From last time:

**Theorem**

*Number of ways of permuting n distinguishable objects:*

\[ n \cdot (n-1) \cdot \ldots \cdot 1 = n! \]

**Theorem**

*Number of ways of arranging n objects among which \( n_1 \) are alike, \( n_2 \) are alike, \ldots, \( n_r \) are alike:*

\[ \frac{n!}{n_1!n_2!\ldots n_r!} \]
Now suppose we have $n$ objects, and we want to choose a subcollection of $k$ of them. This is different from permuting or arranging, because now the order is not important.

Example

*How many sets of three letters can be chosen from A, B, C, D and E?*
Theorem

Number of ways of choosing $k$ objects out of $n$:

$$\frac{n!}{k!(n-k)!}.$$ 

IDEA: first choose $k$ objects in order 

(number of ways $= n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$),

then divide by the number of ways they could be ordered ($= k!$).
Notation:
\[ \frac{n!}{k!(n-k)!} = \binom{n}{k}. \]

It is pronounced “n choose k”. It is defined for $0 \leq k \leq n$. By convention,

\[ 0! = 1 \quad \text{and} \quad \binom{n}{0} = \binom{n}{n} = 1. \]
Example (Ross Ex 1.4a)

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Example (Ross Ex 1.4b)

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?

What if 2 of the men are feuding and refuse to serve on the committee together?
Example (More creative example; Ross Ex 1.4c)

Consider a set of $n$ antennas of which $m$ are defective and $n - m$ are functional. Assume all of the defectives and all of the functionals are indistinguishable. How many linear orderings are there in which no two defectives are consecutive?
The numbers \(^\binom{n}{k}\) follow some general rules which can be deduced from their meaning.

**Theorem (Ross Equation (4.1))**

If \(1 \leq k \leq n - 1\) then

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

**IDEA:** Both sides are counting the same thing in different ways.
Approximations* (not from Ross, not on hwk or exams)

In applications to probability, it’s often useful to approximate the behaviour of the numbers \( \binom{n}{k} \). Starting point:

**Theorem (Stirling’s formula, 1730)**

*If* \( n \) *is very large then*

\[
\sqrt{2\pi n} \cdot \frac{n^n}{e^n} \sim n!
\]

*(More formally: \( \frac{n!}{\sqrt{2\pi n} \cdot \frac{n^n}{e^n}} \rightarrow 1 \text{ as } n \rightarrow \infty \).)*

**IDEA:** approximate

\[
\log(n!) = \log 1 + \log 2 + \cdots + \log n
\]

by an integral (very carefully).
Stirling’s Formula is used but not explained in Ross. For a proof, see Section II.9 of Feller *An Introduction to Probability Theory and Its Applications, Vol I.*

The approximation it gives is within about 0.8% for $10! = 3,628,800$, and 0.08% for $100! \sim 10^{158}$. 
Later we will turn Stirling’s approximation into an approximation for \( \binom{n}{k} \). This approximation has a very important consequence in probability theory: it leads to the Limit Theorems, which we will meet at the end of the course.

For now, let’s just look at some pictures as a foretaste of those results.
For each $n$, we can plot the values of $\binom{n}{k}$ for $0 \leq k \leq n$ in a chart.
Two observations:

1. The bulk of this picture is becoming a tall, narrow spike around $k = n/2$. That is, there are many more ways to choose roughly half of the $n$ objects than, say, one quarter or three quarters of them.

At the end of the course, we will see that this fact has an important interpretation in probability. It is the Law of Large Numbers.
2. If you rescale, the shape of that spike is converging to a well-defined graph. This fact also turns into an important result in probability, called the *Normal Approximation* or *Central Limit Theorem*.

On the right is the graph of the function

\[ e^{-x^2/2} \]

... to be explained later.