HW 6 - ODE’s - Spring 2018

Due date: Tuesday, March 6th

An autonomous system  We consider the system
\[
\begin{align*}
x'(t) &= y(t) + L \left( x(t)^2 + y(t)^2 - 1 \right) x(t) \\
y'(t) &= -x(t) + L \left( x(t)^2 + y(t)^2 - 1 \right) y(t),
\end{align*}
\]
where \( L \) is some constant.

1. Solve the ODE with initial condition \( x(0) = 0, y(0) = 1/2 \) when \( L = 0 \). Now \( L \) is supposed to be positive.

2. What are the stationary points?

3. Show that the unit disk \( \{ x^2 + y^2 \leq 1 \} \) is invariant under the flow (in both directions of the time).

4. What is the time of existence of maximal solutions? You may distinguish three cases depending on the value of \( x^2 + y^2 \) at the initial time. You may also look at the ODE satisfied by the quantity \( x^2 + y^2 \).

5. We denote by \( x_L, y_L \) the solution of the ODE with initial condition \( x(0) = 0, y(0) = 1/2 \). We assume that, when \( L \) is very small, \( x_L, y_L \) admit an expansion
\[
\begin{align*}
x_L(t) &= x_0(t) + L\bar{x}_0(t) + \ldots, & y_L &= y_0(t) + L\bar{y}_0(t) + \ldots,
\end{align*}
\]
where \( x_0, y_0 \) are the solutions found in the first question and the errors
\[
\begin{align*}
x_L(t) - (x_0(t) + L\bar{x}_0(t)), & \quad y_L - (y_0(t) + L\bar{y}_0(t))
\end{align*}
\]
are of order \( L^2 \). Find the unknown functions \( \bar{x}_0, \bar{y}_0 \) (your argument does not have to be rigorous).

Heteroclines trajectories  Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a \( C^1 \) function with a finite number of critical points. We consider the following autonomous ODE
\[
Y' = \nabla f(Y).
\]

1. Give at least one (non-trivial) condition on \( f \) which would ensure that all the maximal solutions are global, i.e. defined on \( (-\infty, +\infty) \). We will now assume that all the maximal solutions are global.
2. Show that there are no periodic orbits, except maybe the constant ones. _Hint:_ You can observe that some natural quantity is always monotone along the flow.

3. (_More difficult._) Show that the only bounded orbits are the constant ones and the ones that go from one critical point to another, called _heteroclines orbits._ More precisely, the orbit $O_x$ is an _heterocline orbit_ if

$$
\lim_{t \to -\infty} \Phi^t(x) = x_0, \quad \lim_{t \to +\infty} \Phi^t(x) = x_1,
$$

where $\Phi^t$ is the flow of the ODE and $x_0, x_1$ are two critical points of $f$. 