Fall 2018: Numerical Analysis
Assignment 7 (due December 13, 2018)

1 extra credit point again for cleanly plotted and labeled figures (see also rules on the first assignment and my post on Piazza).

1. **[Interpolation and optimal norm approximation, 2+1+1+1pt]** For an interval (a, b), $n \in \mathbb{N}$ and disjoint points $x_0, \ldots, x_n$ in $[a, b]$, we define for polynomials p, q

$$\langle p, q \rangle := \sum_{i=0}^{n} p(x_i)q(x_i).$$

   (a) Show that $\langle \cdot\,, \cdot \rangle$ is an inner product for each $\mathcal{P}_k$ with $k \le n$, where $\mathcal{P}_k$ denotes the space of polynomials of degree k or less.

   (b) Why is $\langle \cdot\,, \cdot \rangle$ not an inner product for $k > n$?

   (c) Show that the Lagrange polynomials L_i corresponding to the nodes $x_0, \ldots, x_n$ are orthonormal with respect to the inner product $\langle \cdot\,, \cdot \rangle$.

   (d) For a continuous function $f : [a, b] \to \mathbb{R}$, compute its optimal approximation in $\mathcal{P}_n$ with respect to the inner product $\langle \cdot\,, \cdot \rangle$ and compare with the interpolation of f.

2. **[Euler and trapezoidal methods, 2+2pt]** Consider the following method for solving of $y' = f(y)$:

$$y_{n+1} = y_n + h\left[\theta f(y_{n+1}) + (1-\theta) f(y_n)\right], 0 \le \theta \le 1, \tag{1}$$

   (a) Compute the truncation error T_n of (1). Assuming sufficient smoothness of y and f, for what value of θ is $|T_n|$ the smallest? What does this mean about the accuracy of the method?

   (b) Consider the initial value problem (IVP) $y' = t(1 - e^y)$ with $y(t_0) = 1$ and $t_0 = 0$. Compute y_2 using the forward Euler method ($\theta = 0$ in (1)) with mesh size h (give your answer in terms of h).

   (c) Consider the initial value problem $y' = -y^2$ with $y(t_0) = 1$ and $t_0 = 0$. Compute y_1 using the trapezoidal method ($\theta = \frac{1}{2}$ in (1)) with mesh size h (give your answer in terms of h).

   (d) Implement the forward Euler method and the trapezoidal method. Provide your code.

3. **[Implicit and Explicit Euler, 2+1+2+(2+1)pt]** Consider the linear IVP $y' = -100y$ with $y(0) = y_0$.

   (a) Write down Euler's method with step size h and find an explicit formula for y_n in terms of y_0. Repeat for the backward Euler method.

   (b) For the two approximations, if we take $h = 0.1$, what happens to y_n as $n \to \infty$? Which method is more consistent with the limit of the actual solution $y(t)$ as $t \to \infty$?

(c) Implement the forward Euler method for the IVP $y' = -100y + y^2$ in MATLAB. Use the time interval $[0, 1]$, the initial condition $y(0) = 1$. Compute and plot the solutions for step sizes $h = 0.1, 0.02, 0.001$.

(d) **[Extra credit]** Implement the backward Euler method. At each step, you will need to solve $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$, which constitutes a nonlinear scalar equation for y_{n+1}. This can be done using Newton's method (Chapter 1!), i.e., we define $g(z) = z - y_n - hf(x_{n+1}, z)$ and then, to compute the y_{n+1}, we iterate (here, the superindex k is the Newton iteration index:

$$y^0_{n+1} = y_n + hf(x_n, y_n)$$
$$y^{k+1}_{n+1} = y^k_{n+1} - \frac{g(y^k_{n+1})}{g'(y^k_{n+1})}.$$

A few (e.g., 5) iterations are usually sufficient to obtain a good result for y_{n+1}.

(e) **[Extra credit]** Repeat (c), but using the implementation from (d). Discuss the differences in behavior compared to (c).

Please hand in your code.

4. **[ODEs, 3+7 points]** You are given the Initial Value Problem (IVP)

$$y'(x) = -y(x)\,(x + 1), \quad y(0) = 3.$$

(a) Verify that $y(x) = 3e^{-\frac{x}{2}(2+x)}$ satisfies the IVP.

(b) You are given the following implementation (in pseudo code)

```
f(x,y)=-y(x+1);
h=0.1;
N=10;
xk=0;
yk=3;
for k = 1 to N
    xk=xk+h;
    yk = yk + h*f(xk+h/2,yk+h/2*f(xk,yk));
end
```

Additionally you observe the following error $e_N = |y(1) - y_N|$ at $x = 1$:

h	e_N
0.50000	0.035184
0.25000	0.008257
0.12500	0.001910
0.06250	0.000456
0.03125	0.000111

(i) Is the implemented method implicit or explicit? Explain your answer.

(ii) Is the implemented method a one-step method or a multi-step method? Explain your answer.

(iii) What does the error output suggest about the order of the method? Explain your answer.

5. **[Error behavior, (2)+2+2+1pt]** Consider the IVP $y' = f(x,y)$, for $f(x,y) = x\sin(y)$ and $y(0) = \pi/2$ for $x \in [0,3] =: I$. We showed in class that for the forward Euler method, the following error estimate holds:

$$|e_n| \leq \frac{M_2}{2L}\left(e^{L(x_n - x_0)} - 1\right)h,$$

where $e_n = y(x_n) - y_n$, L, the Lipschitz constant of f, $M_2 = \max_{x\in I}|y''(x)|$ and h the step size.

(a) **[Extra credit]** Verify that $y(x) = \pi - \arctan\left(\frac{2\exp(\frac{x^2}{2})}{\exp(x^2)-1}\right)$ solves the IVP.

(b) Show that the constants in the estimate can be choses as $L = 3$ and $M_2 = 10$ on I.[1]

(c) Using the estimate, what is the step size h that guarantees that the error at $x = 3$ is less than $\epsilon := 10^{-2}$? Using your implemenation of the forward Euler Method and the analytical solution given in (a), what step size do you actually need to obtain the desired tolerance?

(d) Using the step sizes $h = \frac{1}{2^j}$ for $j = 1, 2, \ldots, 5$, report the errors at $x = 3$. Use the theoretical estimate to explain the error behavior, i.e., how does the error change as h is decreased?

[1] Compare with the derivations in Example 12.2 in the book, and with the class notes.