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Elliptic optimal control problems with L^1-control cost and applications for the placement of control devices

Georg Stadler

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Abstract Elliptic optimal control problems with L^1-control cost are analyzed. Due to the nonsmooth objective functional the optimal controls are identically zero on large parts of the control domain. For applications, in which one cannot put control devices (or actuators) all over the control domain, this provides information about where it is most efficient to put them. We analyze structural properties of L^1-control cost solutions. For solving the non-differentiable optimal control problem we propose a semismooth Newton method that can be stated and analyzed in function space and converges locally with a superlinear rate. Numerical tests on model problems show the usefulness of the approach for the location of control devices and the efficiency of our algorithm.

Keywords Optimal control · Nonsmooth regularization · Optimal actuator location · Placement of control devices · Semismooth Newton · Active set method

1 Introduction

In this paper, we analyze elliptic optimal control problems with L^1-control cost and argue their use for the placement of actuators (i.e., control devices). Due to the non-differentiability of the objective functional for L^1-control cost (in the sequel also called L^1-regularization), the structure of optimal controls differs significantly from what one obtains for the usual smooth regularization. If one cannot or does not want to distribute control devices all over the control domain, but wants to place available devices in an optimal way, the L^1-solution gives information about the optimal location of control devices. As model problems, we consider the following constrained

G. Stadler (✉)
Institute for Computational Engineering & Sciences, The University of Texas at Austin, Austin, TX 78712, USA
e-mail: georgst@ices.utexas.edu

elliptic optimal control problems with L^1-control cost.

$$\begin{cases} \text{minimize} & J(y,u) := \frac{1}{2}\|y-y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 + \beta\|u\|_{L^1} \\ & \text{over} \quad (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\ \text{subject to} & Ay = u + f \in \Omega, \\ & a \le u \le b \quad \text{almost everywhere in } \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega$, $y_d, f \in L^2(\Omega)$, $a, b \in L^2(\Omega)$ with $a < 0 < b$ almost everywhere and $\alpha, \beta > 0$. Moreover, $A : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ is a second-order linear elliptic differential operator, and $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^1}$ denote the $L^2(\Omega)$ and $L^1(\Omega)$-norm, respectively. In the sequel, y is called state and u the control variable, and y_d is referred to as desired stated. Note that the novelty in the above problem is the introduction of the L^1-regularization term $\beta\|u\|_{L^1}$.

Nonsmooth regularization for PDE-constrained optimization has mainly been used for inverse problems, see e.g., [2, 5, 26, 28, 33]. In particular, the use of the L^1-norm of the gradient as regularization has led to better results for the recovery of data from noisy measurements than smooth regularization. As mentioned above, our main motivation for the use of nonsmooth regularization for optimal control problems is a different one, namely its ability to provide information about the optimal location of control devices and actuators. Although intuition and experience might help in this design issue, this approach fails when prior experience is lacking or the physical system modelled by the PDE is too complex. Provided only a finite number of control locations is possible, one might use a discrete method for the location problem, but clearly the number of possible configurations grows combinatorially as the number of devices or the number of possible locations increase. To overcome these problems, we propose the use of a L^1-norm control cost. As will be shown in this paper, this results in optimal controls that are identically zero in regions where they are not able to decrease the cost functional significantly (this significance is controlled by the size of $\beta > 0$); we may think of these sets as sets where no control devices need to be put. By these means, using the nonsmooth L^1-regularization term (even if in combination with the squared L^2-norm such as in ($\mathcal{P}$)), one can treat the somewhat discrete-type question of *where* to place control devices and actuators.

An application problem that has partly motivated this research is the optimal placement of actuators on piezoelectric plates [8, 12]. Here, engineers want to know where to put electrodes in order to achieve a certain displacement of the plate. For linear material laws, this problem fits into the framework of our model problem ($\mathcal{P}$). Obviously, there are many other applications in which similar problems arise.

Additional motivation for considering ($\mathcal{P}$) is due to the fact that in certain applications the L^1-norm has a more interesting physical interpretation than the squared L^2-norm. For instance, the total fuel consumption of vehicles corresponds to a L^1-norm term, see [34]. We remark that the L^1-term $\|u\|_{L^1}$ is nothing else than the $L^1(\Omega)$-norm of u, while the L^2-term $\|u\|_{L^2}^2$ (which is the *squared* L^2-norm) is not a norm.

As mentioned above, the use of nonsmooth functionals in PDE-constrained optimization is not standard and has mainly been used in the context of edge-preserving image processing (see [28, 33]) and other inverse problems where nonsmooth data have to be recovered (see e.g., [2, 5]). Interesting comparisons of the properties of various nonsmooth regularization terms in finite and infinite dimensions can be found in [10, 25, 26] and [27]. One of the few contributions using L^1-regularization in optimal control is [34]. Here, a free-flying robot whose dynamical behavior is governed by a system of nonlinear ordinary differential equations is navigated to a given final state. The optimal control is characterized as minimizer of an L^1-functional, which corresponds to the total fuel consumption. Finally, we mention the paper [15] that deals with elliptic optimal control problems with supremum-norm functional.

Clearly, the usage of a nonsmooth cost functional introduces severe difficulties into the problem, both theoretically as well as for a numerical algorithm. As mentioned above, a solution of ($\mathcal{P}$) with $\beta > 0$ obeys properties significantly different from the classical elliptic optimal control model problem

$$\begin{cases} \text{minimize} & J_2(y,u) := \frac{1}{2}\|y - y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L_2}^2 \\ & \text{over} \quad (y,u) \in H_0^1(\Omega) \times L^2(\Omega) \\ \text{subject to} & Ay = u + f \in \Omega, \\ & a \le u \le b \quad \text{almost everywhere in } \Omega, \end{cases} \tag{$\mathcal{P}_2$}$$

with $\alpha > 0$. One aim of this paper is to compare the structure of solutions of ($\mathcal{P}$) to those of ($\mathcal{P}_2$) and to explore their different properties. Moreover, we propose and analyze an algorithm for the efficient solution of ($\mathcal{P}$).

Clearly, setting $\alpha := 0$ in ($\mathcal{P}$) results in the problem

$$\begin{cases} \text{minimize} & J_1(y,u) := \frac{1}{2}\|y - y_d\|_{L^2}^2 + \beta\|u\|_{L^1} \\ & \text{over} \quad (y,u) \in W_0^{1,1}(\Omega) \times L^1(\Omega) \\ \text{subject to} & Ay = u + f \in \Omega, \\ & a \le u \le b \quad \text{almost everywhere in } \Omega. \end{cases} \tag{$\mathcal{P}_1$}$$

Now, the optimal control has to be searched for in the larger space $L^1(\Omega)$. The smoothing property of the elliptic operator A guarantees that the state y corresponding to $u \in L^1(\Omega)$ is an element in $L^2(\Omega)$, provided $n \le 4$. However, for the above problem to have a solution, the inequality constraints on the control are essential. In absence of (one of) the box constraints on u, ($\mathcal{P}_1$) may or may not have a solution. This is due to the fact that $L^1(\Omega)$ is not a reflexive function space.

As a remedy for the difficulties that arise for ($\mathcal{P}_1$), in the sequel we focus on ($\mathcal{P}$) with small $\alpha > 0$. Note that whenever $\beta > 0$, the cost functional in ($\mathcal{P}$) obeys a kink at points where $u = 0$ independently from $\alpha \ge 0$. In particular, $\alpha > 0$ does not regularize the non-differentiability of the functional $J(\cdot,\cdot)$. However, α influences the regularity of the solution and also plays an important role for the analysis of the numerical method we use to solve ($\mathcal{P}$). This algorithm is based on the combination of semismooth Newton methods, a condensation of Lagrange multipliers and certain complementarity functions. It is related to the dual active set method [16, 17, 19] and the primal-dual active set method [3, 20]. Its fast local convergence can be proven in function space, which allows certain statements about the algorithm's dependence (or independence) of the fineness of the discretization [22]. We remark that the analysis of algorithms for PDE-constrained optimization in function space has recently gained a considerably amount of attention; we refer for instance to [20, 21, 30–32, 35, 36].

This paper is organized as follows. In the next section, we derive necessary optimality conditions for ($\mathcal{P}$) using Lagrange multipliers. In Sect. 3, we study structural properties of solutions of ($\mathcal{P}$). The algorithm we propose for solving elliptic optimal control problems with L^1-control cost is presented and analyzed in Sect. 4. Finally, in the concluding section, we report on numerical tests, where we discuss structural properties of the solutions as well as the performance of our algorithms.

2 First-order optimality system

In this section, we derive first-order necessary optimality conditions for ($\mathcal{P}$). For that purpose, we replace ($\mathcal{P}$) by a reduced problem formulation. This reduction to a problem that involves the control variable u only is possible due to the existence of the inverse $A^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ of the differential operator A. The reduced problem is

$$\begin{cases} \text{minimize } \hat{J}(u) := \frac{1}{2}\|A^{-1}u + A^{-1}f - y_d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_{L^2}^2 + \beta\|u\|_{L^1} \\ \qquad\qquad \text{over} \quad u \in U_{ad} := \{u \in L^2(\Omega) : a \le u \le b \text{ a.e. in } \Omega\}. \end{cases} \qquad (\hat{\mathcal{P}})$$

This is a convex optimization problem posed in the Hilbert space $L^2(\Omega)$. Its unique solvability follows from standard arguments [14, 30], and its solution $\bar{u} \in U_{ad}$ is characterized (see e.g., [7, 11, 23]) by the variational inequality

$$\big(A^{-\star}(A^{-1}\bar{u} + A^{-1}f - y_d) + \alpha\bar{u}, u - \bar{u}\big) + \varphi(u) - \varphi(\bar{u}) \ge 0 \quad \text{for all } u \in U_{ad}, \quad (2.1)$$

where $A^{-\star}$ denotes the inverse of the transposed operator, i.e., $A^{-\star} = (A^\star)^{-1}$; moreover, $\varphi(v) := \beta \int |v(x)|\,dx = \beta\|v\|_{L^1}$. In the sequel, we denote by $\partial\varphi(\bar{u})$ the subdifferential of φ at $\bar{u}$, i.e., $\partial\varphi(\bar{u}) = \{u \in L^2(\Omega) : \varphi(v) - \varphi(\bar{u}) \ge (u, v - \bar{u}) \text{ for all } v \in L^2(\Omega)\}$. It follows from results in convex analysis that, for $\bar{\lambda} \in \partial\varphi(\bar{u})$, (2.1) implies

$$\big(A^{-\star}(A^{-1}\bar{u} + A^{-1}f - y_d) + \alpha\bar{u} + \bar{\lambda}, u - \bar{u}\big) \ge 0 \quad \text{for all } u \in U_{ad}. \qquad (2.2)$$

The differential inclusion $\bar{\lambda} \in \partial\varphi(\bar{u})$ yields, in particular, that

$$\bar{\lambda} \in \Lambda_{ad} := \{\lambda \in L^2(\Omega) : |\lambda| \le \beta \text{ a.e. in } \Omega\}. \qquad (2.3)$$

A pointwise (almost everywhere) discussion of the variational inequality (2.2) such as in [30, p. 57] allows to show that there exist nonnegative functions $\bar{\lambda}_a, \bar{\lambda}_b \in L^2(\Omega)$ that act as Lagrange multipliers for the inequality constraints in U_{ad}. Moreover, evaluating the differential inclusion $\bar{\lambda} \in \partial\varphi(\bar{u})$ relates $\bar{\lambda}$ to the sign of $\bar{u}$ (see also [11, 23, 29]). This leads to the optimality system for the reduced problem ($\hat{\mathcal{P}}$) summarized in the next theorem.

Theorem 2.1 *The optimal solution* $\bar{u}$ *of* ($\hat{\mathcal{P}}$) *is characterized by the existence of* $(\bar{\lambda}, \bar{\lambda}_a, \bar{\lambda}_b) \in \Lambda_{ad} \times L^2(\Omega) \times L^2(\Omega)$ *such that*

$$A^{-\star}(A^{-1}\bar{u} + A^{-1}f - y_d) + \alpha\bar{u} + \bar{\lambda} + \bar{\lambda}_b - \bar{\lambda}_a = 0, \tag{2.4a}$$

$$\bar{\lambda}_b \geq 0, \quad b - \bar{u} \geq 0, \quad \bar{\lambda}_b(b - \bar{u}) = 0, \tag{2.4b}$$

$$\bar{\lambda}_a \geq 0, \quad \bar{u} - a \geq 0, \quad \bar{\lambda}_a(\bar{u} - a) = 0, \tag{2.4c}$$

$$\bar{\lambda} = \beta \quad \textit{a.e. on } \{x \in \Omega : \bar{u} > 0\}, \tag{2.4d}$$

$$|\bar{\lambda}| \leq \beta \quad \textit{a.e. on } \{x \in \Omega : \bar{u} = 0\}, \tag{2.4e}$$

$$\bar{\lambda} = -\beta \quad \textit{a.e. on } \{x \in \Omega : \bar{u} < 0\}. \tag{2.4f}$$

Above, (2.4b–2.4c) are the complementarity conditions for the inequality constraints in U_{ad}. Moreover, $\bar{\lambda} \in \Lambda_{ad}$ together with (2.4d–2.4f) is an equivalent expression for $\bar{\lambda} \in \partial\varphi(\bar{u})$.

Next, we derive an optimality system for ($\mathcal{P}$) using (2.4), i.e., the optimality conditions for ($\hat{\mathcal{P}}$). We introduce the adjoint variable $\bar{p}$ by

$$\bar{p} := -A^{-\star}(A^{-1}\bar{u} + A^{-1}f - y_d). \tag{2.5}$$

Then, (2.4a) becomes

$$-\bar{p} + \alpha\bar{u} + \bar{\lambda} + \bar{\lambda}_b - \bar{\lambda}_a = 0. \tag{2.6}$$

Applying the operator $A^\star$ to (2.5) and using the state variable $\bar{y} := A^{-1}(\bar{u} + f)$, we obtain the adjoint equation

$$A^\star\bar{p} = y_d - \bar{y}. \tag{2.7}$$

Next, we study the complementarity conditions (2.4b–2.4f). Surprisingly, it will turn out that we can write these conditions in a very compact form, namely as one (nondifferentiable) operator equation. To do so, first we condense the Lagrange multipliers $\bar{\lambda}$, $\bar{\lambda}_a$ and $\bar{\lambda}_b$ into one multiplier

$$\bar{\mu} := \bar{\lambda} - \bar{\lambda}_a + \bar{\lambda}_b. \tag{2.8}$$

Due to $a < 0 < b$ almost everywhere, this condensation can be reversed, i.e., for $\bar{\mu}$ given, $\bar{\lambda}$, $\bar{\lambda}_a$ and $\bar{\lambda}_b$ can be calculated from

$$\begin{cases} \bar{\lambda} = \min(\beta, \max(-\beta, \bar{\mu})), \\ \bar{\lambda}_a = -\min(0, \bar{\mu} + \beta), \\ \bar{\lambda}_b = \max(0, \bar{\mu} - \beta). \end{cases} \tag{2.9}$$

The condensation (2.8) allows to reformulate the system (2.4b–2.4f) together with $\bar{\lambda} \in \Lambda_{ad}$ using, for $c > 0$ the nonsmooth equation

$$\begin{aligned} C(\bar{u}, \bar{\mu}) := \bar{u} &- \max(0, \bar{u} + c(\bar{\mu} - \beta)) - \min(0, \bar{u} + c(\bar{\mu} + \beta)) \\ &+ \max(0, (\bar{u} - b) + c(\bar{\mu} - \beta)) \\ &+ \min(0, (\bar{u} - a) + c(\bar{\mu} + \beta)) = 0. \end{aligned} \tag{2.10}$$

Above, the min- and max-functions are to be understood pointwise. In the next lemma we clarify the relationship between (2.10) and (2.4b–2.4f).

Lemma 2.2 *For* $(\bar{u}, \bar{\lambda}, \bar{\lambda}_a, \bar{\lambda}_b) \in (L^2(\Omega))^4$, *the following two statements are equivalent*:

1. *The quadruple* $(\bar{u}, \bar{\lambda}, \bar{\lambda}_a, \bar{\lambda}_b)$ *satisfies the conditions* (2.4*b*–2.4*f*), *and* $\bar{\lambda} \in \Lambda_{ad}$.
2. *There exists a function* $\bar{\mu} \in L^2(\Omega)$ *such that* $(\bar{u}, \bar{\mu})$ *satisfies* (2.10) *and* $\bar{\lambda}, \bar{\lambda}_a, \bar{\lambda}_b$ *can be derived using* (2.9).

Proof We give an explicit prove of the above equivalence, and start with $1 \Rightarrow 2$. We set $\bar{\mu} := \bar{\lambda} + \bar{\lambda}_b - \bar{\lambda}_a$ and need to show that $(\bar{u}, \bar{\mu})$ satisfies $C(\bar{u}, \bar{\mu}) = 0$. To do so, we separately discuss subsets of Ω where $\mu(x) > \beta$, $\mu(x) = \beta$, $|\mu(x)| < \beta$, $\mu(x) = -\beta$ and $\mu(x) < -\beta$. The argumentation below is to be understood in a pointwise almost everywhere sense.

- $\bar{\mu} > \beta$: From (2.9) follows that $\bar{\lambda} = \beta$, $\bar{\lambda}_a = 0$ and $\bar{\lambda}_b > 0$. Thus, from (2.4b) we have $\bar{u} = b$. Therefore,

$$C(\bar{u}, \bar{\mu}) = \bar{u} - (\bar{u} + c(\bar{\mu} - \beta)) + ((\bar{u} - b) + c(\bar{\mu} - \beta)) = 0.$$

- $\bar{\mu} = \beta$: It follows from (2.9) that $\bar{\lambda} = \beta$ and $\bar{\lambda}_a = \bar{\lambda}_b = 0$. The conditions (2.4b) and (2.4d) imply that $0 \le \bar{u} \le b$ and thus

$$C(\bar{u}, \bar{\mu}) = \bar{u} - (\bar{u} + c(\bar{\mu} - \beta)) = 0.$$

- $|\bar{\mu}| < \beta$: In this case, $\bar{\lambda} = \bar{\mu}$ and $\bar{\lambda}_a = \bar{\lambda}_b = 0$. From (2.4e) we obtain $\bar{u} = 0$ and $C(\bar{u}, \bar{\mu}) = 0$ is trivially satisfied.
- The verification of $C(\bar{u}, \bar{\mu}) = 0$ for the two remaining sets where $\mu = -\beta$ or $\mu < -\beta$ is similar to the cases $\mu = \beta$ and $\mu > \beta$.

This ends the first part of the proof. Now, we turn to the implication $2 \Rightarrow 1$. We suppose given $(\bar{u}, \bar{\mu}) \in (L^2(\Omega))^2$ that satisfy $C(\bar{u}, \bar{\mu}) = 0$ and derive $\bar{\lambda}, \bar{\lambda}_a$ and $\bar{\lambda}_b$ from (2.9). By definition, it follows that $\bar{\lambda} \in \Lambda_{ad}$ and that $\bar{\mu} = \bar{\lambda} - \bar{\lambda}_a + \bar{\lambda}_b$ holds. To prove the conditions (2.4b–2.4f), we again distinguish different cases:

- $\bar{u} + c(\bar{\mu} - \beta) > b$: In this case, only the two max-terms in (2.10) contribute to the sum. We obtain $0 = C(\bar{u}, \bar{\mu}) = \bar{u} - b$ and thus $\bar{u} = b$. From $\bar{u} + c(\bar{\mu} - \beta) > b$ follows $\bar{\mu} > \beta$, which implies that $\bar{\lambda} = \beta$, $\bar{\lambda}_b > 0$ and $\bar{\lambda}_a = 0$ and the conditions (2.4b–2.4f) are satisfied.
- $0 < \bar{u} + c(\bar{\mu} - \beta) \le b$: Here, only the first max-term in (2.10) is different from zero. Hence, $0 = C(\bar{u}, \bar{\mu}) = \bar{\mu} - \beta$. This implies $\bar{\mu} = \beta$ and $0 < \bar{u} \le b$. Clearly, $\bar{\lambda} = \beta$ and $\bar{\lambda}_a = \bar{\lambda}_b = 0$, and again (2.4b–2.4f) hold.
- $|\bar{u} + c\bar{\mu}| \le c\beta$: In this case, all the max and min-terms in (2.10) vanish, which implies that $\bar{u} = 0$. This shows that $|\bar{\mu}| \le \beta$, and thus $\bar{\lambda} = \bar{\mu}$ and $\bar{\lambda}_a = \bar{\lambda}_b = 0$, which proves (2.4b–2.4f).
- The remaining two cases $\bar{u} + c(\bar{\mu} + \beta) < a$ and $a \le \bar{u} + c(\bar{\mu} + \beta) < 0$ are analogous to the first two cases discussed above.

Since for every point of Ω exactly one of the above five conditions holds, this finishes the proof of the implication $2 \Rightarrow 1$ and ends the proof. $\square$

In the next theorem we summarize the first-order optimality conditions for ($\mathcal{P}$).

Theorem 2.3 *The solution* $(\bar{y}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)$ *of* ($\mathcal{P}$) *is characterized by the existence of* $(\bar{p}, \bar{\mu}) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that*

$$A\bar{y} - \bar{u} - f = 0, \tag{2.11a}$$

$$A^\star p + \bar{y} - y_d = 0, \tag{2.11b}$$

$$-\bar{p} + \alpha\bar{u} + \bar{\mu} = 0, \tag{2.11c}$$

$$\begin{aligned} &\bar{u} - \max(0, \bar{u} + c(\bar{\mu} - \beta)) - \min(0, \bar{u} + c(\bar{\mu} + \beta)) \\ &\quad + \max(0, (\bar{u} - b) + c(\bar{\mu} - \beta)) + \min(0, (\bar{u} - a) + c(\bar{\mu} + \beta)) = 0, \end{aligned} \tag{2.11d}$$

with $c > 0$.

Note that, from (2.11) one obtains an optimality system for ($\mathcal{P}_2$) simply by setting $\beta = 0$ in (2.11): While the equations (2.11a–2.11c) remain unchanged, (2.11d) becomes

$$\bar{\mu} + \max(0, \bar{\mu} + c^{-1}(\bar{u} - b)) + \min(0, \bar{\mu} + c^{-1}(\bar{u} - a)) = 0.$$

This formulation has been used for the construction of an algorithm for bilaterally control constraint optimal control problems of the form ($\mathcal{P}_2$), see [24].

3 Properties of solutions of ($\mathcal{P}$)

This section is concerned with analyzing structural properties of solutions of ($\mathcal{P}$) and with comparing them to solutions of ($\mathcal{P}_2$). For simplicity of the presentation, in this section we dismiss the control constraints in ($\mathcal{P}$) and ($\mathcal{P}_2$), i.e., we choose $a := -\infty$ and $b := \infty$ and thus $U_{ad} = L^2(\Omega)$. We think of $\alpha > 0$ being fixed and study the dependence of the optimal control on β. To emphasize this dependence, in the rest of this section we denote the solution of ($\mathcal{P}$) by $(\bar{y}_\beta, \bar{u}_\beta)$ and the corresponding dual variables by $(\bar{p}_\beta, \bar{\mu}_\beta)$. Using these assumptions, (2.11) becomes

$$A\bar{y}_\beta - \bar{u}_\beta - f = 0, \tag{3.1a}$$

$$A^\star \bar{p}_\beta + \bar{y}_\beta - y_d = 0, \tag{3.1b}$$

$$-\bar{\mu}_\beta + \bar{p}_\beta - \alpha\bar{u}_\beta = 0, \tag{3.1c}$$

$$\bar{u}_\beta - \max(0, \bar{u}_\beta + \alpha^{-1}(\bar{\mu}_\beta - \beta)) - \min(0, \bar{u}_\beta + \alpha^{-1}(\bar{\mu}_\beta + \beta)) = 0. \tag{3.1d}$$

For later use, we remark that using (3.1c) we can replace $\bar{\mu}_\beta$ in (3.1d). The choice $c := \alpha^{-1}$ then leads to

$$\bar{u}_\beta - \max(0, \alpha^{-1}(\bar{p}_\beta - \beta)) - \min(0, \alpha^{-1}(\bar{p}_\beta + \beta)) = 0 \tag{3.2}$$

as an equivalent expression for (3.1c) and (3.1d). The first lemma states that, if β is sufficiently large, the optimal control is $\bar{u}_\beta \equiv 0$.

Lemma 3.1 *If* $\beta \geq \beta_0 := \|A^{-\star}(y_d - A^{-1}f)\|_{L^\infty}$, *the unique solution of* $(\mathcal{P})$ *is* $(\bar{y}_\beta, \bar{u}_\beta) = (A^{-1}f, 0)$.

Proof For the proof we use the reduced form $(\hat{\mathcal{P}})$ of $(\mathcal{P})$. For arbitrary $u \in L^2(\Omega)$ we consider

$$\begin{aligned}\hat{J}(u) - \hat{J}(0) &= \frac{1}{2}\|A^{-1}u\|_{L^2}^2 - (y_d - A^{-1}f, A^{-1}u)_{L^2} + \beta\|u\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ &\geq \frac{1}{2}\|A^{-1}u\|_{L^2}^2 - \|u\|_{L^1}\|A^{-\star}(y_d - A^{-1}f)\|_{L^\infty} + \beta\|u\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2 \\ &= \frac{1}{2}\|A^{-1}u\|_{L^2}^2 + (\beta - \beta_0)\,\|u\|_{L^1} + \frac{\alpha}{2}\|u\|_{L^2}^2.\end{aligned}$$

Clearly, the latter expression is nonnegative if $\beta \geq \beta_0$. Thus, for $\beta \geq \beta_0$, $\hat{J}(u) - \hat{J}(0) \geq 0$ for all $u \in U_{ad}$, which proves that the optimal control is $\bar{u}_\beta \equiv 0$. Using (2.11a) the corresponding state is obtained as $A^{-1}f$. □

An analogous result with respect to the parameter α in $(\mathcal{P}_2)$ does not hold, that is, in general optimal controls for $(\mathcal{P}_2)$ will only approach zero as α tends to infinity. Lemma 3.1 is also a consequence of the fact that the L^1-term in the objective functional can be seen as exact penalization (see e.g., [4]) for the constraint $u = 0$.

To gain more insight in the structure of solutions of $(\mathcal{P})$ and in the role of the cost weight parameters α and β, we next discuss the behavior of $\bar{u}_\beta$ as β changes (while $\alpha > 0$ is kept fixed), i.e., we study the solution mapping

$$\Phi : [0, \infty) \to L^2(\Omega), \quad \Phi(\beta) := \bar{u}_\beta.$$

We first focus on continuity properties of Φ.

Lemma 3.2 *The mapping* Φ *is Lipschitz continuous.*

Proof Let $\beta, \beta' \geq 0$ and denote the solution variables corresponding to β and β' by $(\bar{y}_\beta, \bar{u}_\beta, \bar{p}_\beta, \bar{\mu}_\beta)$ and $(\bar{y}_{\beta'}, \bar{u}_{\beta'}, \bar{p}_{\beta'}, \bar{\mu}_{\beta'})$, respectively. From (2.11a–2.11c) we obtain

$$A^{-\star}A^{-1}u + \alpha u - A^{-\star}y_d + A^{-\star}A^{-1}f + \mu = 0 \tag{3.3}$$

for both $(u, \mu) = (\bar{u}_\beta, \bar{\mu}_\beta)$ and $(u, \mu) = (\bar{u}_{\beta'}, \bar{\mu}_{\beta'})$. Deriving the difference between these two equations and taking the inner product with $\bar{u}_\beta - \bar{u}_{\beta'}$ results in

$$\|A^{-1}(\bar{u}_\beta - \bar{u}_{\beta'})\|_{L^2}^2 + \alpha\|(\bar{u}_\beta - \bar{u}_{\beta'})\|_{L^2}^2 = \left(\bar{\mu}_\beta - \bar{\mu}_{\beta'}, \bar{u}_{\beta'} - \bar{u}_\beta\right)_{L^2}. \tag{3.4}$$

We now estimate the right hand side of (3.4) pointwise (almost everywhere). From the complementarity conditions, we deduce that the following cases can occur:

- $\bar{\mu}_\beta = \beta, \bar{u}_\beta \geq 0, \bar{\mu}_{\beta'} = \beta', \bar{u}_{\beta'} \geq 0$: Here, we obtain

$$(\bar{\mu}_\beta - \bar{\mu}_{\beta'})(\bar{u}_{\beta'} - \bar{u}_\beta) = (\beta - \beta')(\bar{u}_{\beta'} - \bar{u}_\beta) \leq |\beta - \beta'||\bar{u}_{\beta'} - \bar{u}_\beta|.$$

- $\bar{\mu}_\beta = \beta, \bar{u}_\beta \geq 0, |\bar{\mu}_{\beta'}| < \beta', \bar{u}_{\beta'} = 0$: In this case, we find $\bar{u}_{\beta'} - \bar{u}_\beta = -\bar{u}_\beta \leq 0$ and thus

$$(\bar{\mu}_\beta - \bar{\mu}_{\beta'})(\bar{u}_{\beta'} - \bar{u}_\beta) \leq (\beta - \beta')(\bar{u}_{\beta'} - \bar{u}_\beta) \leq |\beta - \beta'||\bar{u}_{\beta'} - \bar{u}_\beta|.$$

- $\bar{\mu}_\beta = \beta, \bar{u}_\beta \geq 0, \bar{\mu}_{\beta'} = -\beta', \bar{u}_{\beta'} \leq 0$: From the sign structure of the variables one obtains the estimate $(\bar{\mu}_\beta - \bar{\mu}_{\beta'})(\bar{u}_{\beta'} - \bar{u}_\beta) \leq 0$.
- $|\bar{\mu}_\beta| < \beta, \bar{u}_\beta = 0, |\bar{\mu}_{\beta'}| < \beta', \bar{u}_{\beta'} = 0$: Here, trivially $(\bar{\mu}_\beta - \bar{\mu}_{\beta'})(\bar{u}_{\beta'} - \bar{u}_\beta) = 0$ holds.
- There are five more cases that can occur. Since they are very similar to those above, their discussion is skipped here and we only remark that in all remaining cases the pointwise estimate

$$(\bar{\mu}_\beta - \bar{\mu}_{\beta'})(\bar{u}_{\beta'} - \bar{u}_\beta) \leq |\beta - \beta'||\bar{u}_{\beta'} - \bar{u}_\beta|$$

holds as well. Taking the L^2-norm over Ω, this results in

$$\left(\bar{\mu}_\beta - \bar{\mu}_{\beta'}, \bar{u}_{\beta'} - \bar{u}_\beta\right)_{L^2} \leq |\Omega|^{1/2}|\beta - \beta'|\|\bar{u}_{\beta'} - \bar{u}_\beta\|_{L^2}. \tag{3.5}$$

Combing (3.4) with (3.5) yields

$$\|\bar{u}_{\beta'} - \bar{u}_\beta\|_{L^2} \leq \frac{1}{\alpha}|\Omega|^{1/2}|\beta - \beta'|,$$

and thus proves Lipschitz continuity of Φ. □

Clearly, from the above lemma we get $L^2(\Omega)$-boundedness of the sequence

$$\frac{1}{\beta - \beta'}(\bar{u}_\beta - \bar{u}_{\beta'}) \quad \text{as } \beta' \to \beta. \tag{3.6}$$

Let $\dot{u}_\beta$ denote a weak limit of a subsequence. We now show that, for almost all β,

$$\frac{1}{\beta - \beta'}(\bar{u}_\beta - \bar{u}_{\beta'}) \rightharpoonup \dot{u}_\beta \quad \text{weakly in } L^2(\Omega). \tag{3.7}$$

Let $\{v_k : k = 1, 2, \ldots\}$ be a countable dense subset of $L^2(\Omega)$. Then, for all k, the maps $\varphi_k : \beta \mapsto (v_k, \bar{u}_\beta)$ are Lipschitz continuous and due to Rademacher's theorem differentiable almost everywhere. Since a countable union of null sets is again a null set, all the maps φ_k are simultaneously differentiable almost everywhere, and, from (3.7) their derivative is $\varphi_k' : \beta \mapsto (v_k, \dot{u}_\beta)$. The density of $\{v_k : k = 1, 2, \ldots\}$ implies that (3.7) holds for almost all β.

Let us take $\beta > 0$ such that (3.7) holds. As $\beta' \to \beta$, it follows from (3.1a) that $(\bar{y}_\beta - \bar{y}_{\beta'})/(\beta - \beta') \to \dot{y}_\beta \in H_0^1(\Omega)$ weakly in $H_0^1(\Omega)$ and thus strongly in $L^2(\Omega)$.

Analogously, due to (3.1b) the same holds true for $(\bar{p}_\beta - \bar{p}_{\beta'})/(\beta - \beta') \to \dot{p}_\beta \in H_0^1(\Omega)$. We now introduce the functions

$$g^-(\beta) := \frac{1}{\alpha}(\bar{p}_\beta + \beta), \qquad g^+(\beta) := \frac{1}{\alpha}(\bar{p}_\beta - \beta). \tag{3.8}$$

Then,

$$\begin{aligned} \frac{1}{\beta - \beta'}(g^-(\beta) - g^-(\beta')) &\to \frac{1}{\alpha}\left(\dot{p}_\beta + 1\right) =: \dot{g}^-(\beta), \\ \frac{1}{\beta - \beta'}(g^+(\beta) - g^+(\beta')) &\to \frac{1}{\alpha}\left(\dot{p}_\beta - 1\right) =: \dot{g}^+(\beta) \end{aligned}$$

both strongly in $L^2(\Omega)$. Let us define the disjoint sets

$$\begin{aligned} \mathcal{S}_\beta^+ &= \{x \in \Omega : g^+(\beta) > 0 \text{ or } (g^+(\beta) = 0 \wedge \dot{g}^+(\beta) \geq 0)\}, \\ \mathcal{S}_\beta^- &= \{x \in \Omega : g^-(\beta) < 0 \text{ or } (g^-(\beta) = 0 \wedge \dot{g}^-(\beta) \leq 0)\} \end{aligned}$$

and denote by $\chi_\mathcal{S}$ the characteristic function for a set $\mathcal{S} \subset \Omega$. Using (3.8) and separately arguing for sets with $g^+(\beta) > 0$, $g^+(\beta) = 0$ and $g^+(\beta) < 0$, it follows that

$$\begin{aligned} \int_\Omega \frac{1}{\beta - \beta'}\left(\max(0, g^+(\beta)) - \max(0, g^+(\beta'))\right) v\, dx &\to \int_\Omega \frac{1}{\alpha}\left(\dot{p}_\beta - 1\right) v \chi_{\mathcal{S}_\beta^+}\, dx, \\ \int_\Omega \frac{1}{\beta - \beta'}\left(\min(0, g^-(\beta)) - \max(0, g^-(\beta'))\right) v\, dx &\to \int_\Omega \frac{1}{\alpha}\left(\dot{p}_\beta + 1\right) \chi_{\mathcal{S}_\beta^-} v\, dx \end{aligned}$$

for $v \in H_0^1(\Omega)$. Hence, from (3.2) we obtain

$$\dot{u}_\beta - \frac{1}{\alpha}\left(\dot{p}_\beta - 1\right)\chi_{\mathcal{S}_\beta^+} - \frac{1}{\alpha}\left(\dot{p}_\beta + 1\right)\chi_{\mathcal{S}_\beta^-} = 0, \tag{3.10}$$

resulting in

$$\dot{u}_\beta = \begin{cases} 0 & \text{a.e. on } \Omega \setminus (\mathcal{S}_\beta^- \cup \mathcal{S}_\beta^+), \\ \frac{1}{\alpha}(\dot{p}_\beta - \chi_{\mathcal{S}_\beta^+} + \chi_{\mathcal{S}_\beta^-}) & \text{a.e. on } \mathcal{S}_\beta^- \cup \mathcal{S}_\beta^+. \end{cases} \tag{3.11}$$

Note that the disjoint sets $\mathcal{S}_\beta^-, \mathcal{S}_\beta^+$ depend on $\dot{p}_\beta$. However, if $\bar{p}_\beta = 0$ only on a set of measure zero, this dependence can be neglected and $\mathcal{S}_\beta^-, \mathcal{S}_\beta^+$ can be derived as soon as $\bar{p}_\beta$ is available. Even then, calculating $\dot{u}_\beta$ requires to solve a linear system on Ω and not only on $\mathcal{S}_\beta^- \cup \mathcal{S}_\beta^+$, since $\dot{p}_\beta = A^{-\star}A^{-1}\dot{u}_\beta$.

For the reader's convenience, we conclude this section with a summary of the results obtained above. These properties are reflected in our numerical findings in Sect. 5, where we also discuss how the choice of β influences the attainability of the desired state.

- For β sufficiently large, (3.1) has the trivial solution $\bar{u}_\beta = 0$. It can be seen easily that the same is true in the presence of control bounds as in ($\mathcal{P}$).

- The solution mapping $\Phi : \beta \to \bar{u}_\beta$ is Lipschitz continuous and almost everywhere differentiable. The derivatives $\dot{u}_\beta$ have jumps along the boundaries of the sets $\mathcal{S}_\beta^-$, $\mathcal{S}_\beta^+$. The height of these jumps tends to grow as α decreases, see (3.11).
- For $n \leq 3$, parts of Ω where $\bar{u}_\beta$ changes sign are separated by regions with positive measure, in which $\bar{u}_\beta$ is identically zero. This follows from standard regularity results for elliptic equations [13], which show that $\bar{p}_\beta \in H^2(\Omega)$: The Sobolev embedding theorem [1, p. 97] yields that $H^2(\Omega)$ embeds into the space of continuous functions, which implies that $\bar{p}_\beta$ is continuous. Now, the assertion follows from (3.2).

4 Semismooth Newton method

In this section we present a numerical technique to find the solution of ($\mathcal{P}$) or, equivalently, of the reformulated first-order optimality conditions (2.11). Obviously, an algorithm based on (2.11) has to cope with the pointwise min- and max-terms in (2.11d). One possibility to deal with these non-differentiabilities is utilizing smooth approximations of the max- and min-operators. This leads to a smoothed L^1-norm term in ($\mathcal{P}$) and thus has the disadvantage that the typical properties of solutions of ($\mathcal{P}$) (e.g., the splitting into sets with $u = 0$ and $u \neq 0$) are lost. Hence, we prefer solving (2.11) directly instead of dealing with a smoothed version. Since we focus on fast second-order methods, we require an appropriate linearization of the nonlinear and nonsmooth system (2.11). We use a recent generalized differentiability concept, which, for the convenience of the reader is summarized below. We point out that this concept of semismoothness and generalized Newton methods holds in a function space setting. Such an infinite-dimensional analysis has several advantages over purely finite-dimensional approaches: The regularity of variables often explains the behavior of algorithms dealing with the discretized problem; Moreover, the well-posedness of a method in infinite dimensions is the basis for the investigation of mesh-independence properties.

4.1 Semismoothness in function space

Below we give a brief summary of generalized differentiability in function space. For more details and proofs of these results we refer to [6, 20, 31]. Let X, Y be Banach spaces, $D \subset X$ be open and $\mathcal{F} : D \to Y$ be a nonlinear mapping. Then, we call the mapping $\mathcal{F}$ generalized differentiable on the open subset $U \subset D$ if there exists a mapping $\mathcal{G} : U \to \mathcal{L}(X, Y)$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|_X} \|\mathcal{F}(x+h) - \mathcal{F}(x) - \mathcal{G}(x+h)h\|_Y = 0 \qquad (4.1)$$

for every $x \in U$. The above introduced mapping $\mathcal{G}$, which need not be unique, is referred to as *generalized derivative*. Note that in (4.1) $\mathcal{G}$ is evaluated at the point $x + h$ rather than at x and thus might change as $h \to 0$. Assume now we intend to find a root $\bar{x}$ of a semismooth mapping

$$\mathcal{F}(x) = 0 \qquad (4.2)$$

employing a Newton iteration. Then, the following local convergence result holds:

Theorem 4.1 *Suppose that $\bar{x} \in D$ is a solution of (4.2) and that $\mathcal{F}$ is semismooth in an open neighborhood U of $\bar{x}$ with generalized derivative $\mathcal{G}$. If $\mathcal{G}(x)^{-1}$ exists for all $x \in U$ and $\{\|\mathcal{G}(x)^{-1}\| : x \in U\}$ is bounded, the Newton iteration*

$$x^0 \in U \quad \textit{given}, \qquad x^{k+1} = x^k - \mathcal{G}(x^k)^{-1}\mathcal{F}(x^k)$$

is well-defined and, provided x^0 is sufficiently close to $\bar{x}$, converges at superlinear rate.

Since we want to use a Newton scheme for the solution of (2.11), we need semismoothness of the pointwise max- and min-operators. It can be shown that this holds true provided a two-norm property is satisfied. To be precise, the pointwise max- and min-operators $\mathcal{F}_{\max}, \mathcal{F}_{\min} : L^r(\Omega) \to L^s(\Omega)$ defined by $\mathcal{F}_{\max}(v) = \max(0, v)$ and $\mathcal{F}_{\min}(v) = \min(0, v)$ for $v \in L^r(\Omega)$, respectively, are generalized differentiable for $1 \le s < r \le \infty$. The mappings

$$\mathcal{G}_{\max}(v)(x) = \begin{cases} 1 & \text{if } v(x) \ge 0, \\ 0 & \text{if } v(x) < 0, \end{cases} \qquad \mathcal{G}_{\min}(v)(x) = \begin{cases} 1 & \text{if } v(x) \le 0, \\ 0 & \text{if } v(x) > 0 \end{cases} \tag{4.3}$$

can be used as generalized derivatives of $\mathcal{F}_{\max}$ and $\mathcal{F}_{\min}$ at v. We point out that the norm gap (i.e., $r < s$) is essential for generalized differentiability of $\mathcal{F}_{\max}$ and $\mathcal{F}_{\min}$.

4.2 Application to ($\mathcal{P}$)

From (2.11c), we infer that $\bar{\mu} = \bar{p} - \alpha\bar{u}$. Inserting this identity in (2.11d) and choosing $c := \alpha^{-1}$ results in

$$\begin{aligned} &\bar{u} - \alpha^{-1}\max(0, \bar{p} - \beta) - \alpha^{-1}\min(0, \bar{p} + \beta) \\ &\quad + \alpha^{-1}\max(0, \bar{p} - \beta - \alpha b) + \alpha^{-1}\min(0, \bar{p} + \beta - \alpha a) = 0. \end{aligned} \tag{4.4}$$

Now, due to this choice of c (see also [20]), only $\bar{p}$ appears inside the pointwise max- and min-operators, which, compared to $\bar{u}, \bar{\mu} \in L^2(\Omega)$ obeys additional regularity. To make this more precise, we introduce the operator $\mathcal{S} := -A^{-\star}A^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ and denote by $h := -A^{-\star}(A^{-1}f - y_d) \in H_0^1(\Omega)$. Then, (2.5) can be written as

$$\bar{p} = \mathcal{S}\bar{u} + h.$$

Let us consider the mapping

$$\mathcal{T} : L^2(\Omega) \to L^s(\Omega) \quad \text{with } s \in \begin{cases} (2, \infty] & \text{for } n = 1, \\ (2, \infty) & \text{for } n = 2, \\ (2, \frac{2n}{(n-2)}] & \text{for } n \ge 3, \end{cases} \tag{4.5}$$

defined by $\mathcal{T}u = p = \mathcal{S}u + h$. Strictly speaking, $\mathcal{T}u = \mathcal{I}(\mathcal{S}u + h)$ with $\mathcal{I}$ denoting the Sobolev embedding (see e.g., [1]) of $H_0^1(\Omega)$ into $L^s(\Omega)$ with s as defined in (4.5). From the above considerations, it follows that $\mathcal{T}$ is well-defined and continuous.

Since it is affine, it is also Fréchet differentiable from $L^2(\Omega)$ to $L^s(\Omega)$. Replacing $\bar{p}$ in (4.4) by $\mathcal{T}\bar{u}$ motivates to define $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned}\mathcal{F}(u) := u - \alpha^{-1}\max(0, \mathcal{T}u - \beta) - \alpha^{-1}\min(0, \mathcal{T}u + \beta)\\ + \alpha^{-1}\max(0, \mathcal{T}u - \beta - \alpha b) + \alpha^{-1}\min(0, \mathcal{T}u + \beta - \alpha a).\end{aligned} \tag{4.6}$$

This allows to express the optimality system (2.11) in the compact form $\mathcal{F}(u) = 0$. We are now able to argue generalized differentiability of the function $\mathcal{F}$ and derive a generalized Newton iteration for the solution of $\mathcal{F}(u) = 0$ and thus for ($\mathcal{P}$).

Theorem 4.2 *The function $\mathcal{F}$ as defined in* (4.6) *is generalized differentiable in the sense of* (4.1). *A generalized derivative is given by*

$$\mathcal{G}(u)(v) = v - \alpha^{-1}\chi_{(\mathcal{I}_- \cup \mathcal{I}_+)}(\mathcal{S}v), \tag{4.7}$$

where $\mathcal{I}_-$, $\mathcal{I}_+$ are disjoint sets defined by

$$\begin{aligned}\mathcal{I}_- &= \{x \in \Omega : \alpha a < \mathcal{T}u + \beta \le 0 \text{ a.e. in } \Omega\},\\ \mathcal{I}_+ &= \{x \in \Omega : 0 \le \mathcal{T}u - \beta < \alpha b \text{ a.e. in } \Omega\}.\end{aligned} \tag{4.8}$$

Proof After the above discussion, the prove is an application of the general theory from Sect. 4.1 to (4.6). For showing that the conditions required for the application of Theorem 4.1 are satisfied, we restrict ourselves to the first max-term in (4.6). Analogous reasoning yields generalized differentiability of the remaining terms in (4.6) and thus of $\mathcal{F}$. From the smoothing property of the affine operator $\mathcal{T}$ we obtain for each $u \in L^2(\Omega)$ that $\mathcal{T}u \in L^s(\Omega)$ with some $s > 2$. It follows that

$$\mathcal{F}_1 : u \to \max(0, \mathcal{T}u - \beta) = \max(0, \mathcal{S}u + h - \beta)$$

is semismooth if considered as mapping from $L^2(\Omega)$ into $L^2(\Omega)$; Moreover, for its generalized derivative we obtain $\mathcal{G}_1(u)(v) = \chi_{\mathcal{A}}(\mathcal{S}v)$, where $\chi_{\mathcal{A}}$ denotes the characteristic function for the set $\mathcal{A} = \{x \in \Omega : \mathcal{T}u - \beta \ge 0 \text{ a.e. in } \Omega\}$, compare with (4.3). A similar argumentation for the remaining max- and min-operators in (4.6) shows that the whole function $\mathcal{F}$ is generalized differentiable. Merging the characteristic functions yields that a generalized derivative is given by (4.7), which ends the proof. □

We next state our algorithm for the solution of ($\mathcal{P}$).

Algorithm 1 (Semismooth Newton)

1. Initialize $u^0 \in L^2(\Omega)$ and set $k := 0$.
2. Unless some stopping criterion is satisfied, compute the generalized derivative $\mathcal{G}(u^k)$ as given in (4.7) and derive δu^k from

$$\mathcal{G}(u^k)\delta u^k = -\mathcal{F}(u^k). \tag{4.9}$$

3. Update $u^{k+1} := u^k + \delta u^k$, set $k := k + 1$ and return to Step 1.

Now, Theorem 4.1 applies and yields the following convergence result for Algorithm 1:

Theorem 4.3 *Let the initialization u^0 be sufficiently close to the solution $\bar{u}$ of ($\mathcal{P}$). Then the iterates u^k of Algorithm 1 converge superlinearly to $\bar{u}$ in $L^2(\Omega)$. Moreover, the corresponding states y^k converge superlinearly to $\bar{y}$ in $H_0^1(\Omega)$.*

Proof To apply Theorem 4.1, it remains to show that the generalized derivative (4.7) is invertible and that the norms of the inverse linear mappings are bounded. Let $u, v \in L^2(\Omega)$ and $(\cdot\,,\cdot)_{\mathcal{S}}$ denote the L^2-product over $\mathcal{S} \subset \Omega$. Then, using that $\mathcal{S} = -A^{-\star}A^{-1}$ and $\mathcal{I}_-$ and $\mathcal{I}_+$ as defined in (4.8)

$$\begin{aligned}(\mathcal{G}(u)(v), v)_\Omega &= (v - \alpha^{-1}\chi_{(\mathcal{I}_-\cup\mathcal{I}_+)}(v), v)_\Omega \\ &= (v, v)_\Omega + \alpha^{-1}(A^{-\star}A^{-1}v, v)_{(\mathcal{I}_-\cup\mathcal{I}_+)} \geq (v, v)_\Omega\end{aligned}$$

independently from $\mathcal{I}_-$, $\mathcal{I}_+$ and thus from u. This shows that for all u, $(\mathcal{G}(u)(\cdot), \cdot)$ defines a coercive (and continuous) bilinear form on $L^2(\Omega) \times L^2(\Omega)$. Using the Lax-Milgram lemma, it follows that the generalized derivative $\mathcal{G}(u)$ is invertible and that $\|\mathcal{G}(u)^{-1}\| \leq 1$ for all u, which ends the proof. □

We conclude this section with the statement of a different, more explicit form for the Newton step (4.9) in Algorithm 1. This alternative formulation obeys the form of an active set strategy. The method's relations to the "dual active set method" (see [9, 16, 17, 19]) and the "primal-dual active set strategy" [3, 20, 24] are discussed as well.

4.3 Interpretation as active set method

Utilizing (4.7), the explicit statement of the Newton step (4.9) becomes

$$\begin{aligned}\delta u^k - \alpha^{-1}\chi_{(\mathcal{I}_-^k\cup\mathcal{I}_+^k)}(\mathcal{S}\delta u^k) &= -u^k + \alpha^{-1}\chi_{\mathcal{I}_-^k}(\mathcal{T}u^k + \beta) \\ &\quad + \alpha^{-1}\chi_{\mathcal{I}_+^k}(\mathcal{T}u^k - \beta) + \chi_{\mathcal{A}_a^k}a + \chi_{\mathcal{A}_b^k}b,\end{aligned} \tag{4.10}$$

where the disjoint sets $\mathcal{A}_a^k, \mathcal{I}_-^k, \mathcal{A}_o^k, \mathcal{I}_+^k$ and $\mathcal{A}_b^k$ are given by

$$\mathcal{A}_a^k = \{x \in \Omega : \mathcal{T}u^k + \beta \leq \alpha a \text{ a.e. in } \Omega\}, \tag{4.11a}$$

$$\mathcal{I}_-^k = \{x \in \Omega : \alpha a < \mathcal{T}u^k + \beta \leq 0 \text{ a.e. in } \Omega\}, \tag{4.11b}$$

$$\mathcal{A}_o^k = \{x \in \Omega : |\mathcal{T}u^k| < \beta \text{ a.e. in } \Omega\}, \tag{4.11c}$$

$$\mathcal{I}_+^k = \{x \in \Omega : 0 \leq \mathcal{T}u^k - \beta < \alpha b \text{ a.e. in } \Omega\}, \tag{4.11d}$$

$$\mathcal{A}_b^k = \{x \in \Omega : \alpha b \leq \mathcal{T}u^k - \beta \text{ a.e. in } \Omega\}. \tag{4.11e}$$

Observe that Algorithm 1 only involves the control u^k explicitly. The corresponding iterates for the state, the adjoint state and the multiplier are, in terms of u^k, given by

$$p^k = \mathcal{T}u^k, \quad y^k = A^{-1}(u^k + f) \quad \text{and} \quad \mu^k = \mathcal{T}u^k - \alpha u^k, \tag{4.12}$$

compare with the definition of $\mathcal{T}$ on page 170, and with (2.11a–2.11c).

We now discuss (4.10) separately on the sets (4.11). To start with, on $\mathcal{A}_a^k$ we obtain $\delta u^k = -u^k + a$. Using $\delta u^k = u^{k+1} - u^k$, this results in setting $u^{k+1} = a$ on $\mathcal{A}_a^k$. Next we turn to $\mathcal{I}_-^k$. From (4.10), we obtain

$$\delta u^k - \alpha^{-1}\mathcal{S}\delta u^k = -u^k + \alpha^{-1}(\mathcal{T}u^k + \beta)$$

and, again using $\delta u^k = u^{k+1} - u^k$ that $u^{k+1} - \alpha^{-1}\mathcal{T}u^{k+1} = \alpha^{-1}\beta$. Multiplying with α and using (4.12) yields $\mu^{k+1} = -\beta$ on $\mathcal{I}_-^k$. Continuing the evaluation of (4.10) separately on the remaining sets $\mathcal{A}_o, \mathcal{I}_+, \mathcal{A}_b$ and using (4.12) shows that $u^{k+1} = 0$ on $\mathcal{A}_o^k$, $\mu^{k+1} = \beta$ on $\mathcal{I}_+^k$ and $u^{k+1} = b$ on $\mathcal{A}_b^k$. Thus, we can restate Algorithm 1 as active set method.

Algorithm 2 (Active set method)

1. Initialize $u^0 \in L^2(\Omega)$, compute y^0, p^0 and μ^0 as in (4.12) and set $k := 0$.
2. Unless some stopping criterion is satisfied, derive the sets $\mathcal{A}_a^k, \mathcal{I}_-^k, \mathcal{A}_o^k, \mathcal{I}_+^k$ and $\mathcal{A}_b^k$ following (4.11).
3. Solve for $(u^{k+1}, y^{k+1}, p^{k+1}, \mu^{k+1})$:

$$\begin{aligned}
&Ay^{k+1} - u^{k+1} - f = 0,\\
&A^\star p^{k+1} - y_d + y^{k+1} = 0,\\
&-p^{k+1} + \alpha u^{k+1} + \mu^{k+1} = 0,\\
&u^{k+1} = \begin{cases} a & \text{on } \mathcal{A}_a^k,\\ 0 & \text{on } \mathcal{A}_o^k,\\ b & \text{on } \mathcal{A}_b^k,\end{cases} \qquad \mu^{k+1} = \begin{cases} -\beta & \text{on } \mathcal{I}_-^k,\\ \beta & \text{on } \mathcal{I}_+^k.\end{cases}
\end{aligned}$$

4. Set $k := k + 1$ and return to Step 2.

Note that the sets in (4.11) are defined using the adjoint state $p^k = \mathcal{T}u^k$ only. Using the fact that $-p^k + \alpha u^k + \mu^k = 0$ for all k, these sets can also be defined using $\alpha u^k + \mu^k$ (or, more generally, even $cu^k + \mu^k$ for some $c > 0$). These different approaches explain two existing names for methods such as Algorithm 2 that derive active sets and solve a sequence of problems with equality constraints on the active and without constraints on the inactive sets. On the one hand, Algorithm 2 is a *dual active set method* as introduced in [19]. Here, "dual" refers to the adjoint (or dual) state p. At the same time, it is an *primal-dual active set method* as introduced in [3]. In this case, "primal" refers to the control variable u, and "dual" to the corresponding multiplier μ. Both methods have been used and deeply analyzed for optimization problems with equality, and inequality bound constraints. Note that nonsmooth optimization problems such as ($\mathcal{P}$) lead to inequality constrained dual problems. This explains the active set form of Algorithm 2 for solving ($\mathcal{P}$), which however is not a straightforward application of these active set methods. Let us now briefly review the development of the dual and the primal-dual active set method.

In [19], the dual active set method has been introduced for bound constrained control of an ordinary differential equations. Local fast convergence is proved using generalized equations. Later, the dual active set method has been analyzed and extended in various directions. We refer for example to [16], where, for problems in $\mathbb{R}^n$ the method is shown to converge in a finite number of steps if an appropriate line search strategy is used. This approach is used for the solution of network optimization problems in [18]. The method has also been extended to linear programming problems in [17], see also [9] for implementation issues.

The primal-dual active set strategy has been introduced in [3] for control constrained elliptic optimal control problems. The method is a dual active set method without line search. Its relation to semismooth Newton methods in $\mathbb{R}^n$ as well as in function space as found in [20] can be used to prove fast local convergence. Conditional global convergence results are based on typical properties of optimization problems with constraints given by partial differential equations such as smoothing or monotonicity, see [3, 20, 24]. We remark that for problems with bilateral constraints, choosing $cu^k + \mu^k$ with c different from α allows to control if points can change from one active set to the other one within one iteration [29]. Such unwanted behavior might lead to cycling of the iterates or to very small steps if some linesearch procedure is used. Moreover, c can be used to balance possible different orders of magnitudes of u^k and μ^k, which becomes an issue if the involved linear systems are solved only approximately.

5 Numerical examples

We end this paper with a numerical study. Our aim is twofold: Firstly, we examine the influence of the L^1-norm on the structure of solutions of ($\mathcal{P}$) and numerically verify our theoretical findings in Sect. 3. Secondly, we study the performance of our algorithm for the solution of ($\mathcal{P}$).

As initialization u^0 for Algorithm 1 (or equivalently Algorithm 2) we choose the solution of (2.11) with $\bar{\mu} = 0$, that is, the solution of ($\mathcal{P}$) with $\beta = 0$ and $U_{ad} = L^2(\Omega)$. We terminate the algorithm if the sets $\mathcal{A}_a^k, \mathcal{I}_-^k, \mathcal{A}_o^k, \mathcal{I}_+^k, \mathcal{A}_b^k$ coincide for two consecutive iterations or as soon as the discrete analogue of $\|u^{k+1} - u^k\|_{L^2}$ drops below the tolerance $\varepsilon = 10^{-10}$. If the linear systems in Algorithm 2 are solved exactly, the first stopping criterion yields the exact solution of (the discretized version of) ($\mathcal{P}$).

Subsequently, we focus on the following test problems. We use $\Omega = [0, 1]^2$ and, unless otherwise specified, $A = -\Delta$. For discretization we apply the standard 5-point stencil.

Example 1 The data for this example are as follows: $a \equiv -30$, $b \equiv 30$, $y_d = \sin(2\pi x)\sin(2\pi y)\exp(2x)/6$, $f = 0$ and $\alpha = 10^{-5}$. We study the influence of the parameter β on the solution. For $\beta = 0.001$, the optimal control $\bar{u}$ with corresponding condensed multiplier $\bar{\mu}$, the optimal state and the splitting into the active/inactive sets are shown in Fig. 1.

Example 2 This example uses the same data as Example 1, but without control bounds (i.e., $a \equiv -\infty$, $b \equiv \infty$) and with $\alpha = 10^{-7}$. Example 2 is used together with Example 1 to discuss the attainability of the desired state y_d depending on β.

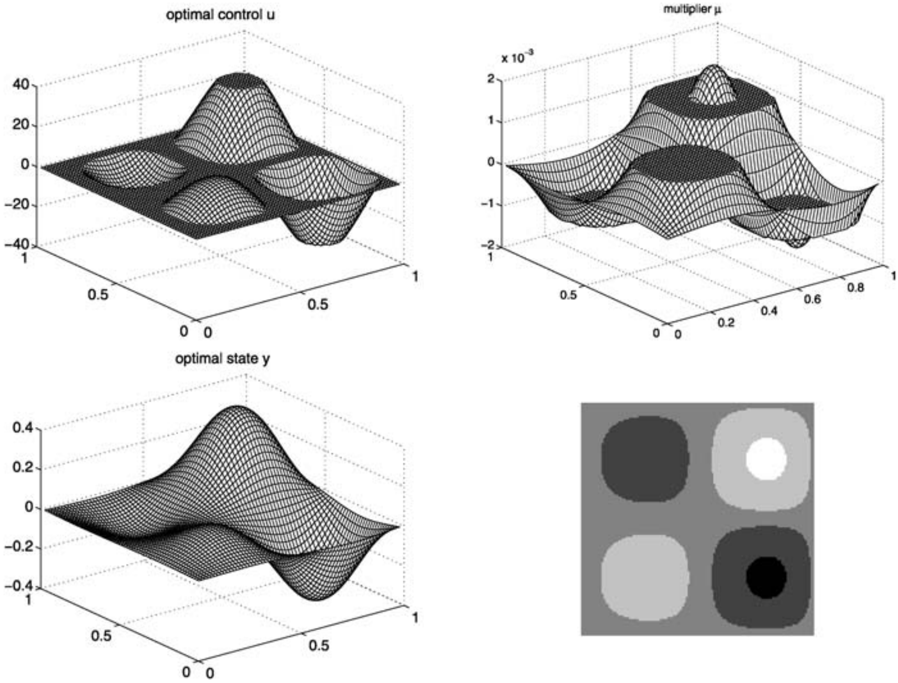

Fig. 1 Example 1: Optimal control $\bar{u}$ (*upper left*), corresponding multiplier $\bar{\mu}$ (*upper right*), optimal state $\bar{y}$ (*lower left*), and visualization of the splitting (*lower right*) into $\mathcal{A}_a$ (*in black*), $\mathcal{I}_-$ (*in dark grey*), $\mathcal{A}_o$ (*in middle grey*), $\mathcal{I}_+$ (*in light grey*) and $\mathcal{A}_b$ (*in white*)

Example 3 Example 3 is constructed in order to obtain sets $\mathcal{A}_a, \mathcal{I}_-, \mathcal{A}_o, \mathcal{I}_+, \mathcal{A}_b$ that have a more complex structure at the solution. Moreover, the upper control bound b is zero on a part of Ω with positive measure. The exact data are $a \equiv -10$,

$$b = \begin{cases} 0 & \text{for } (x, y) \in [0, 1/4] \times [0, 1], \\ -5 + 20x & \text{for } (x, y) \in [1/4, 1] \times [0, 1], \end{cases}$$

$y_d = \sin(4\pi x)\cos(8\pi y)\exp(2x)$, $f = 10\cos(8\pi x)\cos(8\pi y)$, $\alpha = 0.0002$ and $\beta = 0.002$. The solution $(\bar{y}, \bar{u})$, as well as the corresponding Lagrange multiplier are shown in Fig. 2. Moreover, we visualize the splitting in the sets (4.11) at the solution.

Example 4 For this example we use the differential operator $A = \nabla \cdot (a(x, y)\nabla)$, with $a(x, y) = y^2 + 0.05$. The remaining data are given by $y_d \equiv 0.5$, $f = 0$ and $\alpha = 0.0001$. We do not assume box constraints for the control, i.e., $a \equiv -\infty$ and $b \equiv \infty$. In Fig. 3, the optimal control $\bar{u}$ and the corresponding multiplier $\bar{\mu}$ are shown for $\beta = 0.005$.

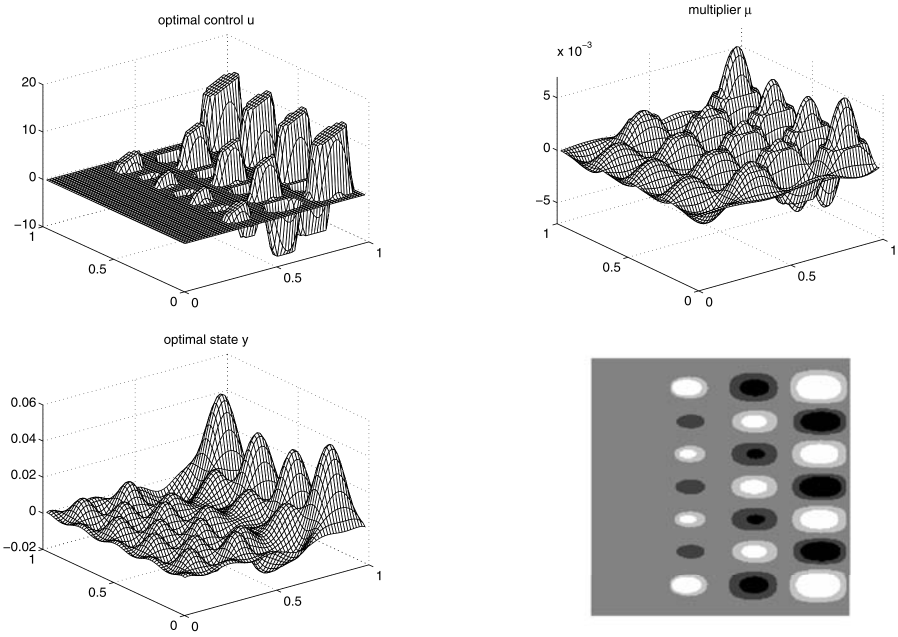

Fig. 2 Example 3: Optimal control $\bar{u}$ (*upper left*), corresponding multiplier $\bar{\mu}$ (*upper right*), optimal state $\bar{y}$ (*lower left*), and visualization of the splitting (*lower right*) into $\mathcal{A}_a$ (*in black*), $\mathcal{I}_-$ (*in dark grey*), $\mathcal{A}_o$ (*in middle grey*), $\mathcal{I}_+$ (*in light grey*) and $\mathcal{A}_b$ (*in white*)

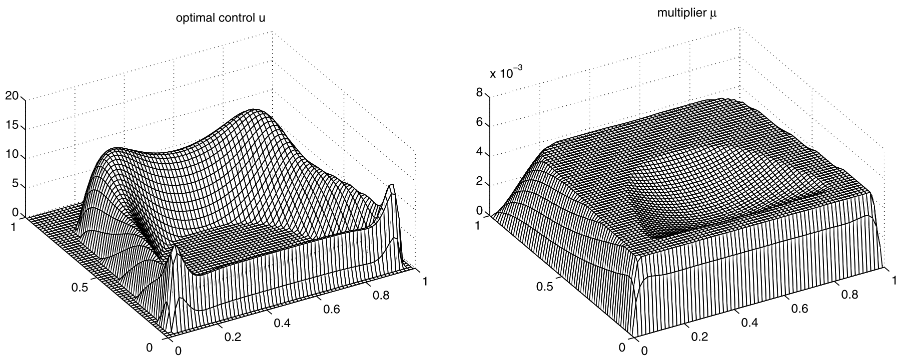

Fig. 3 Example 4: Optimal control $\bar{u}$ (*left*) and corresponding multiplier $\bar{\mu}$ (*right*)

5.1 Qualitative discussion of the results

Complementarity conditions We start with visually verifying the complementarity conditions for Example 1, see Fig. 1, upper row. First note that $\bar{u} = 0$ on a relatively large part of Ω. Further, observe that on this part $|\bar{\mu}| \leq \beta$ holds, as expected. On subdomains with $0 \leq u \leq b$, $\mu = \beta$; Moreover, $\bar{u} = b$ corresponds to sets where $\mu \geq \beta$,

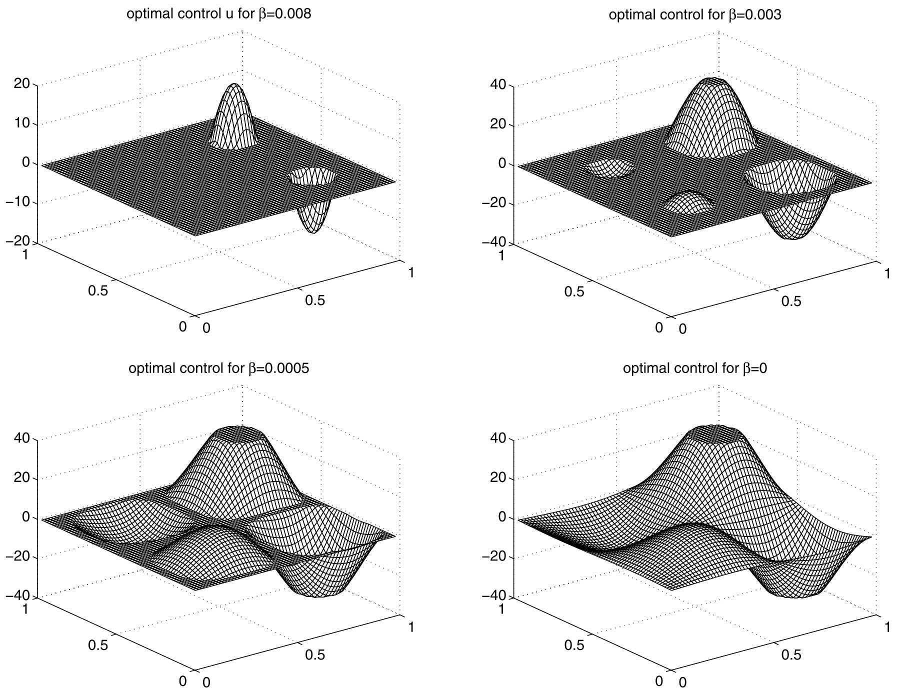

Fig. 4 Example 1: Optimal control $\bar{u}$ for $\beta = 0.008$ (*upper left*), $\beta = 0.003$ (*upper right*), $\beta = 0.0005$ (*lower right*) and $\beta = 0$ (*lower left*)

as desired. Similarly, one can verify visually that the complementarity conditions hold for $\bar{u} \leq 0$ as well.

The role of β—placement of control devices To show the influence of the parameter β on the optimal control $\bar{u}$, we solve Example 1 for various values of β while keeping α fixed. For $\beta = 0.02$ or larger, the optimal control $\bar{u}$ is identically zero, compare with Lemma 3.1. As β decreases, the size of the region with $\bar{u}$ different from zero increases. In Fig. 4 we depict the optimal controls for $\beta = 0.008$, 0.003, 0.0005 and 0. To realize the optimal control for $\beta = 0.003$ in an application, four separated relatively small control devices are needed, since $\bar{u}$ is zero on large parts of the domain. This means that no distributed control device acting on the whole of Ω is necessary. The attainability of the desired state y_d depending on β can be seen from Table 1. Here, for Examples 1 and 2 we give for various values of β the L^2-norm $\|y_d - \bar{y}\|$ as well as the percentage of Ω where $\bar{u} \neq 0$. Note that the solution for $\beta = 0$ is the solution for the classical smooth optimal control problem ($\mathcal{P}_2$) and that the solution for $\beta = 0.02$ is $\bar{u} \equiv 0$.

Next, we turn to Example 4. In Fig. 5, we visualize for different values of β those parts of Ω, where control devices need to be placed (i.e., where the optimal control is nonzero). It can be seen that the control area shrinks for increasing β. Moreover, for different β also the shape of the domain with nonzero control changes significantly.

Table 1 Examples 1 and 2: L^2-norm $\|y_d - \bar{y}\|$ and percentage $|\bar{u} \neq 0|/|\Omega|$ of area where $\bar{u} \neq 0$ for various β

	β	0	0.0001	0.0005	0.003	0.008	0.02
Example 1	$\|y_d - \bar{y}\|$	7.52e−2	7.79e−2	9.17e−2	1.79e−1	2.65e−1	2.91e−1
	$\|\bar{u} \neq 0\|/\|\Omega\|$	100%	75.0%	51.6%	18.4%	3.4%	0%
Example 2	$\|y_d - \bar{y}\|$	4.90e−4	1.37e−2	4.15e−2	1.43e−1	2.52e−1	2.91e−1
	$\|\bar{u} \neq 0\|/\|\Omega\|$	100%	38.5 %	19.4%	2.9%	0.4%	0%

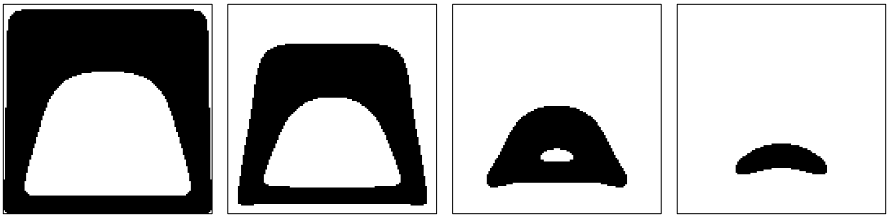

Fig. 5 Example 4: Visualization of sets with nonzero optimal control $\bar{u}$ (*black*) for $\beta = 0.001, 0.01, 0.05, 0.1$ (*from left to right*); Only in the *black parts* of Ω control devices are needed

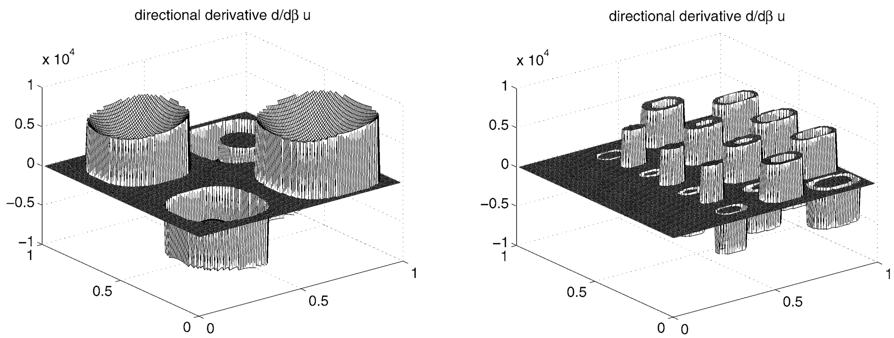

Fig. 6 Derivatives $\dot{u}_\beta$ for $\beta = 0.001$ in Example 1 (*left*) and $\beta = 0.002$ in Example 3 (*right*)

Derivatives with respect to β Here, we derive derivatives of $\bar{u}$ with respect to β as discussed in Sect. 3. To be precise, we use an extended version of the results obtained there, since we also allow for box constraints on the control. Without proof we remark that, for strongly active box constraints (i.e., where the Lagrange multiplier satisfies $|\bar{\mu}| > \beta$), $\dot{u}_\beta = 0$ holds. We checked that for our choice of β, $\bar{p}_\beta \neq 0$. Thus, the sets $\mathcal{S}_-$ and $\mathcal{S}_+$ can be derived from $\bar{p}_\beta$. For $\beta = 0.001$, the derivative $\dot{u}_\beta$ is shown on the left of Fig. 6. Observe that the sensitivity for changes in β in only nonzero on $\mathcal{I}_\beta^- \cup \mathcal{I}_\beta^+$, since in this example numerical strict complementarity holds. As observed at the end of Sect. 3, $\dot{u}_\beta$ is noncontinuous with jumps of approximate magnitude $\alpha^{-1} = 10^4$. On the right hand side of Fig. 6, we depict $\dot{u}_\beta$ for Example 3, where $\beta = 0.002$ and

Table 2 Number of iterations for Example 1 with $\alpha = 0.0001$ for meshsize h and various values for β

h	β				
	0.008	0.003	0.001	0.0005	0
1/32	5	5	4	4	3
1/64	5	5	4	4	4
1/128	5	5	5	4	4
1/256	6	6	5	5	4

Table 3 Convergence of u^k in Example 4, $h = 1/128$

k	$\|u^k - \bar{u}\|$	$\|u^k - \bar{u}\|/\|u^{k-1} - \bar{u}\|$
1	4.588585e+00	–
2	5.045609e+00	1.099600e−00
3	2.148787e−01	4.258726e−02
4	2.054260e−01	9.560091e−01
5	5.556892e−02	2.705058e−01
6	6.047484e−04	1.088285e−02
7	0.000000	0.000000

$\alpha = 0.0002$. Again, the height of the jump along the boundaries of $\mathcal{S}_-$ and $\mathcal{S}_+$ is of magnitude $\alpha^{-1} = 5000$.

5.2 Performance of the algorithm

Number of iterations In Table 2 we show the number of iterations required for the solution of Example 1 for various values of β. For all mesh-sizes h and choices of β the algorithm yields an efficient behavior. Note also the stable behavior for various meshsizes h. The parameter β does not have a significant influence on the performance of the algorithm. Thus, the computational effort for solving problem ($\mathcal{P}$) is comparable to the one for the solution of ($\mathcal{P}_2$).

Though the solution of Example 2 obeys a more complex structure of active sets, the algorithm detects the solution after 3 iterations for meshes with $h = 1/32, \dots, 1/256$.

Convergence rate For Example 4, we study the convergence of the iterates u^k to the optimal control $\bar{u}$. In Table 3, we show the L^2-norm of the differences $u^k - \bar{u}$ and the quotients $\|u^k - \bar{u}\|_{L^2}/\|u^{k-1} - \bar{u}\|_{L^2}$ for $\alpha = 0.0001$ and $\beta = 0.005$. Observe that after a few iterations, the quotients decrease monotonically. This numerically verifies the local superlinear convergence rate proven in Theorem 4.1.

Attempts to speed up the algorithm when solving ($\mathcal{P}$) for various β If one is interested in placing control devices in an optimal way, one needs to solve ($\mathcal{P}$) for several values of β in order to find a control structure that is realizable with the available resources. We considered two ideas to speed up the algorithm for the case that a solution is needed for various β. The first one used an available solution for β as initialization

Table 4 Example 3: Number of iterations for nested iteration and for direct solution on finest grid

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-8}	CPU-time ratio
#iterations	3	2	2	6	4	3	3	11	0.39

for the algorithm to solve ($\mathcal{P}$) for β'. The second one used $u_\beta + (\beta' - \beta)\dot{u}_\beta$ as initialization for ($\mathcal{P}$) with β'. We tested both approaches when solving Example 4 for various β. Unfortunately, we did not observe a significant speedup of the iteration.

Speeding up the algorithm using a nested iteration Here, we use a prolongated solution on a rough grid as initialization on a finer grid. We start with a very rough grid and iteratively repeat the process (solve–prolongate–solve on next finer grid–$\cdots$) until the desired mesh-size is obtained. In Table 4, we give the results obtained for Example 4 with $\beta = 0.001$, where $h = 1/256$ for the finest grid. Using the nested strategy, only 3 iterations are needed on the finest grid, compared to 11 when the iteration is started on that grid. Since the effort on the rougher meshes is small compared to the finer ones, using the nested approach speeds up the solution process considerably (only 39% of CPU time is needed).

5.3 Conclusions for the placement of control devices

As seen above, the structure of the optimal controls depends in a nontrivial and nonmonotone way on the value of β. In practice, one will need to find a value for β such that $\bar{y}$ is sufficiently close to y_d while only sparsely distributed control devices are used (i.e., the optimal control is nonzero only on a rather small portion of Ω). This might require some manual tuning of β and thus a couple of solves of ($\mathcal{P}$) for different β. Nevertheless, the approach used in this paper should be much more efficient than using discrete optimization techniques to find optimal locations for control devices.

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