Testing the parameter of an exponential distribution Let $X_1, \ldots, X_n$ be independent random variables distributed according to the exponential distribution $f(x; \theta) := \theta e^{-\theta x}$. We believe that $\theta \geq 1$, this is our null hypothesis $H_0$, and the alternative hypothesis is $H_1: \{\theta < 1\}$.

Design a test of size 0.1. That is: build a test statistic, and an appropriate rejection region, and justify what you are doing.

Here is a data set with 15 data points. Do you keep or reject $H_0$? Give the $p$-value, and any comment that you consider relevant about the methodology.

$0.52, 0.33, 0.30, 3.27, 0.66, 0.20, 1.42, 0.49, 0.07, 0.09, 0.30, 0.84, 0.41, 0.62, 1.44$

Noise in the dictatorship Let us consider the “dictatorial plus noise” situation, where we have couples (feature $X$ is a real number, label $Y$ is a real number) and where the distribution of $Y$ knowing $X$ is given by

$$Y = d + \varepsilon \mathcal{N}(0, 1),$$

where $d$ is the “dictatorial constant”, $\varepsilon > 0$ and $\mathcal{N}(0, 1)$ represents a standard normal random variable.

1. For any real numbers $a$, and $b$, let $Z_{a,b}$ be a random variable with distribution $\mathcal{N}(a, b^2)$. Compute the following quantity

$$\mathbb{E} \left[ |Z_{a,b}|^2 \right]$$

2. Recall the two rules that we are using when we write

$$a + b\mathcal{N}(0, 1) = \mathcal{N}(a, b^2)$$

3. We choose the cost function $c$ as $c(x, y) = |x - y|^2$. Given a predictor $f: \mathbb{R} \to \mathbb{R}$, compute the risk

$$\mathbb{E} [c(f(X), Y)],$$

and show that it is minimal when $f$ is the constant function equal to the “dictatorial constant” $d$.

4. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a data set. Our learning rule will be to use the data set to get an estimator $\hat{d}_n$ of $d$, and then to predict $\hat{d}_n$ all the time. We take for $\hat{d}_n$ the empirical mean. Compute the average risk, i.e.

$$\mathbb{E} \left[ c(\hat{d}_n, Y) \right],$$

and compare it to the result of question 3.