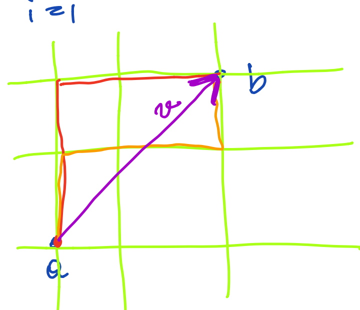
Def: Norms $\|\cdot\| : V \to \mathbb{R}_+$, | V is linear space over $\mathbb{R}$; V can be $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$

- $\|v\| = 0 \iff v = 0 \quad \forall v \in V$
- $\|\lambda v\| = |\lambda|\,\|v\| \quad \forall v \in V,\ \lambda \in \mathbb{R}$
- $\|v+w\| \le \|v\| + \|w\| \quad \forall v, w \in V$, "triangle inequality"

Examples: Norms in $\mathbb{R}^n$:

- $\|v\|_2 = \left(\sum_{i=1}^n v_i^2\right)^{\frac{1}{2}}$ "2-norm", "Euclidean norm"
- $\|v\|_1 = \sum_{i=1}^n |v_i|$ "1-norm"



$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\|v\|_1 = 4$ (blocks)

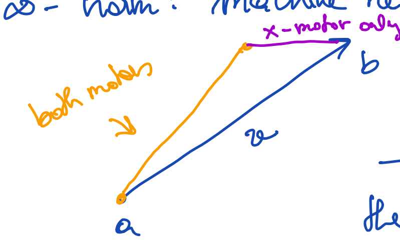
$\|v\|_2 = \sqrt{4+4} = \sqrt{8}$

$\|v\|_\infty = 2$

- $\|v\|_\infty = \max_{1\le i\le n} |v_i|$ "∞-norm"

Ex: $v = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ $\|v\|_2 = \sqrt{1^2+2^2+3^2}$, $\|v\|_1 = 6$, $\|v\|_\infty = 3$

Example for ∞-norm: machine head with different motors in each coordinate directions



$\to$ distance depends on the larger of both coordinates

$\to \|\ \|_\infty$-norm

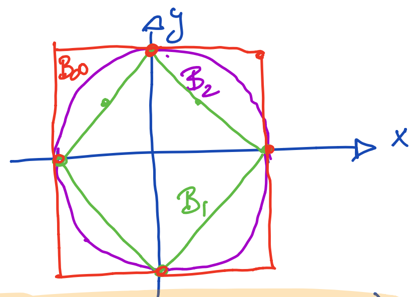
Norms for $V = \mathbb{R}^{n\times m}$:

$A \in \mathbb{R}^{n\times m}$: $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$ Frobenius norm

Unit circles / spheres for $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$:

$$B_1 = \{ v \in \mathbb{R}^n,\ \|v\|_1 \le 1 \}$$
$$B_2 = \{ v \in \mathbb{R}^n,\ \|v\|_2 \le 1 \}$$
$$B_\infty = \{ v \in \mathbb{R}^n,\ \|v\|_\infty \le 1 \}$$

in 2D:



Matrix norms induced by vector norms (or subordinate)

For a norm $\|\cdot\|$ on $\mathbb{R}^n$, the induced (or subordinate) matrix norm on $\mathbb{R}^{n\times n}$ is:

$$A \in \mathbb{R}^{n\times n}: \underbrace{\|A\|}_{\text{matrix norm}} := \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|}$$

vector norms in $\mathbb{R}^n$, e.g. $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$

equivalent: $\|A\|_* := \max_{\|v\|=1} \|Av\|$

Proof that $\|A\| = \|A\|_*$:

$$\|A\| = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|} \ge \max_{\|v\|=1} \frac{\|Av\|}{\cancel{\|v\|}_{=1}} = \|A\|_*$$

$$v \neq 0: \quad \frac{\|Av\|}{\|v\|} = \left\| A \frac{v}{\|v\|} \right\| \le \max_{\|w\|=1} \|Aw\| \quad \text{for all } v$$

$\xrightarrow[\text{over all } v]{\text{take max}}$ $$\|A\| = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|} \le \max_{\|w\|=1} \|Aw\| = \|A\|_*$$

$\longrightarrow$ $\|A\| = \|A\|_*$ for all $A \in \mathbb{R}^{n\times n}$

The induced norm satisfies the norm axioms, and it is compatible with the vector norm, i.e.:

$$\underbrace{\|Av\|}_{\text{vector norms}} \le \overbrace{\|A\|}^{\text{matrix norm}}\,\|v\| \quad \text{for all } v \in \mathbb{R}^n$$

<u>Thm:</u> Induced matrix norm for $\|\cdot\|_\infty$ vector norm is

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

"largest row sum"

<u>Ex:</u> $A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} \longrightarrow \|A\|_\infty = 7$

<u>Thm:</u> Induced matrix norm for $\|\cdot\|_1$ vector norm is

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

"largest column sum" $\quad \|A\|_1 = 6$

<u>Thm:</u> Induced matrix norm for the $\|\cdot\|_2$ vector norm can be computed as follows:

Compute the eigenvalues of A^TA : $\lambda_1, \ldots, \lambda_n$

$$\|A\|_2 = \max_{1\le i\le n} \sqrt{\lambda_i}$$

<u>Special case:</u> If A is symmetric, then $A^TA = A^2$ and the eigenvalues of A^2 are squares of eigenvalues of A

$$\longrightarrow \|A\|_2 = \max_{1 \le i \le n} |\lambda_i(A)|$$

$\|A\|_2 = 5.1167$

A pos. semidef if $x^T A x \geq 0$ for all x