1) It is non-negative, and we can check that
\[ \int_{-\infty}^{+\infty} f(x; \sigma) \, dx = \int_{-\infty}^{+\infty} \frac{1}{M} \sigma e^{-\sigma^4 x^4} \, dx \]

\[ = \int_{-\infty}^{+\infty} \frac{1}{M} e^{-u^4} \, du = \frac{M}{M} = 1, \] so \( f(x; \sigma) \) is a pdf.

2) Likelihood function
\[ L_n(\sigma) = \prod_{i=1}^{n} \frac{1}{M} \sigma e^{-x_i \sigma^4} \]

Log-likelihood
\[ \log L_n(\sigma) = -ny\sigma^4 + n \log \sigma \quad \sigma^4 \]

Derivative = 0:
\[ \frac{n}{\sigma^4} - 4 \sum_{i=1}^{n} \sigma^3 x_i = 0 \iff \sigma^4 = \frac{n}{4 \sum_{i=1}^{n} x_i^3} \]

So MLE
\[ \hat{\sigma} = \left( \frac{1}{\frac{1}{4} \sum_{i=1}^{n} x_i^3} \right) \]

3) Fisher information
\[ I_1(\sigma) = -\int_{-\infty}^{+\infty} \frac{\partial^2 \log f(x; \sigma)}{\partial \sigma^2} f(x; \sigma) \, dx \]

Compute first: \( \log f(x; \sigma) = -\log M + \log \sigma - \sigma^4 x^4 \)
\[ \frac{\partial}{\partial \sigma} \quad = \frac{1}{\sigma} - 4 \sigma^3 x^4 \]
\[ \frac{\partial^2}{\partial \sigma^2} \quad = -\frac{1}{\sigma^2} - 12 \sigma^2 x^4 \]
So \( I(\delta) = \int_{-\infty}^{\infty} \left( \frac{1}{\delta^2 + 12 \sigma^2 x^4} \right) \frac{1}{M} \ e^{-\frac{x^4}{M}} \ dx \)

\((U = \delta x)\) \[= \int_{-\infty}^{\infty} \left( \frac{1}{\delta^2 + \frac{12 u^4}{\delta^2}} \right) \frac{1}{M} \ e^{-\frac{u^4}{M}} \ du\]

\[= \frac{1}{\delta^2} \left( \int_{-\infty}^{\infty} \frac{12 u^6 e^{-\frac{u^4}{M}} \ du}{M} \right) \]

Some constant \(A\)

4] Likelihood \( L_n(\mu) = \frac{1}{M^n} \gamma^n e^{-\sigma^4 \sum_{i=1}^{n} (x_i - \mu)^4} \)

Log-likelhood \( \log L_n(\mu) = -n \log M + n \log \sigma - \sigma^4 \sum_{i=1}^{n} (x_i - \mu)^4 \)

Derivative = 0?

\[-\sigma^4 \cdot 4 \sum_{i=1}^{n} (x_i - \mu)^3 = 0 \]

So \( \sum_{i=1}^{n} (x_i - \mu)^3 = 0 \) is the "MLE equation," but not easy to solve analytically.

5] Same as above

\[ I_2(\mu) = - \int_{-\infty}^{\infty} \frac{\partial^2 \log g(x; \mu)}{\partial \mu^2} g(x; \mu) \ dx \]

\[= \int_{-\infty}^{\infty} \frac{12 \sigma^4 (x - \mu)^2}{M} \ e^{-\frac{x^4}{M}} \ dx \]

\( (U = \gamma (x - \mu)) \)

\[= \int_{-\infty}^{\infty} \frac{12 \sigma^4 (x - \mu)^2}{M} \ e^{-\frac{U^4}{M}} \ du \]

\(= \sigma^8 \cdot B \)

Where \( B = \frac{12}{M} \int_{-\infty}^{\infty} e^{-\frac{u^4}{M}} \ du \)

\( \) Some constant.
6) b) is the clearest one.

\[ \delta \approx 1 \]

\[ \delta \ll 1 \]

When \( \delta \) is large, the pdf is sharply peaked around \( \mu \),
when \( \delta \) is small, the pdf is "flatter."

So \( \delta \) large \( \rightarrow \) more "personality"
\( \rightarrow \) more information.

a) is simply the fact that the pdf for various \( \mu \) are obtained
by translating the graph, but they have the same shape,
here the same amount of "personality."

7) Likelihood

\[ f_n(\sigma) = \frac{1}{\mu^n} \frac{1}{\sigma^n} e^{-\frac{\sum x_i^4}{\sigma}} \]

\[ \log f_n(\sigma) = -n \log \mu - n \log \sigma - \frac{1}{\sigma^4} \sum x_i^4 \]

Derivative = 0?

\[ -\frac{n}{\sigma} + \frac{4}{\sigma^5} \sum x_i^4 = 0 \]

So \( \sigma = \left( \frac{4}{n} \sum x_i^4 \right)^{1/4} \)

8) "Equivariance"

Write \( \hat{\sigma}_n - \sigma \approx \sqrt{n} N(0, I(\sigma)). \)

9) Same computation as before, essentially

10) The Fisher information is indeed not equivariant

Assume

\[ \sigma = \Phi(\delta) \]

is a transformation of the parameters.

\[ \hat{\sigma}_n = \Phi(\hat{\delta}_n) \]

for MLE, because MLE is \( \Phi \)-invariant.