. Let's use the parametrization $\varphi(t) = (\cos t, \sin t)$ over $t \in [0, 2\pi)$. Note that $\varphi'(t) = (-\sin t, \cos t)$, and that $\|\varphi'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$ for all t.

Part (a)

$$\int_\gamma f\, ds = \int_0^{2\pi} f(\cos t, \sin t)\|\varphi'(t)\|\, dt = \int_0^{2\pi} \cos^2 t + \sin t\, dt.$$

This is just $\int_0^{2\pi} \cos^2 t\, dt$ which is probably like π or something.

Part (b)

$$\int_\gamma F \cdot ds = \int_0^{2\pi} F(\cos t, \sin t) \cdot \varphi'(t)\, dt = \int_0^{2\pi} (\cos t \sin t, \cos t - \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t \cos t + \cos^2 t - \sin t \cos t\, dt.$$

This evaluates to something.

Part (c)

I don't like this notation so I like to think of this as

$$\int_\gamma x\, dy = \int_\gamma (0, x) \cdot (dx, dy) = \int_\gamma (0, x) \cdot ds.$$

So this is

$$= \int_0^{2\pi} (0, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} \cos^2 t\, dt,$$

which is probably like π or something. ■

1.5 Surface Integrals

Now we're going to integrate over higher-dimensional things! For simplicity we'll stick to 2-dimensional manifolds in $\mathbb{R}^3$. Recall that a 2-dimensional manifold is, roughly speaking, a set $M \subseteq \mathbb{R}^3$ which is smoothly parametrized by a function $\varphi : U \to M$, with $U \subseteq \mathbb{R}^2$. φ is called a *chart*.

Definition 1.1 (Surface Integral)

Let $f : \mathbb{R}^3 \to \mathbb{R}$. The *surface integral* of f over M is given by

$$\int_M f\, dS := \int_U f(\varphi(u,v))\sqrt{\det[D\varphi(u,v)^T D\varphi(u,v)]}\, du\, dv,$$

where $\varphi : U \to M$ is a chart for M.
Other notations include $\int_M f\, dA$, $\int_M f\, d\sigma$, $\int_M f\, d\Sigma$, and $\int_M f\, d\mathcal{H}^2$.

The $\sqrt{\det[D\varphi(u,v)^T D\varphi(u,v)]}$ term (the "Jacobian") is the nasty part. Some people write it as $|||\varphi(u,v)|||$. A useful result for taming it is the *Cauchy-Binet formula*. In 2-dimensions, this formula says that

$$\begin{aligned}\det\left[\begin{pmatrix} a & b & c \\ d & e & f\end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f\end{pmatrix}\right] &= \begin{vmatrix} a & b \\ d & e\end{vmatrix}^2 + \begin{vmatrix} a & c \\ d & f\end{vmatrix}^2 + \begin{vmatrix} b & c \\ e & f\end{vmatrix}^2 \\ &= (ae-bd)^2 + (af-cd)^2 + (bf-ce)^2 \\ &= \|(a,b,c)\times(d,e,f)\|^2.\end{aligned}$$

So it's common to define/write

$$\int_M f\, dS := \int_U f(\varphi(u,v))\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\| du\, dv.$$

This makes some visual sense in that $\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\|$ is measuring the area of a parallelogram with "sides" $\frac{\partial\varphi}{\partial u}(u,v)$ and $\frac{\partial\varphi}{\partial v}(u,v)$, and is hence a "differential surface element" that in some sense "measures how curvy/distorted M is at (u,v)".

1.5.1 Integrating Over a Sphere

For a suitably decent function f, let us find a formula for the surface integral

$$\int_{\partial B(0,R)} f\, dS,$$

where $\partial B(0,R)$ is the surface of the ball of radius R centered at the origin. We can chart out this surface in coordinates via

$$\varphi(\phi,\theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi), \qquad 0<\phi<\pi, 0<\theta<2\pi.$$

We find that

$$D\varphi(\phi,\theta) = \begin{pmatrix} R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ -R\sin\phi & 0\end{pmatrix},$$

and so

$$|||D\varphi(\phi,\theta)||| = \sqrt{\begin{matrix}(R^2\sin\phi\cos\phi\cos^2\theta + R^2\sin\phi\cos\phi\sin^2\theta)^2 \\ +(R^2\sin\theta\sin^2\phi)^2 \\ +(-R^2\cos\theta\sin^2\phi)^2\end{matrix}}$$
$$= R^2\sqrt{\sin^2\phi\cos^2\phi + \sin^4\phi}$$
$$= R^2|\sin\phi| = R^2\sin\phi.$$

(Sanity check: Why does R^2 make sense, vis-a-vis, say, R or R^3?)

Thus

$$\int_{\partial B(0,R)} f\,dS = \int_0^{2\pi}\int_0^{\pi} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\cdot R^2\sin\phi\,d\phi\,d\theta.$$

(Normal people probably switch θ and ϕ around but I honestly don't care lmao i just rederive this every time)

1.5.2 Other Surfaces

Some manifolds, like the boundary of a cylinder, can't really be parametrized with a single chart. In that case, you divide the manifold into surfaces that can be (easily) parametrized.

Example 1.4: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x,y,z) = x^2 + yz$. Compute the surface integral $\int_C f\,dS$, where C is the curved surface of the upside-down circular cone with base $B_2(0,1)\times\{1\}$ and vertex $(0,0,0)$.

Solution. Let's parameterize C with the chart $\varphi : (0,1)\times(0,2\pi)$ given by $\varphi(r,\theta) := (r\cos\theta, r\sin\theta, r)$. Then

$$\int_C f\,dS = \int_0^1\int_0^{2\pi} f(r\cos\theta, r\sin\theta, r)|||\varphi(r,\theta)|||\,d\theta\,dr.$$

We now compute the Jacobian $|||\varphi(r,\theta)|||$. First we note that

$$D\varphi(r,\theta) = \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 1 & 0\end{pmatrix},$$

so by the Cauchy-Binet formula we see that

$$\det(D\varphi(y)^T D\varphi(y)) = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}^2 + \begin{vmatrix}\cos\theta & -r\sin\theta \\ 1 & 0\end{vmatrix}^2 + \begin{vmatrix}\sin\theta & r\cos\theta \\ 1 & 0\end{vmatrix}^2$$
$$= r^2 + r^2\sin^2\theta + r^2\cos^2\theta = 2r^2,$$

thus

$$|||\varphi(r,\theta)||| = \sqrt{\det(D\varphi(y)^T D\varphi(y))} = \sqrt{2}r.$$

It follows that

$$\int_C f\,dS = \int_0^1 \int_0^{2\pi} (r^2\cos^2\theta + r^2\sin\theta)\cdot\sqrt{2}r\,d\theta\,dr = \sqrt{2}\int_0^1 \pi r^3\,dr = \boxed{\frac{\sqrt{2}\pi}{4}}.$$

■

2 Day 2: Vector fields, Stokes, and Series

2.1 Vector Fields

For curious analysts: All instances of "smooth" in this section can be replaced with "Lipschitz".

Basic Facts:

- A *vector field* is a function that looks like $F : E^{(\subseteq \mathbb{R}^n)} \to \mathbb{R}^n$.
- F is *conservative* if there exists a *potential function*, $f : \mathbb{R}^n \to \mathbb{R}$, for which
$$\nabla f = F.$$
- f is a sort of "anti-derivative" for F, and hence satisfies the following version of the FTC: For $a, b \in \mathbb{R}^n$, we have
$$\int_C F \cdot dr = f(b) - f(a)$$
for any path C from a to b.

Example 2.1 (Winter 2017 #3): Consider the integral

$$I = \int_\Gamma \frac{x}{x^2 + y^2}\, dx + y\frac{1 - x^2 - y^2}{x^2 + y^2}\, dy$$

integrated over a path Γ.

(a) Show that I does not depend on the path Γ chosen to connect two fixed points.

(b) Compute I if Γ is a path joining $A = (0, 1)$ to $B = (1, 1)$.

2.1.1 Divergence

Facts:

- The *divergence* of a vector field F is a function $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}$ given by
$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i}.$$

- In 3D, this is
$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$
- Intuitively, $\operatorname{div} F(x, y, z)$ quantifies how much "F expands space at (x, y, z)".
- $\operatorname{div} F$ is often written as $\nabla \cdot F$.
- (*Bonus: This is not a contrived quantity, in fact it is actually quite "natural" because, surprisingly, divergence is independent of the (orthonormal) coordinate system chosen.*)

The following is the fundamental fact that pretty much explains why divergence is so important.

Theorem 2.1 (Divergence Theorem)

Let $U \subseteq \mathbb{R}^3$ be a bounded open set with smooth boundary. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then
$$\int_U \operatorname{div} F \, d(x, y, z) = \int_{\partial U} F \cdot \nu \, dS,$$
where for a point $p \in \partial U$, $\nu(p)$ denotes the *unit outward normal* to ∂U at p.

Notes:

- The motto for the Divergence Theorem is basically this: "If you have a liquid in a container, then the total pressure inside the liquid is equal to the total force the liquid exerts on the container." In layman's terms, "if thing go in then thing go out".
- **When to use:** This theorem gives you a great way to convert some nasty surface integrals to "normal" integrals. So if you hate surface integrals, the Divergence Theorem can save you.
- The surface integral $\int_{\partial U} F \cdot \nu \, dS$ may be phrased as "flux".
- (Bonus) By applying the Divergence Theorem, you can deduce the following variants of "integration by parts":
  - $\displaystyle\int_\Omega f \operatorname{div} G \, dx = \int_{\partial\Omega} f(G \cdot \nu) \, dS - \int_\Omega \nabla f \cdot G \, dx$
  - $\displaystyle\int_\Omega F \cdot \nabla g \, dx = \int_{\partial\Omega} (F \cdot \nu) g \, dS - \int_\Omega (\operatorname{div} F) g \, dx$ (literally the same as the previous one)

  It does not seem like these will appear on your exam, but being familiar with these is quite crucial for studying PDE.

2.1.2 Curl

The "amount" that a vector field "twists" space is a surprisingly revealing quantity.

- In $\mathbb{R}^2$, the *curl* of F is a function $\operatorname{curl} F : \mathbb{R}^2 \to \mathbb{R}$, with

$$\operatorname{curl} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

  How to remember the order:

  - *Method 1:* You *differentiate* a *function*, so the first column is *derivatives* and the second column is the *functions* in the "determinant"

$$\operatorname{curl} F = \begin{vmatrix} \partial_x & F_1 \\ \partial_y & F_2 \end{vmatrix}.$$

  - *Method 2:* The prototypical "twisty" vector field to test against is $(-y, x)$. If you sketch this you'll find that this "twists counter-clockwise", so you should expect that $\operatorname{curl}(-y, x) > 0$.

- In $\mathbb{R}^3$, the *curl* of F is a vector field $\operatorname{curl} F : \mathbb{R}^3 \to \mathbb{R}^3$, with

$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

  How to remember the order:

  - *Method 1:* Same mnemonic as in $\mathbb{R}^2$, but you attach a third column with the basis vectors as in

$$\operatorname{curl} F = \begin{vmatrix} \partial_x & F_1 & \hat{i} \\ \partial_y & F_2 & \hat{j} \\ \partial_z & F_3 & \hat{k} \end{vmatrix}.$$

  - *Method 2:* The above determinant is basically the "cross product" $\nabla \times F$, so if you know how to take a cross product then you know how to compute 3D curl.
  - *Method 3:* Test against the vector field $(-y, x, 0)$. You should expect that the components of $\operatorname{curl}(-y, x, 0)$ are non-negative.

A first important result regarding curl:

Theorem 2.2 (Green's Theorem)

Let $U \subseteq \mathbb{R}^2$ be a connected open set with (piecewise) smooth boundary. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field. Then

$$\int_U \operatorname{curl} F \, d(x,y) = \int_{\partial U} F \cdot ds,$$

where ∂U is viewed as a path oriented counter-clockwise, and

$$\operatorname{curl} F := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Notes:

- This follows from the Divergence Theorem (why?).
- This is also an instance of Stoke's Theorem (why?).
- By applying Green's to certain vector fields (namely $(0, x)$, $(-y, 0)$, and $(\frac{-y}{2}, \frac{x}{2})$), you can find the area of any region by just examining its boundary.
- **When to use:**
  - Can be useful for turning line integrals into possibly "nicer" area integrals, especially when the curl of F is a nice quantity.
  - Is useful for finding the area of a region whose *boundary* is easy to parametrize, whereas the region itself is hard to parametrize.

Theorem 2.3 (Stoke's Theorem)

Let $M \subseteq \mathbb{R}^3$ be a smooth 2d manifold with smooth boundary ∂M. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\int_M \operatorname{curl} F \cdot \nu \, dS = \int_{\partial M} F \cdot ds,$$

where the orientation of ∂M and the choice of ν are selected so that if ν is pointing "up", then ∂M is travelling "counter-clockwise" when viewed from above.

Notes:

- "Intuition:" https://www.smbc-comics.com/comic/2014-02-24
- **When to use:** Often this is used to turn nasty line integrals into "nice" surface integrals, which are usually only "nice" when the curl ends up being a really simple quantity like 0.

2.2 Examples with Divergence, Green's, Stoke's

I am going to use the same notation as the exam. Use context to interpret the problems!

Example 2.2 (Fall 2007 #2): Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{F}(x, y, z) = (2x - y^2 - x^3, 3y - y^3, -x - z^3)$.
For a closed surface S in $\mathbb{R}^3$, consider $\int_S \mathbf{F}.\mathbf{n}dA$, the flux of $\mathbf{F}$ through S. Here $\mathbf{n}$ is chosen to be an outward normal. For what choice of S will $\int_S \mathbf{F}.\mathbf{n}dA$ be *maximal*? Explain your answer and compute $\int_S \mathbf{F}.\mathbf{n}dA$ in that case.

Example 2.3 (Fall 2008 #4): An object moves in the force field $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ starting at the origin and ending at some point $A(\xi, \eta, \zeta)$ that lies on the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. What is the maximum possible value of the work done $W = \oint \vec{F} \cdot d\vec{r}$?

Example 2.4 (Winter 2009 #2): Compute

$$\oint_L (y - z)dx + (z - x)dy + (x - y)dz$$

where L is the curve given by the intersection of the two surfaces

$$\begin{cases} x^2 + y^2 + z^2 = a^2, a > 0 \\ x + y + z = 0 \end{cases}$$

with counterclockwise orientation viewed from the positive x-axis.

Example 2.5 (Winter 2018 #1b): Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field such that, for every continuously differentiable function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ with compact support,

$$\int_{\mathbb{R}^3} \mathbf{f}(x) \cdot \nabla\varphi(x)\, dx = 0.$$

Show that the divergence of $\mathbf{f}$ is zero.

Example 2.6 (Winter 2019 #4): Let C be the closed curve formed by the intersection of the surface $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2\}$ with the plane $z = \frac{\sqrt{3}}{2} y$. Choose an orientation for C and compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for the vector field $\mathbf{F}(x, y, z) := (x^2 + z, \sin y, \cos z)$.

Example 2.7 (Fall 2021 #3):

(a) [some nonsense that i don't want you worrying about]

(b) Let

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}, \qquad \mathbf{F} = \nabla f.$$

What is the flux of $\mathbf{F}$ through the surface of the unit sphere?

(c) Lastly, consiedr the vector fields:

$$\mathbf{G} = -r \sin \varphi \mathbf{i} + r \cos \varphi \mathbf{j} + \mathbf{k},$$

$$\mathbf{H} = \nabla \times \mathbf{G},$$

where (r, θ, φ) are the usual spherical coordinates for a point in $\mathbb{R}^3$:

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arccos \frac{z}{r}, \qquad \varphi = \arctan \frac{y}{x}.$$

Compute the integral

$$\beta = \iint_S \mathbf{H} \cdot d\mathbf{A},$$

where S is the top half of the unit sphere, i.e.

$$S = \{x, y, z : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}.$$

Example 2.8 (I made this one up): Let $F(x, y) = (y^3, x - x^3)$. Find the maximum possible value of

$$\int_C F \cdot ds$$

over all simple closed curves C that are oriented counter-clockwise.

Solution. Let U be the region enclosed by C. Then by Green's Theorem,

$$\int_C F \cdot ds = \int_U \operatorname{curl} F \, d(x, y) = \int_U 1 - 3x^2 - 3y^2 \, d(x, y).$$

Evidently this is maximized by selecting the maximal region U for which the integrand $1 - 3x^2 - 3y^2$ is non-negative. After some musing, you can find that $U = B_2(0, 1/\sqrt{3})$ is a great choice. The maximum value in this case is

$$\int_{B_2(0,1/\sqrt{3}} 1 - 3x^2 - 3y^2 \, d(x, y) = 2\pi \int_0^{1/\sqrt{3}} (1 - 3r^2) r \, dr$$

by polar coordinates. This evaluates to something. ■

Example 2.9 (I also made this one up): Let

$$H := \{(x, y, z) : x^2 + y^2 + z^2 = 1, z > 1\}$$

be the upper half of the surface of the unit sphere. Compute

$$\int_H x^4 + y^4 + z^4 \, dS.$$

Solution.

$$\int_H x^4 + y^4 + z^4 \, dS = \int_H (x^3, y^3, z^3) \cdot \nu \, dS$$

$$= -\int_{\{x^2+y^2<1\}} (x^3, y^3, 0) \cdot (0, 0, -1) \, dS + \int_{x^2+y^2+z^2<1, z>0} 3x^2 + 3y^2 + 3z^2 \, d(x, y, z)$$

$$= 3 \int_{\phi=0}^{\pi/2} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (r^2 \sin^2 \phi \cos^2 \theta + r^2 \sin^2 \phi \sin^2 \theta + r^2 \cos^2 \phi) r^2 \sin \phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \sin \phi \, d\theta \, d\phi = 2\pi$$

■

2.3 Series Convergence

2.3.1 Basics

Theorem 2.4 (Stupid Test)

If $\lim_{n\to\infty} |a_n| \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

Theorem 2.5 (p-Test et. al.)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
$\sum_{n=1}^{\infty} a^n$ converges for $|a| < 1$ and diverges for $|a| \geq 1$.

Theorem 2.6 (Direct Comparison Test)

Let $a_n \geq 0$ be a sequence. If you can find a b_n that *eventually dominates* a_n (i.e. there is N such that $b_n \geq a_n$ for all $n \geq N$), such that $\sum_{n=1}^{\infty} b_n < \infty$, then $\sum_{n=1}^{\infty} a_n < \infty$ i.e. converges.
Similarly, if instead you found a b_n for which eventually $0 \leq b_n \leq a_n$ forever, with $\sum_{n=1}^{\infty} b_n = +\infty$, then $\sum_{n=1}^{\infty} a_n = +\infty$ i.e. diverges.

We're going to be implicitly invoking direct comparison quite a lot. I'll explain why this test is so useful/underrated in a bit.

Theorem 2.7 (Limit Comparison Test)

If two sequences are close together, they behave the same way. That is, if $a_n \geq 0$ and $b > 0$ are two sequences for which $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges (so if either converges then the other converges, and if either diverges then the other diverges!).

Note: Most sources instead write "$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L \in (0, \infty)$", but I think this muddies the "best" way to think about using this test.

Limit comparison can be used to "clean up" junk and simplify a series. For example, if we are aiming to ascertain convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1},$$

then limit comparison says we can multiply by a term with limit 1 and the convergence behavior does not change. For instance we can multiply by $\frac{n^3+1}{n^3}$ to "convert" the series into

$$\sum_{n=1}^{\infty} \frac{n}{n^3},$$

which is far easier to reason with. Effectively, what we have done is erased the insignificant "1" term in $n^3 + 1$ and replaced it with n^3. If this is something you wish to do, consider limit comparison.

2.3.2 Positive and Negative Terms

Theorem 2.8 (Alternating Series Test)

If $a_n \geq 0$ is **monotone decreasing** and **tends to** 0, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Theorem 2.9 (Absolute Convergence)

If $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\sum_{n=1}^{\infty} a_n$ converges.

2.3.3 Geometric-y Tests

The classic example of a convergent series is the geometric series,

$$\sum_{n-1}^{\infty} r^n,$$

where $|r| < 1$. Intuitively, if we can ascertain whether a series "decays faster" than geometric, then it should converge. Capitalizing on this idea, we can notice that there are two ways to characterize the geometric sequence $a_n = r^n$:

- The *ratio* between terms, a_{n+1}/a_n, is r with $|r| < 1$.
- The nth *root* of each term, $\sqrt[n]{a_n}$, is r with $|r| < 1$.

So for an arbitrary sequence a_n, it is fairly intuitive that if we do *better* than either of the above properties, then we have convergence.

Please feel free to replace $\limsup$ *with* $\lim$ *if you are uncomfortable with* $\limsup$*.*

- If $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$, then this means that over the tail end of the sequence, the ratio between terms is better than geometric, so it converges.

- If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then this again means that over the tail end of the sequence, the nth root of terms is better than geometric, so it converges.

Of course, these observations are simply the ratio test and root test.

Theorem 2.10 (Ratio Test)

Consider the limit of the ratio between successive terms, $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$, if it exists.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- (If $L = 1$, you know nothing.)

Theorem 2.11 (Root Test)

Consider $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, if it exists.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- (If $L = 1$, you know nothing.)

Remark 1: The hyper-analysis-savvy reader would be delighted to know that you can replace lim with lim sup.

Remark 2: You can use the root test to prove the ratio test.

2.3.4 Integral Test

If you only care about convergence, then numerous sums can be replaced by integrals.

Theorem 2.12 (Integral Test)

In the series $\sum_{n=1}^{\infty} f(n)$, You can replace the sum with an integral and nothing changes convergence-wise (as long as f is non-negative and monotone decreasing).

This is quite niche but if it just so happens that the summand "looks like something that you can integrate", consider this test.

2.3.5 (Semi-Optional) Summation by Parts and the Dirichlet Test

You should also know the Dirichlet test since that actually shows up in the writtens sometimes. But the Dirichlet test can be hard to remember, so I will instead frame it as a specific instance of a well-motivated general method.

First, recall the *integration by parts:* If $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x g(t)\,dt$, then

$$\begin{aligned}\int_0^\infty f(x)G(x)\,dx &= F(\infty)G(\infty) - F(0)G(0) - \int_0^\infty F(x)g(x)\,dx \\ &= F(\infty)G(\infty) - \int_0^\infty F(x)g(x)\,dx.\end{aligned}$$

(Here, $F(\infty) := \lim_{x\to\infty} F(x) = \int_0^\infty f(x)\,dx$ and similarly for $G(\infty)$.) In this way we may thus "move an (anti-)derivative" from one factor to the other, and hence converting fG to Fg.

Amazingly there is a discrete version of this formula, called *summation by parts.* My thesis here is that you can essentially derive it by simply mimicking integration by parts: If $A_n = \sum_{k=1}^n a_k$ and $B_n = \sum_{k=1}^n b_k$, then

$$\sum_{n=1}^{\infty} a_n B_n \overset{?}{=} A_\infty B_\infty - \sum_{n=1}^{\infty} A_n b_n.$$

Unfortunately this isn't *exactly* true, the indices are slightly wrong or something. But I don't give a fuck and neither should you. What is *definitely* true, and what is ultimately important here, is that one side converges iff the other side converges. Thus the motto: **for determining convergence, we may freely "move" a partial sum from one factor to the other.** To be more precise:

Fact: *With the above notation, assuming that "$A_\infty B_\infty$" $:= \lim_{n\to\infty} A_n B_n = 0$, we have that*

$$\sum_{n=1}^{\infty} a_n B_n \text{ converges} \iff \sum_{n=1}^{\infty} A_n b_n \text{ converges.}$$

A prototypical example that we may attack with this approach is the series $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$. We choose the "$a_n$" sequence to be a factor whose partial sums are somehow nice. It turns out that taking $a_n = \sin n$ is a great choice, because amazingly the sequence of partial sums $A_n = \sum_{k=1}^{n} \sin k$ is bounded (why?).

That leaves us with taking the "B_n" sequence to be $B_n = \frac{1}{\sqrt{n}}$. But what is b_n? You can find that the only way to have $B_n = \sum_{k=1}^{n} b_k$ is if we choose $b_1 = 1$ and $b_n = \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$ for $n > 1$. Convince yourself that this works: We "integrate" the $\sin n$ via a partial sum, and "differentiate" the $1/\sqrt{n}$ via differences.

Our bet is that the "integral" of $\sin n$ is tame enough, and in return we expect that $\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$ is somehow much nicer to work with. And this is true!

The summation by parts method tells us that provided that $A_n B_n \to 0$ (and it does, because A_n is bounded and $B_n \to 0$), we have that $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$ converges iff

$$\sum_{n=1}^{\infty} A_n b_n$$

converges. But A_n is bounded with some upper bound $M > 0$, and so

$$\left| \sum_{n=1}^{\infty} A_n b_n \right| \leq \sum_{n=1}^{\infty} M \cdot |b_n| = M \sum_{n=1}^{\infty} b_n = M \cdot B_\infty < \infty,$$

so we have convergence!

If you review what properties were important in this argument, you'll find that $\sin n$ can be replaced with *any* sequence with *bounded partial sums*, and $\frac{1}{\sqrt{n}}$ can be replaced with *any* sequence that *is positive and decreases to 0*. This is Dirichlet's Test.

Theorem 2.13 (Dirichlet Test)

Suppose that:

- a_n is a sequence with bounded partial sums, i.e. $|\sum_{k=1}^{n} a_k| \leq M$ for all n, for some large $M > 0$, and
- $b_n \geq 0$ be a decreasing monotone sequence that tends to 0.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

If you decide to practice this, seek problems from past exams which seem like they are begging for the Dirichlet test (there are at least two such problems!), and try to solve them *without looking at the above theorem*, because the point of this exposition is to get you to either derive it or bypass its necessity on the spot if it happens that you require this test. All you need to remember is the prototypical example, and that **it's ultimately just integration by parts**.

2.4 Convergence Test Tier List and Minor Examples

SS	Direct Comparison
S	Stupid Test, Limit Comparison
A	Alternating Series, Absolute Convergence
B	Ratio, Root, Integral
C	Dirichlet

- A-tier and above are by far the most useful. Direct comparison in particular is extremely underrated and people should abuse it much more often.
- Ratio and Root test are mid but can be useful. Integral test is good in specific instances.
- Dirichlet test is rare.
- I know I said that Limit Comparison was F tier during the actual workshop, but I think if presented "correctly" it can have S-tier usefulness for simplifying series.

Now I'll explain why I say that Direct Comparison is SS-tier. Often you can use it to erase certain terms completely: Consider

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}.$$

As noted earlier, this is a prototypical use-case for Limit Comparison. However, we can actually just use Direct Comparison by noting that $\frac{1}{n^3+1} \leq \frac{1}{n^3}$, and so

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which clearly converges.

This exact approach fails for a series such as

$$\sum_{n=2}^{\infty} \frac{n}{n^3-1},$$

in which case one can argue that Limit Comparison is now easier by "multiplying by $\frac{n^3-1}{n^3}$". But actually Direct Comparison can still be used in the following way: We can note that $\frac{1}{n^3-1} \leq \frac{1}{n^3/2}$ for *all n large enough*; say, $n \geq 10^{100}$. Then

$$\sum_{n=10^{100}}^{\infty} \frac{n}{n^3-1} \leq \sum_{n=10^{100}}^{\infty} \frac{n}{n^3}$$

which converges, thus the original sum converges because **the first one billion terms do not matter**! A finite number of terms, no matter how large, can never affect whether the sum converges!

Here's an example where Direct Comparison trumps Limit Comparison. For the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}},$$

it's tough to get a handle on the asymptotic nature of $2^{\sqrt{n}}$, so Limit Comparison isn't very usable. However, using the philosophy of throwing away the first 1 trillion terms, we may use Direct Comparison by arguing that

$$2^{\sqrt{n}} \geq n^2$$

for all $n \geq 10^{9999999}$ (why?). Hence

$$\sum_{n=10^{9999999}}^{\infty} \frac{1}{2^{\sqrt{n}}} \leq \sum_{n=10^{9999999}}^{\infty} \frac{1}{n^2} < \infty,$$

so the original series converges because the first 1 quadrillion terms do not matter.

2.5 Examples with Series: Convergence Tests

Example 2.10 (Winter 2008 #1): Find all the values $p \in \mathbb{R}$ such that the following series converges:

$$\sum_{k=2}^{\infty} (\log k)^{p \log k}$$

Example 2.11 (Fall 2008 #2): For each of the following, find the range of $x \in \mathbb{R}$ for which the series converges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{n^2+x^{2n}}$

Example 2.12 (Winter 2011 #3): Find the range of the parameter for which the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}\left(\frac{1-2x}{1+x}\right)^n$

(ii) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$

Example 2.13 (Winter 2017 #1): Consider the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} x^n.$$

Determine the radius of convergence R of the series. Determine whether the series conerges or diverges for $x = R$ and $x = -R$.

Example 2.14 (Fall 2019 #4): Let $A_n = \frac{a_1}{1+a_1} + \frac{a_2}{\sqrt{2}+a_2} + \cdots + \frac{a_n}{\sqrt{n}+a_n}$, $a_n > \frac{1}{\sqrt{n}}$, and consider the series,

$$\sum_{n=1}^{\infty} \frac{\cos(\sqrt{3}n + \frac{\pi}{3})}{A_n}.$$

Show that the series is convergent.

2.6 Example with Series: Bare Hands

Sometimes there simply are no big guns you can use to nuke problems. You'll have to tackle these series with nought but your bare hands.

Example 2.15 (Fall 2007 #5b): Does the series $\sum_{n\in S} \frac{1}{n}$ converge, where S consists of those positive integers whose decimal expansion does not contain the digit 1?

Example 2.16 (Winter 2018 #5): Show that the limit

$$\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n}\sin(\log n)$$

does not exist.

3 Day 3: Big Guns

3.1 Convergence of Functions

There are numerous notions of function convergence. For the writtens you should know two: pointwise convergence and uniform convergence.

Definition 3.1 (Pointwise Convergence)

A sequence of functions f_n converges *pointwise* to f if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all x.

Definition 3.2 (Uniform Convergence)

A sequence of functions f_n converges *uniformly* to f if

$$\lim_{n\to\infty} \sup_x |f_n(x) - f(x)| = 0.$$

The pointwise/uniform convergence of *series* are defined in terms of the pointwise/uniform convergence of their *partial sums*.

How to think about uniform convergence:

- To show that $f_n \to f$ uniformly, you want to find a really nice upper bound on $|f_n(x) - f(x)|$ that *doesn't have an x in it.* If the upper bound you find goes to 0 as $n \to \infty$, you win.

$$f_n \to f \text{ uniformly } \iff |f_n(x) - f(x)| \le M_n \text{ with } M_n \xrightarrow{n\to\infty} 0$$

- To show that $\sum_{n=1}^{\infty} f_n$ converges uniformly, you want to find a really nice upper bound on $|\sum_{n=1}^{\infty} f_n(x)|$ that does not depend on x. Though in practice, this is likely a pain to show, so you'd rather use the *Weierstrass M-test.*

Prototypical Example 1: $\frac{\sin x}{n} \to 0$ uniformly (in x), because

$$\left|\frac{\sin x}{n}\right| \le \frac{1}{n},$$

and $1/n$ is an upper bound that has no x in it, and moreover $1/n \to 0$.

M**-Test:** For uniform convergence of *series* in particular, there is a very nice test you can use: If you can find a nice upper bound on $|f_n(x)|$ that *does not involve* x, and

$$\sum_{n=1}^{\infty}(\text{Upper bound on } |f_n|) < \infty,$$

then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly!

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly } \Longleftarrow |f_n(x)| \leq M_n \text{ such that } \sum_{n=1}^{\infty} M_n \text{ converges}$$

Prototypical Example 2: $\sum_{n=1}^{\infty} \frac{\sin x}{n^2}$ converges uniformly, because we have the upper bound

$$\left|\frac{\sin x}{n^2}\right| \leq \frac{1}{n^2},$$

whose sum converges.

IMPORTANT RESULT: A major reason for why we care about uniform convergence is that it *preserves continuity*!

- If f_n is continuous and $f_n \to f$ uniformly, then f is continuous.
- If f_n is continuous and $\sum_n f_n$ converges uniformly, then $\sum_n f_n$ is continuous.

Hence, establising uniform convergence is an excellent method for studying the regularity of some limit or some infinite sum.

Now some examples from the actual writtens.

Example 3.1 (Winter 2021 #2): Define, for each $n \in \mathbb{N}$,

$$u_n(x) = \frac{x}{n^2 + x^2}, \qquad (x \geq 0).$$

(a) Show that $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[0, K]$ for every $K > 0$.

(b) Determine whether $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[0, \infty)$.

Example 3.2 (Fall 2019 #3): For $\alpha \geq 1$, the sequence of functions $\{f_n\}$ is defined by

$$f_n(x) = x^{\alpha} \ln\left(x + \frac{1}{n}\right), \qquad x \in (0, \infty).$$

Show that (a) $\{f_n\}$ is uniformly convergent when $\alpha = 1$; (b) $\{f_n\}$ is not uniformly convergent when $\alpha > 1$.

Solution. **Part (a)**

We claim $f_n \to f$ where $f(x) = x \ln x$. Indeed,

$$|f_n(x) - f(x)| = x\ln(1 + 1/(nx)) \le x \cdot 1/(nx) = 1/n \to 0,$$

where we have applied the (famous?) inequality $\ln(1+x) \le x$.

Part (b)

(Not done in workshop) If f_n converges uniformly, then it must converge uniformly to the pointwise limit, which is $f(x) = x^\alpha \ln(x + 1/n)$. So to *disprove* that $f_n \to f$ uniformly, we want to show that

$$|f_n(x) - f(x)| = x^\alpha \ln\left(1 + \frac{1}{nx}\right)$$

does *not* tend to 0 uniformly. That is,

$$\sup_{x>0} x^\alpha \ln\left(1 + \frac{1}{nx}\right) \not\to 0.$$

Indeed, we claim that the sup is always $+\infty$ for every fixed n. To prove this, we can show that

$$\lim_{x\to+\infty} x^\alpha \ln\left(1 + \frac{1}{nx}\right) = +\infty.$$

We write

$$x^\alpha \ln\left(1 + \frac{1}{nx}\right) = x^{\alpha-1} \cdot x \ln\left(1 + \frac{1}{nx}\right).$$

Now, on one hand,

$$\lim_{x\to\infty} x \ln\left(1 + \frac{1}{nx}\right) = \frac{1}{n} \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{nx}\right) - \ln(1+0)}{\frac{1}{nx} - 0} = \frac{1}{n} \frac{d}{dt} \ln(1+t)\bigg|_{t=0} = 1/n.$$

On the other hand, $x^{\alpha-1} \to +\infty$. This completes the proof. ■

Example 3.3: Consider the series

$$\sum_{n=1}^{\infty} nxe^{-nx^2}$$

for $x \in (0, \infty)$.

(a) Find all intervals $(a, b) \subseteq (0, \infty)$ over which the series converges pointwise.

(a) Find all intervals $(a, b) \subseteq (0, \infty)$ over which the series converges uniformly.

3.2 Swapping

Several questions on the writtens require you to do things like swap a limit and an integral, so we need to cover this. The theorems I list here are in decreasing order of importance, and this list is far from exhaustive.

3.2.1 Swap Limit and Integral

This is so important that the average measure theory class dedicates a ton of time to this. Unfortunately we don't have that luxury so here's a "cheap" result that should be good enough for the exam.

Theorem 3.1 (Swap Limit and Integral)

Suppose a sequence of functions f_n on a bounded interval $[a, b]$ converges *uniformly* to some f. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. We have

$$\left|\int_a^b f_n(x) - f(x)\,dx\right| \le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \sup_{[a,b]} |f_n - f|\,dx = (b-a)\sup_{[a,b]}|f_n - f| \xrightarrow{n\to\infty} 0.$$

□

Comments:

- There are much better results like Dominated Convergence but I won't make you remember them. If you do know them, you should feel free to use them on the exam. I think showing off your knowledge of measure theory cannot possibly hurt your score.
- (Technically we need to assume that the functions involved are nice enough for their integral to "make sense", but whatever.)

Uniform convergence tends to be the "magic wand" that allows you to do most swaps. For swapping a limit and integral in particular, there is another criterion you can use.

Theorem 3.2 (Bounded convergence theorem)

Suppose f_n is a sequence of functions on $[a,b]$ with $f_n \to f$ pointwise, and that there is a common upper bound M for all the f_n's, i.e. $|f_n(x)| \leq M$ for all n and for all x. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

3.2.2 Swap Derivative and Integral

A derivative is just a limit! So you can use the above results. In particular, you can show (using the Mean Value Theorem) that if $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq M$ for all x and t, then

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx.$$

Example 3.4 (Winter 2021 #1): Define

$$f(x) = \int_0^{\pi/2} \frac{\cos t}{t+x}\,dt, \qquad (x > 0).$$

(a) Show that $f \in C^1((0,\infty))$

(b) Show that

$$\lim_{x\to 0} \frac{f(x)}{|\log x|} = 1.$$

Solution. (Not done in workshop)

Method 1: Using uniform convergence

I'll just do Part (a). First we show f is differentiable. Fix $x_0 > 0$. We show $f'(x_0)$ exists. It suffices to show that the limit of

$$\frac{f(x) - f(x_0)}{x - x_0} = \int_0^{\pi/2} \cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0} \right) dt$$

exists as $x \to x_0$. In particular we claim that the limit is $\int_0^{\pi/2} \cos t \cdot \frac{-1}{(t+x_0)^2}\,dt$.

By the "swap integral and limit" theorem, it is sufficient to show that the integrand $\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right)$ converges uniformly (in t!) to $\cos t \cdot \frac{-1}{(t+x_0)^2}$. That is, we ultimately aim to obtain an inequality of the form

$$\left|\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0}\right) - \cos t \cdot \frac{-1}{(t+x_0)^2}\right| \leq M_x$$

where M_x has no t and $M_x \xrightarrow{x \to x_0} 0$.

Firstly, we can use the silly bound $|\cos t| \leq 1$ to get

$$\left|\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0}\right) - \cos t \cdot \frac{-1}{(t+x_0)^2}\right| \leq \left|\left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0}\right) - \frac{-1}{(t+x_0)^2}\right|.$$

Next, we simplify the different quotient using the Mean Value Theorem, to get that

$$\left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0}\right) = \frac{-1}{(t+y_x)^2}$$

for some y_x between x_0 and x.

Now, we do some algebra to find that

$$\left|\frac{-1}{(t+y_x)^2} - \frac{-1}{(t+x_0)^2}\right| = \left|\frac{(t+y_x)^2 - (t+x_0)^2}{(t+y_x)^2(t+x_0)^2}\right| = \frac{|2t(y_x - x_0) + y_x^2 - x_0^2|}{(t+y_x)^2(t+x_0)^2}.$$

We do some more silly bounding in order to simplify and clean this up (ultimately we need to eliminate all the t's). We find that

$$\leq \frac{2t|y_x - x_0| + |y_x - x_0|(y_x - x_0)}{y_x^2 x_0^2} \leq \frac{\pi|y_x - x_0| + |y_x - x_0|(y_x - x_0)}{y_x^2 x_0^2} \to 0$$

as $x \to x_0$ (because $y_x \to x_0$), so the difference quotient $\left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0}\right)$ converges uniformly (in t!) to the derivative $\frac{-1}{(t+x_0)^2}$ (Important: The convergence is uniform in t because the upper bound $\frac{\pi|y_x - x_0| + |y_x - x_0|(y_x - x_0)}{y_x^2 x_0^2}$ has no t.). Thus the integrand

$$\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x - x_0}\right)$$

converges uniformly (in t) to $\cos t \cdot \frac{-1}{(t+x_0)^2}$.

Thus the limit indeed exists and

$$f'(x_0) = \int_0^{\pi/2} \cos t \cdot \frac{-1}{(t+x_0)^2}\, dt.$$

As x_0 was arbitrary, we conclude that f is differentiable everywhere with

$$f'(x) = \int_0^{\pi/2} \cos t \cdot \frac{-1}{(t+x)^2}\, dt.$$

Now we must show that this derivative is continuous. I leave this to you.

Method 2: Using bounded convergence theorem

Fix $x_0 > 0$. To show that $f : (0,\infty) \to \mathbb{R}$ is differentiable at x_0, we'll show instead that $f : [x_0-\delta, x_0+\delta] \to \mathbb{R}$ is differentiable at x_0 where $\delta = x_0/100$. (This may seem useless but there is a clever point to this — as you read this proof, try to figure out what it's doing!)

As in the first solution, we want to show that

$$\int_0^{\pi/2} \cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) dt \xrightarrow{x\to x_0} \int_0^{\pi/2} \cos t \cdot \frac{-1}{(t+x_0)^2}\, dt.$$

Clearly $\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) \xrightarrow{x\to x_0} \cos t \cdot \frac{-1}{(t+x_0)^2}$ for every t (i.e. we have pointwise convergence), so by the bounded convergence theorem it suffices to prove that $\cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right)$ is uniformly bounded. That is, we need to find a constant M (which does NOT depend on x or t) such that

$$\left| \cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) \right| \le M$$

for all $t \in (0,\pi/2)$ and for all $x \in [x_0-\delta, x_0+\delta]$.

First we can write

$$\left| \cos t \cdot \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) \right| \le \left| \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) \right|.$$

Now by the Mean Value Theorem,

$$\left| \left(\frac{\frac{1}{t+x} - \frac{1}{t+x_0}}{x-x_0} \right) \right| = \left| \frac{-1}{(t+y_x)^2} \right|$$

for some y_x between x and x_0. Finally,

$$\left| \frac{-1}{(t+y_x)^2} \right| \le \frac{1}{(0+(x_0-\delta))^2}.$$

Taking $M := \frac{1}{(x_0-\delta)^2} < \infty$ works because it has no t or x. The bounded convergence theorem concludes. ■

3.2.3 Swap Limit and Sum

Theorem 3.3

Let $f_n(t)$ be a function depending on a parameter t. Suppose that the sum $\sum_{n=1}^{\infty} f_n(t)$ converges *uniformly in* t. Then

$$\lim_{t\to t_0} \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} \lim_{t\to t_0} f_n(t).$$

Notes:

- A similar result can be used to swap two limits.
- This can be used to prove that an infinite sum is continuous. To be precise, if f_n is continuous for all n and the sum $\sum_{n=1}^{\infty} f_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ is continuous. (Of course, this is just an instance of "the uniform limit of continuous functions is continuous".)

3.2.4 Swap Derivative and Limit

Theorem 3.4

Let $f_n : I \to \mathbb{R}$ be a sequence of differentiable functions on an interval I. Suppose:

- f_n converges pointwise to a function f, and
- f_n' converges *uniformly* to a function g.

Then f is differentiable with $f' = g$.

Proof. (Not done in workshop but it is a good exercise) We have

$$\int_{x_0}^{x} f_n'(t)\,dt = f_n(x) - f_n(x_0).$$

Now send $n \to \infty$. The left side converges to $\int_{x_0}^{x} g(t)\,dt$ because $f_n' \to g$ uniformly. The right side converges to $f(x) - f(x_0)$ because $f_n \to f$ pointwise. So

$$\int_{x_0}^{x} g(t)\,dt = f(x) - f(x_0).$$

Differentiating in x, we conclude that $f'(x) = g(x)$. □

3.2.5 Swap Derivative and Sum

A direct application of the previous theorem allows us to more closely study the regularity of a series!

Theorem 3.5

Let $f_n : I \to \mathbb{R}$ be a sequence of differentiable functions on an interval I. Suppose:

- $\sum_{n=1}^{\infty} f_n$ converges pointwise, and
- $\sum_{n=1}^{\infty} f_n'$ converges *uniformly*.

Then $\sum_{n=1}^{\infty} f_n$ is differentiable, and

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x).$$

By an inductive argument we have the following corollary: If

- $\sum_{n=1}^{\infty} f_n$ converges pointwise,
- $\sum_{n=1}^{\infty} f_n'$ converges uniformly,
- $\sum_{n=1}^{\infty} f_n''$ converges uniformly,
- $\sum_{n=1}^{\infty} f_n'''$ converges uniformly,
- ...
- and $\sum_{n=1}^{\infty} f_n^{(k)}$ converges uniformly,

then $\sum_{n=1}^{\infty} f_n$ is k-times differentiable!

3.2.6 Swap Integral and Integral

For finite intervals $[a, b]$ and $[c, d]$, when is it true that

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy?$$

I'll give you two possible "tests" you can use:

1. It is true when f is non-negative. (Tonelli's Theorem)
2. It is true when f is bounded. (Budget Fubini's Theorem)

For example, if f is continuous on $[a, b] \times [c, d]$, then f is bounded and so you can swap.

3.2.7 Swap Derivative and Derivative

Easy one: If $f \in C^2(\mathbb{R}^n)$ (i.e. f is twice-differentiable and all second derivatives are continuous), then you can swap partial derivatives, i.e.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

(This is the *Schwarz Theorem*.)

3.2.8 Example: Taylor Series with Integral Remainder

The stupidest example of a Taylor Series is the FTC:

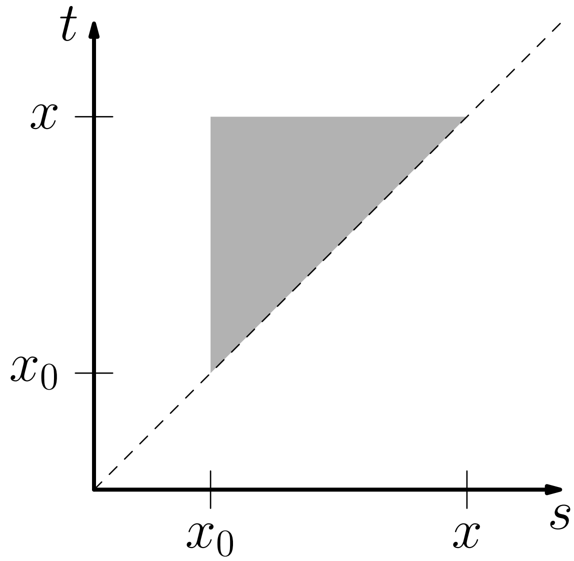
$$f(x) = f(x_0) + \int_{x_0}^x f'(t)\,dt$$

If we want to involve higher order derivatives now, it is natural to use the FTC again:

$$\begin{aligned}
f(x) &= f(x_0) + \int_{x_0}^x f'(t)\,dt \\
&= f(x_0) + \int_{x_0}^x f'(x_0) + \int_{x_0}^t f''(s)\,ds\,dt && \text{(FTC)} \\
&= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x \int_{x_0}^t f''(s)\,ds\,dt \\
&= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x \int_s^x f''(s)\,dt\,ds && (*) \\
&= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - s) f''(s)\,ds.
\end{aligned}$$

We can do this again and again!

Real quick though, let's talk about how we can reason about getting the new bounds after swapping integrals, as in $(*)$. One way is to draw a picture. In the integral $\int_{t=x_0}^{x}\int_{s=x_0}^{t}\cdots\,ds\,dt$, the range of values that s can take on given a fixed t is given by $[x_0, t]$. Plotting this for every s, the region of all permissible values (s, t) looks as follows:



Thus, from the picture, we see that when s is fixed then the range of possible values for t is given by $[s, x]$. Hence

$$\int_{t=x_0}^{x}\int_{s=x_0}^{t}\cdots\,ds\,dt = \int_{s=x_0}^{x}\int_{t=s}^{x}\cdots\,dt\,ds.$$

Alternatively, one can reason out the new bounds by observing that the only restriction to s and t (other than $s, t \in [x_0, x]$) is that $s < t$.

3.2.9 Example: Fourier Transform converts integrability to smoothness

Was not done in the actual workshop but here it is if you're itching for some Fourier analysis.

Example 3.5: Let $f \in C(\mathbb{R})$ be compactly supported. Show that the Fourier transform $\hat{f}$ is smooth.

Proof. Recall that

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx.$$

Now we argue:

- $f(x)e^{-2\pi i\xi x}$ is differentiable in ξ and is bounded.
- Therefore, $\hat{f}(\xi)$ is differentiable, with

$$\hat{f}'(\xi) = \frac{d}{d\xi}\int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx = \int_{\mathbb{R}} \frac{\partial}{\partial\xi} f(x)e^{-2\pi i\xi x}\,dx = \int_{\mathbb{R}} -2\pi i x f(x)e^{-2\pi i\xi x}\,dx.$$

- $-2\pi i x f(x)e^{-2\pi i\xi x}$ is differentiable in ξ and is still bounded.
- So by the same logic, $\hat{f}'(\xi)$ is differentiable again.
- We can repeat this logic forever, so $\hat{f}$ is actually smooth.

(Why could we confidently apply our theorems to complex-valued functions?) □

In general, higher integrability on f gives higher regularity on $\hat{f}$.

3.3 Examples on Swapping

Example 3.6 (Winter 2009 #5): Let $u_n(x) = x^n \log x, x \in (0, 1]$.

(a) Check the convergence and uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ in $(0, 1]$.

(b) Compute $I = \int_0^1 \left(\sum_{n=1}^{\infty} x^n \log x\right) dx$.

Example 3.7 (Fall 2017 #4): Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 + x^2}$$

defines a continuously differentiable function of $x \in \mathbb{R}$.

Proof. First let's show that it is differentiable. By the "swap derivative and sum" theorem, we need only show that the sum of the derivatives converges uniformly. The series in question is

$$\sum_{n=1}^{\infty} \frac{d}{dx}\frac{\sin(nx)}{n^3 + x^2} = \sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2 + x^2} - \frac{2x\sin(nx)}{(n^3 + x^2)^2}.$$

It's not clear what the pointwise limit is, so it is wise to try the Weierstrass M-test. Let's get an upper bound on the nth term that's independent of x:

$$\left|\frac{n\cos(nx)}{n^3+x^2} - \frac{2x\sin(nx)}{(n^3+x^2)^2}\right| \le \frac{n}{n^3} + \left|\frac{2x}{n^3+x^2}\right| \cdot \left|\frac{\sin(nx)}{n^3+x^2}\right| \le \frac{1}{n^2} + \frac{1}{n^{3/2}} \cdot \frac{1}{n^3}$$

This upper bound converges when summed, so the M-test applies and we deduce uniform convergence of the derivatives. Hence the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3+x^2}$ is differentiable and its derivative is

$$\frac{d}{dx}\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3+x^2} = \sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2+x^2} - \frac{2x\sin(nx)}{(n^3+x^2)^2}.$$

But we are not done yet because we actually want to show that the series is *continuously* differentiable. Thus it remains to show that $\sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2+x^2} - \frac{2x\sin(nx)}{(n^3+x^2)^2}$ is continuous. By "the uniform limit of continuous functions is continuous", we just need to show that this series converges uniformly. But we've already shown that, so we're done. ☐

Example 3.8 (Fall 2020 #2): Consider the series $f(x) := \sum_{n=1}^{\infty} x^n \sin(x^n x)$ where x is a real variable.

(a) Is there a (positive length) interval over which f is well-defined and continuous?

(b) Is there a (positive length) interval over which f is differentiable? If so, is it infinitely differentiable anywhere in this interval?

3.4 Lagrange Multipliers

Theorem 3.6 (Lagrange Multipliers)

Let $f, g \in C^1(\mathbb{R}^n)$. Suppose the maximum (or minimum) of f subject to the constraint $g(\vec{x}) = 0$ is obtained at some $\vec{x}_0$. Assume further that $\nabla g(\vec{x}_0) \neq 0$. Then $\vec{x}_0$ satisfies

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0) \qquad (*)$$

for some $\lambda \in \mathbb{R}$.

If you just apply $(*)$ without actually checking all the hypotheses (particularly the tricky $g(\vec{x}_0) \neq 0$), you'll *probably* be fine, but a picky grader like me may deduct. Feel free to ignore the precise hypotheses if they make you uncomfortable. It's more important that you understand how to execute a solution involving Lagrange Multipliers.

IMPORTANT: For full credit, you MUST show that a maximum exists before you can apply Lagrange Multpliers.

Otherwise, Lagrange Multipliers may only detect a *local* maximum, which does not correspond to the global maximum. This classic folly is best exemplified with, say, the function

$$f(x) = (x^2 - 1)^2.$$

Setting $f'(x) = 0$ is woefully insufficient for identifying the maximum of f. Indeed, it turns out that f is not bounded and hence has no maximum, so solving $f'(x) = 0$ (which is the "baby" version of Lagrange multipliers) does not detect the global maximum because it simply does not exist.

Fortunately, the way to argue that a maximum/minimum exists is not too hard: You simply (1) observe that f is continuous (if it's not then what are you even doing lol), and (2) argue that the domain over which we are minimizing, i.e. $\{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = 0\} \cap \{\text{Domain of } f \text{ and other restrictions}\}$, is *compact*, i.e. closed and bounded.

Example 3.9 (Winter 2009 #2): Let

$$\Sigma = \{x \in \mathbb{R}^3 : x_1x_2 + x_2x_3 + x_2x_3 = 1\}$$

and

$$f(x) = x_1^2 + x_2^2 + \frac{9}{2}x_3^2.$$

(a) Show that Σ is a smooth surface in $\mathbb{R}^3$. *(Note from me: requires knowledge from the next section of these notes)*

(b) Show that $\inf_{x\in\Sigma} f(x)$ is achieved. Find $\inf_{x\in\Sigma} f(x)$.

Example 3.10 (Winter 2010 #3): Let x, y, z be positive real numbers. Show that the following inequality holds:

$$\frac{1}{x^3} + \frac{16}{y^3} + \frac{1}{z^3} \geq \frac{256}{(x+y+z)^3}.$$

(Hint: homogeneity)

Example 3.11 (Winter 2013 #2): Let the nonnegative real numbers x_1, x_2, x_3, x_4 satisfy $x_1 + x_2 + x_3 + x_4 = \pi$.

(a) Show that

$$\sin x_1 \sin x_2 \sin x_3 \sin x_4 \leq \frac{1}{4}.$$

(b) Find all (x_1, x_2, x_3, x_4) that result in equality above.

Solution. Part (b) ends up being very easy (and, I dare say, *trivial*) once Part (a) is done, so I will only show Part (a).

Let $f(x_1, \cdots, x_4) := \sin x_1 \sin x_2 \sin x_3 \sin x_4$ and $g(x_1, \cdots, x_4) = x_1+x_2+x_3+x_4-\pi$. You first **NEED TO ARGUE that a maximum EXISTS!** Otherwise Lagrange Multipliers may only find relative maxima, which doesn't help your case! To argue the existence of a relative maxima, note that f is continuous and moreover the set

$$\{\vec{x} \in \mathbb{R}^4 : g(\vec{x}) = 0, x_i \geq 0 \forall i\}$$

is compact. This is because it is bounded (why?) and closed (because the inverse image of a closed set under the continuous function g is closed).

So we may conclude that f obtains a maximum at some point $\vec{x}$. Lastly we must check $\nabla g(\vec{x}) \neq 0$. But $\nabla g \equiv (1, 1, 1, 1) \neq 0$, so this is definitely not an issue. Therefore we have justified the application of Lagrange Multipliers.

Lagrange Multipliers tells us that $\vec{x} = (x_1, \cdots, x_4)$ satisfies the following system:

$$\begin{cases} \cos x_1 \sin x_2 \sin x_3 \sin x_4 = \lambda \\ \sin x_1 \cos x_2 \sin x_3 \sin x_4 = \lambda \\ \sin x_1 \sin x_2 \cos x_3 \sin x_4 = \lambda \\ \sin x_1 \sin x_2 \sin x_3 \cos x_4 = \lambda \\ x_1 + x_2 + x_3 + x_4 = \pi \end{cases}$$

Solving this system comes down to algebraic finesse. I do not believe there is any one silver bullet that can solve even most systems that come with Lagrange Multipliers, so you have to stay flexible.

Here, it is useful to first eliminate the case where $\lambda = 0$. If $\lambda = 0$, then you can show that $x_i = 0$ for some i, in which case $f(x_1, \cdots, x_4) = 0$. But there clearly exist points where $f > 0$, so then $(x_1, \cdots, x_4)$ cannot be the point at which f obtains the maximum, contradiction.

So $\lambda \neq 0$. Next, we note that this implies that $\sin x_i \neq 0$ and $\cos x_i \neq 0$ for all i (in particular $x_i \notin \{0, \pi/2, \pi\}$). Thus we may divide the first two equations to get

$$\tan x_1 \cot x_2 = 1,$$

or simply $\tan x_1 = \tan x_2$. But $x_1, x_2 \in (0, \pi)$ and tan is injective over $(0, \pi)$ where it is defined. Thus $x_1 = x_2$.

But choosing the first two equations was an arbitrary choice, so by symmetrical reasoning we deduce that $x_1 = x_2 = x_3 = x_4$. Hence $x_i = \pi/4$ for all i, and we conclude that the maximum is given by

$$f(x_1, \cdots, x_4) = (\sin \pi/4)^4 = \frac{1}{4},$$

which is what we wanted to show.

(Funnily enough we never needed to compute λ for this problem, but often enough you will have to.) ■

Example 3.12 (Fall 2018 #3): Given any positive sequence of weights $(w_k)_1^n$, let m and M be the respective minimum and maximum values of the function

$$\sum_{k=1}^{n} w_k x_k^2$$

on the simplex $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_k = 1, x_k \geq 0 \; \forall k\}$.

(a) Find closed form expressions for m and M (in terms of the weights).

(b) What is the smallest possible value of $\frac{M}{m}$, and when (for which weights) is it achieved?

Solution. (Not done in workshop) Let $f(x_1, \cdots, x_n) := \sum_{k=1}^n w_k x_k^2$.

Part (a)

First we identify the minimum. From drawing some pictures (the levels sets are ellipses!), we obtain the ansatz that the minimum should occur in the interior of the simplex. We would like to apply Lagrange multipliers, however LM is not suitable for the surface as-is, since it cannot detect extrema along the boundary of the simplex.

Here's the fix: let us instead minimize f over the *whole hyperplane* $\{x_1 + \ldots + x_n = 1\}$, and we shall pray that the minimum is obtained within the simplex. (According to our ansatz picture, this must be true, so we have no need to fear.) Please convince yourself that minimizing over the larger set is a legitimate maneuver (provided our prayers come true)!

(Unfortunately enlarging the domain to this nice and smooth surface comes at a cost: we lose compactness, so it's no longer obvious that a minimum even exists. Remember: we *need to know extrema exist* before we can apply LM. The (seemingly roundabout) fix here is to argue that the minumum, if it exists, must be obtained within a compact set, such as the simplex that we started with (though this might not be the easiest-to-argue choice). Roughly speaking, this is true because f explodes as $(x_1, \cdots, x_n)$ zooms to infinity, so in theory we ought to be able to restrict to a bounded subdomain. To make this rigorous: Let H be the hyperplane. We know $\inf_H f$ exists because $f \geq 0$. Moreover $\inf_H f \geq w_1 > 0$ as witnessed by taking $(x_1, x_2, x_3 \cdots, x_n) = (1, 0, 0, \cdots, 0)$. Now find $R > 0$ for which $f \geq 0$ outside the ball $B_n(0, R)$. Then it is not hard to show that $\inf_H f = \inf_{H \cap B_n(0,R)} f$. Now $\overline{H \cap B_n(0, R)}$ is compact so f obtains a minimum in this smaller set, which is hence the minimum over H.)

Finally we can apply LM. LM tells us that $\nabla f(x_1, \cdots, x_n) = \lambda(1, \cdots, 1)$, hence this gives the system

$$\begin{cases} 2w_1x_1 = \lambda \\ \cdots \\ 2w_nx_n = \lambda \\ x_1 + \ldots + x_n = 1 \end{cases}.$$

Fortunately this is pretty easy to solve: We find that $x_k = \frac{\lambda}{2w_k}$ for all k, and so

$$\sum_{k=1}^{n} \frac{\lambda}{2w_k} = 1,$$

giving $\lambda = \frac{1}{\frac{1}{2}\sum_{k=1}^{n} \frac{1}{w_k}}$. This hence gives the solution

$$x_k = \frac{w_k^{-1}}{\sum_{j=1}^{n} w_j^{-1}},$$

and now the minimum m is given by

$$m = \sum_{k=1}^{n} w_k x_k^2 = \frac{1}{\left(\sum_{j=1}^{n} w_j^{-1}\right)^2} \sum_{k=1}^{n} w_k \cdot w_k^{-2} = \frac{1}{\sum_{j=1}^{n} w_j^{-1}}.$$

Now we identify the maximum. From once again drawing a picture (and, perhaps, taking $n = 2$ for inspiration), we obtain the ansatz that the maximum is obtained at a vertex of the simplex. In particular, we expect that the maximum is obtained at the vertex for which $x_i = 1$, where i is the index at which the weight w_i is the maximum. If this is true, then the max is given exactly by $M = w_i = \max(w_1, \cdots, w_n)$.

It may be tempting to try Lagrange Multipliers again, and in theory it could work with enough sweat and tears, but it is going to be a blood mess because *we know the maximum is going to occur along the edges of the simplex*, and checking every single k-dimensional edge/face over all k is going to be a nightmare.

The solution to our problems: *just do it.* Bare hands. We have a guess for M, do we? Then we just need to show that

$$\sum_{k=1}^{n} w_k x_k^2 \overset{?}{\leq} \max(w_1, \cdots, w_n).$$

Let's call $w_{\max} := \max(w_1, \cdots, w_n)$. First we wastefully go up with $w_k \leq w_{\max}$ for all k to get

$$\sum_{k=1}^{n} w_k x_k^2 \leq \sum_{k=1}^{n} w_{\max} x_k^2 = w_{\max} \sum_{k=1}^{n} x_k^2.$$

It remains to prove that $\sum_{k=1}^n x_k^2 \le 1$. Well, since $x_k \in [0,1]$ for all k, we have that $x_k^2 \le x_k$, so $\sum_{k=1}^n x_k^2 \le \sum_{k=1}^n x_k = 1$. Voila!

Part (b)

We seek to solve the minimization problem

$$\min\left(\frac{\max(w_1,\cdots,w_n)}{\frac{1}{\sum_{j=1}^n w_j^{-1}}}\right)$$

over all positive weights $\{w_k\}_k$. By homogeneity (we can multiply each weight by the same factor and the value of the expression does not change), we may rescale all the weights to assume without loss of generality that $\max(w_1,\cdots,w_n) = 1$. Then our minimization problem is

$$\min\left(\sum_{j=1}^n \frac{1}{w_j} : 0 < w_j \le 1, \max_j w_j = 1\right).$$

With this framing, it's not hard to argue that the minimum occurs when $w_j = 1$ for all j. So the minimum is n. ■

3.5 Implicit Function Theorem

Motivations

The prototypical example for Implicit Function Theorem is the circle $x^2 + y^2 = 1$. This is *not* a function; this miserably fails the grade-school "vertical line test". However, *near* certain points on the circle, we could write y as an *implicit* function of x in the sense that there is a function ϕ, defined on a small interval, for which $x^2 + \phi(x)^2 = 1$.

A quick look at the "graph" of a circle reveals that this isn't possible around every point. In particular, we could never obtain an implicit function at the points $(-1,0)$ or $(1,0)$ because somehow the circle's "slope" is "vertical" at these points, which intuitively should correspond to some condition in terms of some derivative of $x^2 + y^2$. However, in my opinion, the exact condition is *not at all intuitive*: It turns out that if we wish to write y as an implicit function of x at a point (x_0, y_0), the condition in question is that the partial derivative *in the **output variable**, y,* must be non-zero at (x_0, y_0).

If you're ever feeling lost, always test your memory and ideas on the circle $x^2 + y^2 = 1$. For instance, if you don't entirely trust me, let's check that I'm right. The partial derivative in y of $x^2 + y^2$ is

$$\frac{\partial}{\partial y} x^2 + y^2 = 2y,$$

and this is nonzero at any point (x, y) on the circle, as long as $y \neq 0$. This demonstrates that indeed, we are unable to obtain an implicit function at the points $(-1, 0)$ and $(1, 0)$ precisely because these are exactly the points at which $y = 0$.

Facts

- Let $g(x, y)$ be at least C^1. Let (x_0, y_0) be such that $g(x_0, y_0) = 0$. When does the equation $g(x, y) = 0$ procure an implicit function $y = \phi(x)$, i.e. a ϕ such that $\phi(x_0) = y_0$ and $g(x, \phi(x)) = 0$ for x near $x = x_0$? A sufficient condition is that *the derivative of g in the "output" variable*, at (x_0, y_0), does not vanish. That is,
$$\frac{\partial g}{\partial y}(x_0, y_0) \neq 0.$$
- The regularity of the implicit function ϕ is that of g. If g is C^2, then ϕ is C^2. If g is smooth, then ϕ is smooth.
- This extends to more "input" variables in a natural way: If $g(x^1, x^2, y)$ is at least C^1, $g(x_0^1, x_0^2, y_0) = 0$, and $\frac{\partial g}{\partial y}(x_0^1, x_0^2, y) \neq 0$, then there is a function $\phi(x^1, x^2)$ defined near (x_0^1, x_0^2) such that $g(x^1, x^2, \phi(x^1, x^2)) = 0$.
- Once again, the regularity of ϕ in this case is the same as the regularity of g.
- (You can have more "output" variables too and it's not too nasty but I won't cover this. It's easiest to imagine the case where there's just one "output" variable, since you can picture this as being able to represent your function as a "graph".)

3.5.1 Examples

Example 3.13 (Fall 2011 #3): Show that $e^x - e^y + xy = 0$ defines near $(0, 0)$ an implicit function $y = \phi(x)$ in C^3 and compute its expansion to order 3 at $(0, 0)$.

Solution. Let $g(x, y) = e^x - e^y + xy = 0$. We have $g(0, 0) = 0$ so we can indeed talk about trying to y as an implicit function of x given $g(x, y) = 0$ at $(x, y) = (0, 0)$.

To determine if this can be done, we need only compute the derivative of g in the "output variable", which is y. This is
$$\frac{\partial g}{\partial y}(x, y) = -e^y + x.$$
Evaluating at $(0, 0)$ gives $\frac{\partial g}{\partial y}(0, 0) = -1$. Since this is $\neq 0$, the Implicit Function Theorem applies, telling us that near $(0, 0)$ we may write y as an implicit function $\phi(x)$ of x. Moreover the regularity of ϕ is inherited from that of g — that is, $\phi \in C^\infty$. So certainly ϕ is C^3.

Now we compute the expansion of ϕ at $x = 0$. The first term is easy since we already know $\phi(0) = 0$. Now, the way to compute derivatives of ϕ (at $x = 0$) is to rely on the fact that $g(x, \phi(x)) = 0$ for all x near 0. Differentiating in x and applying the chain rule, we get

$$0 = \frac{d}{dx}0 = \frac{d}{dx}g(x, \phi(x)) = \frac{\partial g}{\partial x}(x, \phi(x)) + \frac{\partial g}{\partial y}(x, \phi(x))\phi'(x) \qquad (*).$$

Plugging in $x = 0$ gives $0 = \frac{\partial g}{\partial x}(0,0) + \frac{\partial g}{\partial y}(0,0)\phi'(0)$. We find that $\frac{\partial g}{\partial x}(0,0) = 1$, and we computed previously that $\frac{\partial g}{\partial y}(0,0) = -1$. So we conclude that $\phi'(0) = 1$.

Now we want one more derivative, so we differentiate $(*)$ in x to get

$$0 = \frac{\partial^2 g}{\partial x^2}(x, \phi(x)) + \frac{\partial^2 g}{\partial x \partial y}(x, \phi(x))\phi'(x) + \frac{\partial^2 g}{\partial x \partial y}(x, \phi(x))\phi'(x) + \frac{\partial^2 g}{\partial y^2}(x, \phi(x))\phi'(x)^2 + \frac{\partial g}{\partial y}(x, \phi(x))\phi''(x).$$

That inevitably went into the margin so hopefully you're not printing these notes out. Plugging in $x = 0$ gives

$$0 = \frac{\partial^2 g}{\partial x^2}(0,0) + 2\frac{\partial^2 g}{\partial x \partial y}(0,0)\phi'(0) + \frac{\partial^2 g}{\partial y \partial (}0,0)\phi'(0)^2 + \frac{\partial g}{\partial y}(0,0)\phi''(0).$$

We can compute $\frac{\partial^2 g}{\partial x^2}(0,0) = 1$, $\frac{\partial^2 g}{\partial y^2}(0,0) = -1$, and $\frac{\partial^2 g}{\partial x \partial y}(0,0) = 1$. So

$$0 = 1 + 2\phi'(0) - \phi'(0)^2 - \phi''(0).$$

Recalling that $\phi'(0) = 1$, we conclude that $\phi''(0) = 2$, and so Taylor's theorem gives the expansion of ϕ as

$$\phi(x) = 0/0! + x/1! + 2x^2/2! + o(x^2) = x + x^2 + o(x^2)$$

for all x near $x = 0$.

("Practical exercise:" Find a way to verify that this answer is correct.) ■

Example 3.14 (Winter 2016 #3): Let C be the set of solutions of $\cos x + \cos y - y = 1 - \frac{\pi}{2}$.

(a) Show that C can be represented as the graph of an implicit function $y = \phi(x)$ near $(0, \pi/2)$ and compute the first three terms (the 1, x, and x^2 terms) of the Taylor expansion of ϕ at $x = 0$.

(b) Let $r : C \to \mathbb{R}$ be the function $r(x, y) = x^2 + y^2$. Is $(0, \pi/2)$ a local maximum or a local minimum of r on C?

3.5.2 What about the Inverse Function Theorem?

(Not done in workshop and it does not seem to really appear on the writtens, but still it is generally good to be familiar with the connection.)

When does the function $f : \mathbb{R} \to \mathbb{R}$ have an inverse near some $x = x_0$? Well, we're essentially taking the equation $y = f(x)$ and asking whether this makes x and implicit function of y! So let

$$g(y, x) := f(x) - y.$$

To apply Implicit Function Theorem, we need to differentiate g in the output variable, which in this case is x. So what we want is

$$0 \neq \frac{\partial g}{\partial x}(f(x_0), x_0) = f'(x_0).$$

That's it! Makes total sense too. This generalizes to more dimensions using similar reasoning.

4 Day 4: Bare Hands

With the exception of some niche results, we've essentially exhausted all the content you need to know. There are no more tools. All that is left for us is to do is to take on the world with our bare hands.

4.1 Fix Epsilon Greater Than Zero...

Example 4.1 (Fall 2009 #1): Let the series $\sum_{n=1}^{\infty} x_n$ be absolutely convergent. Show that, for any increasing sequence $(a_n)_1^{\infty}$ of positive numbers with $a_n \to \infty$, we have

$$\lim_{N\to\infty} \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n = 0.$$

Proof. Fix $\varepsilon > 0$. Find N_ε so large that $\sum_{n=N_\varepsilon}^{\infty} |x_n| < \varepsilon$. Then for all $N > N_\varepsilon$,

$$\left| \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n \right| \leq \frac{1}{a_N} \sum_{n=1}^{N_\varepsilon - 1} a_n |x_n| + \frac{1}{a_N} \sum_{n=N_\varepsilon}^{N} a_n |x_n| \leq \left(\frac{1}{a_N} \sum_{n=1}^{N_\varepsilon - 1} a_n |x_n| \right) + \varepsilon. \qquad (*)$$

Seeing that $\left(\frac{1}{a_N} \sum_{n=1}^{N_\varepsilon - 1} a_n |x_n| \right) \xrightarrow{N\to\infty} 0$, there are now two ways to finish.

Method 1

Instead of taking $N > N_\varepsilon$, we take $N > \max(N_\varepsilon, M_\varepsilon)$, where M_ε is chosen so that

$$\frac{1}{a_N} \sum_{n=1}^{N_\varepsilon - 1} a_n |x_n| < \varepsilon.$$

Then for all such N, we have, by $(*)$, that

$$\left| \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n \right| \leq \varepsilon + \varepsilon = 2\varepsilon,$$

which is enough.

Method 2

From $(*)$, we send $N \to \infty$ on both sides. Though, we do not know if the limit of the LHS exists a priori, so we take the limsup instead to get

$$\limsup_{N\to\infty} \left| \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n \right| \leq 0 + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the limsup is 0, and so in particular the limit exists and is 0. $\square$

Pay special attention to the idea of splitting the sum into two parts: the "head" and the "tail". The head is irrelevant, and the hypotheses applied to tame the tail. The next problem is a very good exercise for practicing this.

Example 4.2 (Fall 2008 #1, Winter 2011 #4): Let $(a_n)_1^\infty$ and $(b_n)_1^\infty$ be two positive sequences such that $\sum b_n = \infty$ and $a_n/b_n \to c$ for some $c \geq 0$. Define $A_n = a_1 + \cdots + a_n$ and $B_n = b_1 + \cdots + b_n$, $n \geq 1$. Show that $A_n/B_n \to c$.

Now let us fix $\varepsilon > 0$ in other contexts.

Example 4.3 (Winter 2014 #2): Assume f and g are two continuous positive functions on $[a, b]$. Determine

$$\lim_{n\to\infty} \left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n}.$$

Proof. Try drawing some pictures and/or reflecting on previous experiences in analysis, and you may come to the guess that the limit is $M := \max_{[a,b]} f$. A good approach/mindset for tackling this is to prove an inequality of the form

$$\left(\text{Something that } \xrightarrow{n\to\infty} M\right) \leq \left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n} \leq \left(\text{Something that } \xrightarrow{n\to\infty} M\right).$$

An upper bound is not so hard:

$$\left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n} \leq \left[\int_a^b M^n g(x)\,dx\right]^{1/n} = M\left[\int_a^b g(x)\,dx\right]^{1/n} \xrightarrow{n\to\infty} M.$$

For a lower bound, we wish somehow to focus the integral on only the part of f near the max, which we'll say is attained at some $x_0 \in [a, b]$. Hence, fix $\varepsilon > 0$. Now we may find $\delta > 0$ so that $f \geq M - \varepsilon$ on $(x_0 - \delta, x_0 + \delta)$. This lets us get a lower bound:

$$\left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n} \geq \left[\int_{x_0-\delta}^{x_0+\delta} f(x)^n g(x)\,dx\right]^{1/n} \geq \left[\int_{x_0-\delta}^{x_0+\delta} (M-\varepsilon)^n g(x)\,dx\right]^{1/n}$$

$$= (M-\varepsilon)\left[\int_{x_0-\delta}^{x_0+\delta} g(x)\,dx\right]^{1/n} \xrightarrow{n\to\infty} M - \varepsilon.$$

We conclude that

$$M - \varepsilon \leq \liminf_{n\to\infty}\left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n} \leq \limsup_{n\to\infty}\left[\int_a^b f(x)^n g(x)\,dx\right]^{1/n} \leq M.$$

Now send $\varepsilon \to 0^+$. $\square$

We did not *need* to bring in liminf and limsup (and it is not hard to avoid their use), though it certainly makes for a cleaner argument. In any case, you should be familiar with liminf/limsup and their basic properties.

Try some more exercises.

Example 4.4 (Fall 2015 #5): Let

$$\phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}, \qquad t > 0, \qquad x \in \mathbb{R}.$$

(i) Show that $\int_{-\infty}^{\infty} \phi(x,t)\, dx = 1$.

(ii) Let $f(x)$ be a continuous function on $\mathbb{R}$ such that $f(x) \to 0$ as $x \to \pm\infty$. Set

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x-y,t) f(y)\, dy, \qquad t > 0, \qquad x \in \mathbb{R}.$$

Show that $u(x,t)$ solves the heat equation

$$\frac{\partial u}{\partial t} = u_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0,$$

with initial condition

$$\lim_{t \downarrow 0} u(x,t) = f(x).$$

Note from me: The instructive part of interest, with respect to this section of the notes, is showing that $\lim_{t\to 0} u(x,y) = f(x)$.

Example 4.5 (Winter 2016 #2):

(a) Suppose that $a_i > 0$ for all i, that $\sum_{i=1}^{\infty} a_i$ converges, and that $\lim_{t\to\infty} \frac{a_i}{a_{i+1}} = 1$. Prove that

$$\lim_{N\to\infty} \frac{a_N + a_{N+2} + a_{N+4} + \dots}{a_N + a_{N+1} + a_{N+2} + \dots} = \frac{1}{2}.$$

(b) Prove that

$$\lim_{n\to\infty} n \cdot \sum_{i=2n}^{\infty} \frac{(-1)^i}{i} = \frac{1}{4}.$$

Example 4.6 (Spring 2018 #4):

(a) Show that there exists constants $0 < A \le B < \infty$ such that, for every $y, z \in \mathbb{R}$ with $y \neq z$,

$$\frac{A}{|y-x|^{1/3}} \le \int_{\mathbb{R}} \frac{1}{|x-y|^{2/3}} \frac{1}{|x-z|^{2/3}}\, dx \le \frac{B}{|y-z|^{1/3}}.$$

(Advice: Do not attempt to compute the integral directly. Instead break it into pieces and estimate the pieces separately.)

(Note from me: yeah that advice didn't come from me either sorry. it came with the test. but i do agree with it!)

(b) Find the exponent $\alpha > 0$ such that there exists constants $0 < A < B < \infty$ such that, for every $y, z \in \mathbb{R}^2$ with $y \neq z$,

$$\frac{A}{|y-z|^{\alpha}} \le \int_{\mathbb{R}^2} \frac{1}{|x-y|^{3/2}} \frac{1}{|x-z|^{3/2}}\, dx \le \frac{B}{|y-z|^{\alpha}}.$$

If you have done Part (a) properly, you do not have to give a full justification.

Example 4.7 (Spring 2021 #4): Suppose that $f : [0, \infty) \to \mathbb{R}$ is square-integrable, that is, $\int_0^\infty |f(x)|^2\, dx < \infty$. Prove that

$$\lim_{t\to\infty} t^{-1/2} \int_0^t |f(x)|\, dx = 0.$$

Let's do this last example together, as it is quite technical.

Example 4.8 (Fall 2018 #2): Suppose $f_n : \mathbb{R} \to \mathbb{R}$, $n \ge 0$, is a sequence of continuous functions converging to f uniformly on every compact set. Consider any continuous function ϕ on $\mathbb{R}$. Show that $\phi \circ f_n$ converges to $\phi \circ f$ uniformly on every compact set, too. What if the f_n are not necessarily continuous?

Proof. Take a compact set K and we'll show the claim for K. Fix $\varepsilon > 0$. Then for $n \ge$ (Something we'll figure out later), we would like to show that

$$|\phi(f_n(x)) - \phi(f(x))| < \varepsilon$$

for all $x \in K$. Intuitively this should be true because $f_n(x)$ and $f(x)$ are close for all $x \in K$. So after applying ϕ, the resulting values should still be close because ϕ is continuous. However, since x isn't fixed, we can't work with continuity of ϕ at a fixed point. This tells us that we need to use some sort of *uniform continuity*.

The next issue is that ϕ need not be uniformly continuous. However, **a continuous function over a compact set is uniformly continuous**, so if we can restrict the domain of ϕ to a compact subset $[-M, M]$ containing all the values we could ever care about, we'll be done.

Let's worry about that later. Given the $\varepsilon > 0$, we may choose $\delta > 0$ witnessing that ϕ is uniformly continuous over $[-M, M]$. That is, if $|s - t| < \delta$ then $|\phi(s) - \phi(t)| < \varepsilon$. What do we plug in for s and t? Evidently they should be $f_n(x)$ and $f(x)$, which are close because $f_n \to f$ uniformly on K! So let us take $n \geq N_\varepsilon$ large enough so that for all such n, we have $|f_n(x) - f(x)| < \delta$ for all $x \in K$.

Putting everything together, we have for all $x \in K$ and all $n \geq N_\varepsilon$ that

$$|\phi(f_n(x)) - \phi(f(x))| < \varepsilon,$$

that is, we have the uniform convergence over K.

Now let us return to worry about restricting ϕ to a compact set. Can we find M so large that all the values we care about, i.e. $f_n(x)$ and $f(x)$ for $x \in K$, will be contained in $[-M, M]$? The answer is yes because $f_n \to f$ uniformly over K.

To be explicit, we can take $M_1 := \sup_K |f|$, so that $f(x) \in [-M_1, M_1]$ for all $x \in K$. Next, to take advantage of the uniform convergence, find N' so large that for all $n \geq N'$, we have $|f_n(x) - f(x)| \leq 1$ for $n \geq N'$. Then $f_n(x) \in [-M_1 - 1, M_1 + 1]$ for all $x \in K$, $n \geq N'$.

We have "tamed the tail", and it remains to use that "the head is irrelevant". Take

$$M = \max\left(\sup_K |f_1|, \cdots, \sup_K |f_{N'}|, M_1 + 1\right).$$

This works. □

4.2 Sums vs. Integrals

Riemann sums can be very useful for approximating integrals.

Example 4.9 (Winter 2009 #1): Show that

$$\lim_{n\to\infty} \frac{\log n}{n} \sum_{k=2}^{n} \frac{1}{\log k} = \lim_{n\to\infty} \frac{\log n}{n} \int_2^n \frac{1}{\log x}\, dx = 1.$$

Proof. You should see the handwritten notes for (slightly messy) diagrams. The idea is that by drawing a picture of the Riemann sum, we can find that

$$\sum_{k=2}^{n-1} \frac{1}{\log k} - \int_2^n \frac{1}{\log x}\, dx \leq \log 2.$$

To prove this rigorously, we can use the fact that

$$\frac{1}{\log k} - \int_k^{k+1} \frac{1}{\log x}\, dx \le \frac{1}{\log k} - \int_k^{k+1} \frac{1}{\log(k+1)} = \frac{1}{\log k} - \frac{1}{\log(k+1)}$$

and then sum over k. In all, this gives a bounded error between the sum and the integral, and multiplying this error by $\frac{\log n}{n}$ gives a quantity that goes to 0. This shows the first equality.

For the second equality, try applying L'Hopital's rule. □

The next exercise is good for practicing this sort of methodology.

Example 4.10 (Fall 2008 #3): Show that

$$\frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{N}}{N} - \frac{2}{3}\sqrt{N} \to 0.$$

(**Hint:** Choose appropriate Riemann sums and estimate the approximation error.)

4.3 Integral Asymptotics

Example 4.11 (Fall 2016 #1): Prove that the improper Riemann integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ is conditionally convergent.

Proof. First we show that $\displaystyle\lim_{T\to\infty} \int_0^\infty \frac{\sin x}{x}\, dx$ converges.

If you look at the solution on the wiki, you will find that they use some sort of trick to prove this. But I think tricks are the morally wrong way to prove this sort of claim about the integral converging, since they somehow ignore the "true" reason for the convergence.

Namely, by drawing a diagram, you should be reminded of the alternating series test for sums. If we take

$$a_n := \int_{\pi n}^{\pi(n+1)} \frac{|\sin x|}{x}\, dx,$$

then intuitively (but this does require some proof as we will handle later), we should have that

$$\lim_{T\to\infty} \int_0^\infty \frac{\sin x}{x}\, dx \stackrel{?}{=} \sum_{n=0}^{\infty} (-1)^n a_n.$$

Since the sum on the right converges by the alternating series test (check this!), we want to say that the limit on the LHS converges as well. Unfortunately, this is a bit hasty. Since a_n

is defined by "taking steps of length π", we actually only know that

$$\lim_{n\to\infty}\int_0^{\pi n}\frac{\sin x}{x}\,dx=\sum_{n=0}^{\infty}(-1)^n a_n,$$

where n on the LHS runs over positive integers!

One way to turn the LHS back into a limit over $T\to\infty$ is to argue that if $\pi n\le T\le \pi(n+1)$, then $\int_0^T \frac{\sin x}{x}\,dx$ lies in between $\int_0^{\pi n}\frac{\sin x}{x}\,dx$ and $\int_0^{\pi(n+1)}\frac{\sin x}{x}\,dx$ (why?). Then we may show that

$$\lim_{T\to\infty}\int_0^{\infty}\frac{\sin x}{x}\,dx=\lim_{n\to\infty}\int_0^{\pi n}\frac{\sin x}{x}\,dx$$

by fixing $\varepsilon>0$ and considering all T large enough. I'll leave the gory details to you, but you should try and work it out!

Next we must show that $\displaystyle\int_0^{\infty}\frac{|\sin x|}{x}\,dx=+\infty$. By drawing a picture, the intuition should be that we have the divergence because of the divergence of the harmonic series. Note that we have the bound

$$\int_{\pi n}^{\pi(n+1)}\frac{|\sin x|}{x}\,dx\ge\int_{\pi n+\pi/6}^{\pi n+5\pi/6}\frac{|\sin x|}{x}\,dx\ge\int_{\pi n+\pi/6}^{\pi n+5\pi/6}\frac{1/2}{x}\,dx$$

$$\ge\int_{\pi n+\pi/6}^{\pi n+5\pi/6}\frac{1/2}{\pi n+5\pi/6}\,dx=\frac{1}{3\pi(\pi n+5\pi/6)}.$$

So

$$\int_0^{\infty}\frac{|\sin x|}{x}\,dx=\sum_{n=0}^{\infty}\int_{\pi n}^{\pi(n+1)}\frac{|\sin x|}{x}\,dx\ge\sum_{n=0}^{\infty}\frac{1}{3\pi(\pi n+5\pi/6)}$$

$$=\frac{1}{3\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+5/18}\ge\frac{1}{3\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1}=\frac{1}{3\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}=+\infty.$$

□

Try your hand at some problems that are solved very similarly.

Example 4.12 (Winter 2011 #5): Show that the improper integral $\displaystyle\int_{-\infty}^{\infty}\cos(x\log|x|)\,dx$ is convergent.

Example 4.13 (Winter 2015 #3 Part (b)): For which values of α does

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}s^{\alpha}\sin\left(\frac{1}{s}\right)\,ds$$

exist?

Example 4.14 (Winter 2016 #4):

(a) For what real numbers α does the limit

$$\lim_{t\to\infty}\int_1^t x^\alpha \sin(x)\,dx$$

converge?

(b) For what real numbers α does the limit

$$\lim_{t\to\infty}\int_1^t x^\alpha \sin(e^x)\,dx$$

converge?

4.4 Convexity

The following problem seems intimidating, but it is solved well with a good picture.

Example 4.15 (Fall 2014 #3): Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $f'(x)$ is strictly increasing, with $\lim_{x\to-\infty} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = \infty$, and $f(0)\neq 0$.

(a) Prove that $\forall \xi \neq 0$, there exists an η such that $f(\xi+\eta) = f(\xi) + f(\eta)$.

(b) Prove that through this point (ξ,η) there is a solution $y = \phi(x)$ of $f(x+y) = f(x) + f(y)$ which is unique in a neighborhood of (ξ,η).

(c) Construct an example to show that when $\xi = 0$ there may be no corresponding η.

I am unable to type out a description of what the picture is, so do refer to either the handwritten notes or the workshop recording. The point is that based on the picture, you can get the idea that the IVT needs to be used, and that moreover the way IVT is justified must come from the limits of f'. This motivates considering $f(\xi+\eta) - f(\eta)$ and sending $\eta \to +\infty$ and $\eta \to -\infty$.

Part (b) is a good review of the Implicit Function Theorem, and Part (c) is not that exciting.

Overall, if you know the definition of convexity then you should pretty much be fine. Try your hand at some problems.

Example 4.16 (Fall 2009 #2): Let f be a convex function on $\mathbb{R}^n$ such that

$$\lim_{r\to\infty} \frac{f(rx)}{r} = 0$$

for all $x \in \mathbb{R}^n$. Prove that f is constant.

Example 4.17 (Winter 2018 #3): Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously twice differentiable.

(a) Suppose $n = 1$. Show that $f'' \geq 0$ if and only if $f(x+h) - 2f(x) + f(x-h) \geq 0$ for every $x, h \in \mathbb{R}$. (Note: you should prove this "from scratch,", i.e., don't invoke any theorem more powerful than the statement you are asked to prove. *(Note from me: yeah that note is not from me that's what it says on the test)*)

(b) Suppose $n \in \{2, 3\}$. Show that the *Hessian* matrix $a_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ has all nonnegative eigenvalues if and only if $f(x+h) - 2f(x) + f(x-h) \geq 0$ for every $x, h \in \mathbb{R}^n$.

Example 4.18 (Spring 2019 #5): Suppose $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth, strictly convex function such that $\lim_{|y|\to\infty} \phi(x, y) = \infty$ for all $x \in \mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \min_{y\in\mathbb{R}} \phi(x, y).$$

Is F convex?

4.5 Miscellanea

Sometimes recursive sequences come up.

Example 4.19 (Winter 2015 #1 Part (b)): Let a_n be a sequence defined as follows: $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Show that a_n is bounded, nondecreasing and that the limit of a_n exists. Find $\lim_{n\to\infty} a_n$.

Solution. First we show that a_n is nondecreasing. That is, we want to show that $\sqrt{2 + a_n} \leq a_n$. Squaring both sides (justified because $a_n \geq 0$ for all n) and rearranging, we see that this is true iff $a_n \in [-1, 2]$. Obviously $a_n \geq -1$. To show $a_n \leq 2$, we use induction (which turns out to be pretty simple).

This argument incidentally also shows that a_n is bounded. Any bounded increasing sequence has a limit, so $L := \lim_{n\to\infty} a_n$ exists. To solve for L, we send $n \to \infty$ in the recurrence to get that

$$L = \sqrt{2 + L}.$$

Solving gives $L = -1, 2$. Obviously $L \neq -1$ so $L = 2$ is the only possible value of the limit, and therefore it is the limit. ■

I leave the rest to you.

This problem is a good exercise.

Example 4.20 (Winter 2012 #3): Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable, and the partial derivatives of the components f_1, f_2 satisfy

$$\max\left(\left|\frac{\partial f_1}{\partial x} - 1\right|, \left|\frac{\partial f_1}{\partial y}\right|, \left|\frac{\partial f_2}{\partial x}\right|, \left|\frac{\partial f_2}{\partial y} - 1\right|\right) < 10^{-10}.$$

Prove that f is a bijection. Note: f is not assumed to be continuously differentiable.

You should attempt the next problem. At the very least, you definitely should read the solution on the wiki. The methodology used here is instructive.

Example 4.21 (Winter 2009 #4): Assume $f : \mathbb{R} \to [0, \infty)$ is C^2 and $|f''(x)| \leq A$ for all x. Show that the inequality

$$f'(x)^2 \leq 2Af(x)$$

holds for all x.
(**Hint:** Taylor's theorem $f(x + t) = f(x) + tf'(x) + R(x, t)$.)

If you're not immediately sure how to do Part (a) of the next problem, you really should do it. The upper bound for Part (b) is also a good exercise. The lower bound of $1 - \frac{\ln N}{N}$ is tougher, don't fret if you can't find the path for obtaining it.

Example 4.22 (Fall 2017 #1): Let N be an odd integer and consider the equation $x^N + x = 1$ where $x \in \mathbb{R}$.

(a) Show that this equation has no solution when N is negative and has a unique solution which lies on $(0, 1)$ when N is positive.

(b) Show that when $N \geq 3$ the solution lies in the interval $(1 - \frac{\ln N}{N}, 1 - \frac{1}{N+1})$.

If you're not immediately sure how to do the next one, you should do it.

Example 4.23 (Spring 2019 #3): Suppose f is a continuous function on $\mathbb{R}$ such that $f(x + \frac{1}{n}) = f(x)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Show that f equals a constant. Does this conclusion hold without continuity?

The following is another example of a recursive sequence that you can try, though it's a bit more cumbersome than the example I did. You can do it if you want practice.

Example 4.24 (Fall 2019 #1): The sequence $\{a_n\}$ is defined by $a_{n+1} = 3(a_n - 1)^{1/2}$, $a_1 = 2$. Prove that $\{a_n\}$ is convergent, and find the limit.

4.6 Toughies

Part (a) is a good exercise. Part (b) may require an insight or some cleverness, but I'd say it's doable if you bash your head against it sufficiently. You don't need any powerful theorems.

Example 4.25 (Winter 2013 #1): Let $\sum_{k=1}^{\infty} a_k$ be a convergent series. Show that

(a) $\displaystyle\frac{1}{n^2}\sum_{k=1}^{n} k|a_k| \to 0$, and

(b) $\displaystyle\frac{1}{n}\sum_{k=1}^{n} ka_k \to 0$.

I don't quite remember how I solved the next one, and also it's 2 AM while I'm writing this so I can't really give good advice. Fortunately, from some skimming, it appears that the student-submitted solution on the wiki is well-written and motivated. I'd suggest reading it if you're stuck.

Example 4.26 (Winter 2013 #3): Let a real-valued $f \in C^2([0,1])$ satisfy $f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ for all $x \in [0,1]$.

(a) Show that
$$\left|\int_0^1 f(x)\,dx\right| \leq \frac{1}{12}.$$

(b) Find all f that result in equality above.

This last one is doable with all the knowledge we have covered in this workshop thus far, but it is not easy. If you want a hint, please use rot13.com to decode the following messages:

- Hint 1: hfr n gencrmbvny Evrznaa fhz.
- Hint 2: gur nafjre vf s bs mreb bire gjb cyhf s bs bar bire gjb. gb obhaq gur "reebe", gel hfvat gnlybe frevrf jvgu vagrteny erznvaqre. v qba'g yvxr gur jvxv'f nccebnpu.

Example 4.27 (Winter 2014 #3): For f of class C^2 on $[0,1]$, find a such that

$$\int_0^1 f(t)\,dt - \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = \frac{a}{n} + o\left(\frac{1}{n}\right).$$

There are some other problems that I'd characterize as "tough" in the more recent exams, but I will leave them for you to discover. Remember to practice, and best of luck.