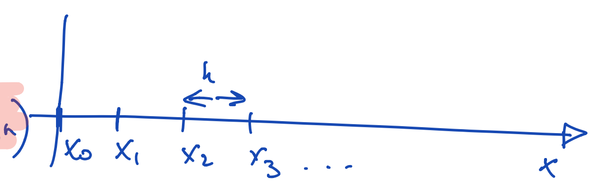
IVPs:

$$y' = f(x,y), \quad y(x_0) = y_0$$

Explicit / forward Euler:

$$y_{n+1} = y_n + h f(x_n, y_n)$$



Implicit / backwards Euler:

[redacted]

y_{n+1} on both sides $\longrightarrow$ implicit

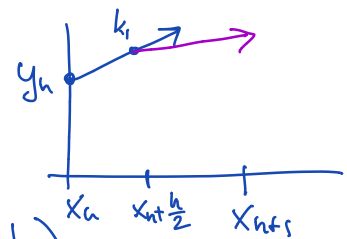
Trapezoidal method:

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big)$$

Runge-Kutta methods

Is it possible to have an explicit rule that has higher-order accuracy compared to explicit Euler? Yes, through intermediate evaluation of f.

$$y_{n+1} = y_n + h\,(a k_1 + b k_2)$$



stages

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \alpha h,\ y_n + \beta h k_1)$$

$$a, b, \alpha, \beta \in \mathbb{R}$$

replaced k_1 in expression for k_2.

we want: $a + b = 1$

$$\phi(x, y, h) = a\, f(x_n, y_n) + b\, f(x_n + \alpha h,\ y_n + \beta h f(x_n, y_n))$$

truncation error:

$$T_n = \frac{y(x_n + h) - y(x_n)}{h} - \phi(x_n, y_n; h)$$

Taylor expansion for $y(x_n + h)$, $\phi(x_n, y_n, h)$

$$y(x_n + h) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + O(h^4)$$

$f(x_n, y(x_n))$

$$y''(x_n) = f_x + f_y y' = f_x + f_y f$$

$$y'''(x_n) = f_{xx} + f_{xy} f + (f_{xy} + f_{yy} f) f + f_y (f_x + f_y f)$$

$$\phi(x_n, y_n, h) = a f + b\Big(f + \alpha h f_x + \beta h f f_y + \tfrac{1}{2}(\alpha h)^2 f_{xx} + \alpha \beta h^2 f f_{xy} + \tfrac{1}{2}(\beta h)^2 f^2 f_{yy} + O(h^3)\Big)$$

$$T_n = \frac{y(x_n + h) - y(x_n)}{h} - \phi(x_n, y_n, h) =$$

$$f + \tfrac{1}{2} h\,(f_x + f f_y) + \tfrac{1}{6} h^2 \big[f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y (f_x + f_y f)\big]$$

$$- \Big\{ a f + b\big[f + \alpha h f_x + \beta h f f_y + \tfrac{1}{2}(\alpha h)^2 f_{xx} + \alpha \beta h^2 f f_{xy} + \tfrac{1}{2}(\beta h)^2 f f_{yy}\big]\Big\} + O(h^3)$$

$f(1-a-b) = 0$ since $a+b=1$

$h\left(\frac{1}{2}(f_x + f f_y) - b\alpha f_x - b\beta f f_y\right) \longrightarrow$ will be zero if

$b\alpha = \frac{1}{2}, \quad b\beta = \frac{1}{2}$

$\left[\Rightarrow \beta = \alpha, \; a = 1 - \frac{1}{2\alpha}, \; b = \frac{1}{2\alpha}\right]$

For any $\alpha \neq 0$, we found a second-order RK scheme

$\underline{\underline{\alpha = \frac{1}{2}}}$ (modified Euler scheme):

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, \; y_n + \frac{h}{2} f(x_n, y_n)\right)$$

2 evaluations of f, explicit, second-order accurate

$\underline{\alpha = 1}$ (improved Euler)

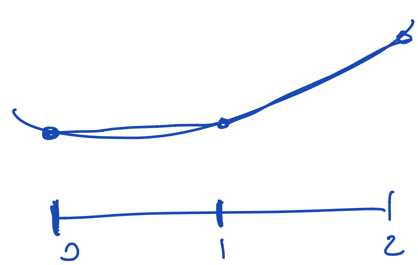
$$y_{n+1} = y_n + \frac{1}{2} h \left[f(x_n, y_n) + f\big(\underbrace{x_n + h}_{x_{n+1}}, \underbrace{y_n + h f(x_n, y_n)}_{\approx y_{n+1}}\big)\right]$$

explicit, 2nd order, 2 evaluations of f

Example: For $f(x) = x^2 - x + 1$ on $[0,2]$ we want to approximate the integral $I := \int_0^2 f(x)\,dx$

(i) Splitting the interval into $[0,1]$, $[1,2]$ and using composite trapezoidal rule

(ii) using Simpson's rule:



Trap: $I_{trap} = \left(\frac{1}{2} f(0) + \frac{1}{2} f(1)\right) 1 + \left(\frac{1}{2} f(1) + \frac{1}{2} f(2)\right) 1 = \frac{1}{2}\left(f(0) + 2 f(1) + f(2)\right)$

$$I_{Simp} = 2 \cdot \left[\frac{1}{6} f(0) + \frac{4}{6} f(1) + \frac{1}{6} f(2)\right]$$

Error for Simpson's rule is

$$|E_2(f)| \leq \frac{(b-a)^5}{2880} M_4 \,, \qquad M_4 = \max_{x \in [a,b]} |f''''(x)|$$

$f(x) = \frac{1}{4} x^4 + \cos(x)$, what is the maximal error you make when approximating $\int_0^{2\pi} f(x)\,dx$ with Simpson?

ODE example:

Error estimate

$$|e_n| \leq 10 \, (e^x - 1) \, h^2$$

- To ensure that the error at $x = \ln(2)$ is less than $\tau = 10^{-5}$, how should you choose h?
- If you consider $x > \ln(2)$ at which you desire the error e_n to be less than τ, do you have to choose h larger, smaller or the same & why?