HW 7 - ODE’s - Spring 2018

Due date: Tuesday, March 27th

Conjugated systems  Let us consider two autonomous ODE’s in $\mathbb{R}^N$

$$Y' = F(Y), \quad Z' = G(Z),$$

Let $\Phi, \Psi$ be the associated flows. We assume, for simplicity, that $F, G$ are globally Lipschitz on $\mathbb{R}^N$, so that the flows are defined everywhere and for all $t$.

Let $p \geq 0$ be an integer. We say$^1$ that the flows are $C^p$-conjugated if there exists a $C^p$-diffeomorphism$^2$ $h : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $t$ in $(-\infty, \infty)$

$$\Phi^t \circ h = h \circ \Psi^t. \quad (1)$$

1. Check that it is an equivalence relation between flows (reflexive, symmetric, transitive).

2. Show that if the flows are $C^0$-conjugated, then $h$ induces a bijection from the orbits of $\Psi$ onto the orbits of $\Phi$.

3. Show that if the flows are $C^1$-conjugated by the diffeomorphism $h$, we have for any $x$ in $\mathbb{R}^N$

$$(Dh)(x)G(x) = (F \circ h(x)), \quad (2)$$

where $(Dh)(x)$ is the Jacobian matrix of $h$ at $x$.

4. Now, we consider two linear ODE’s

$$Y' = AY, \quad Z' = BY.$$

Show that if $A, B$ are similar matrices then the associated flows are $C^p$-conjugated for every $p$.

5. Take $A, B$ to be

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Using the first result of last lecture (“straightening flows”), explain why the flows associated to $A$ and $B$ are locally $C^1$-conjugated away from the real axis, in the following sense: for any point $x_0$ that is not $(0,0)$, and for $t$ small enough, the relation (1) holds for a certain map $h$ defined around $x$.

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$^1$This might not be standard terminology.

$^2$We recall that a $C^p$-diffeomorphism is a bijection of class $C^p$, whose inverse bijection is also of class $C^p$. 
6. Explain why they cannot be locally $C^1$-conjugated around $(0,0)$. You can use the identity (2) (and differentiate it) to show that if the flows associated to $A, B$ are conjugated around $(0,0)$ then $A$ and $B$ must be similar.

7. Let us consider the ODE’s

\begin{align*}
\begin{cases}
x' &= 2x^2y^2 + \sin(x) \\
y' &= y^3 + 2y
\end{cases}, \quad \begin{cases}
x' &= \sin(x) \\
y' &= 2\sin(y)
\end{cases}
\end{align*}

Using Hartman-Grobman’s theorem, show that the flows of these ODE’s are locally $C^0$-conjugated around $(0,0)$.

**Taylor’s method** We study the scalar ODE $y' = f(t,y)$ under the assumption that the map $(t,x) \mapsto f(t,x)$ is of class $C^2$ on $(-\infty, \infty) \times \mathbb{R}$, and that its first and second partial derivatives are all uniformly bounded.

Say we look at the time interval $t \in [0, 1]$, with a given initial condition $y(0) = y^0$.

For $N \geq 1$, we consider the following numerical scheme: $y_{0,N}$ is fixed and for $0 \leq n < N$ we define

$$y_{n+1,N} := y_{n,N} + \frac{1}{N} f\left(\frac{n}{N}, y_{n,N}\right) + \frac{1}{2} \left(\frac{1}{N}\right)^2 f'\left(\frac{n}{N}, y_{n,N}\right).$$

(3)

This amounts to taking a fixed step-size $\frac{1}{N}$ and replacing the first-order approximation of “Explicit Euler” by a second-order approximation.

**Question:** Prove the convergence of this numerical method. In other words, assuming that we take $\lim_{N \to \infty} y_{0,N} = y^0$, show that the quantity

$$\max_{0 \leq n \leq N} \left| y_{n,N} - y\left(\frac{n}{N}\right) \right|$$

tends to zero as $N \to \infty$. What is the speed (order) of convergence? What if we assume $f$ to be $C^p$ and take a Taylor’s approximation of order $p$ in the numerical scheme (3)?

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3You may assume that $y_{0,N}$ is equal to $y^0$ if that makes your life simpler.