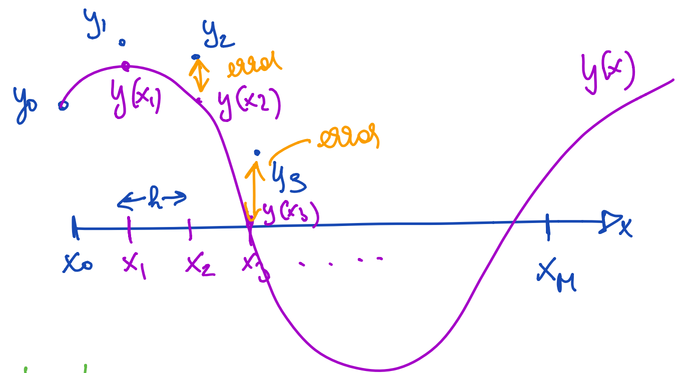
Initial value problems



Approximation

points $x_0, x_1, x_2, \ldots$

$x_n = x_0 + nh \qquad h = \dfrac{x_M - x_0}{M}$

$y_n \approx y(x_n)$

num. approx

real solution!

Euler's method: (explicit)

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)} \qquad n = 0, 1, 2, \ldots$$

One-step method, general form

$$y_{n+1} = y_n + h\,\phi(x_n, y_n, h)$$

exact values, unknown

$$e_n = y(x_n) - y_n \qquad \text{error}$$

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n), h) \qquad \text{truncation error}$$

Theorem: Consider one-step method, where

$|\phi(x, u, h) - \phi(x, v, h)| \leq L_\phi |u - v|$;

Assuming $|y_n - y_0| \leq C$ for $n = 1, 2, \ldots$

$$\boxed{|e_n| \leq \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right)} \qquad n = 1, 2, 3, \ldots$$

larger for large n (exponential growth of error bound)

$T = \max\limits_{0 \leq n \leq N-1} |T_n|$ maximum truncation error.

truncation error for (explicit) Euler:

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$

$$y(x) \in C^2 \longrightarrow \quad y(x_{n+1}) = y(x_n + h) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2!} y''(\xi_n)$$

$x_n \le \xi_n \le x_{n+1}$

move $y(x_n)$ to left side, divide by h

$$\Longrightarrow T_n = \frac{1}{2} h\, y''(\xi_n)$$

$$M_2 = \max_{x \in [x_0, x_n]} |y''(x)| \Longrightarrow |T_n| \le T = \frac{1}{2} h M_2$$

$$\xrightarrow{\text{theorem}} \quad |e_n| \le \frac{1}{2} M_2 \left[\frac{e^{L(x_n - x_0)} - 1}{L} \right] h$$

allows to make the error small.

Example: $y' = \tan^{-1}(y)$, $y(0) = y_0$

Compute L and M_2 explicitly:

$\eta \in (u, v)$

$$|f(x,u) - f(x,v)| = \left| \frac{\partial f}{\partial y}(x, \eta)(u - v) \right| = \left| \frac{\partial f}{\partial y}(x, \eta) \right| |u - v|$$

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = \left| \frac{1}{1+y^2} \right| \le 1 \longrightarrow \underline{\underline{L = 1}}$$

M_2 depends on y'' which is unknown, but we can still estimate it by differentiating $y' = f(x,y) = \tan^{-1}(y) \longrightarrow$

$$y'' = \frac{d}{dx}\left(\tan^{-1} y(x)\right) = \frac{1}{1+y^2} \underbrace{\frac{dy}{dx}}_{y' = \tan^{-1}(y)} = \frac{1}{1+y^2} \tan^{-1}(y)$$

$$\longrightarrow |y''| \le M_2 := \frac{\pi}{2}$$

$$\Longrightarrow |e_n| \le \frac{1}{4} \pi \left(e^{x_n} - 1\right) h$$

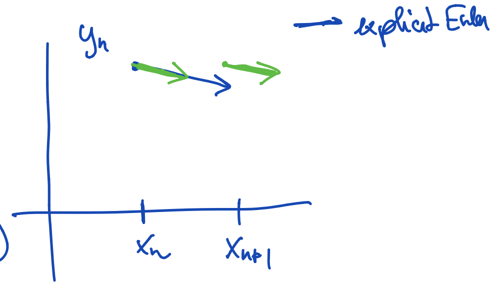
This allows to choose h such that one is guaranteed to have $|e_n| \le tol$ (e.g. 10^{-5})

§12.4 An implicit one-step method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Main difference: The right-hand side depends on y_{n+1} — so we need to solve a nonlinear equation:

$$y_{n+1} - \frac{h}{2} f(x_{n+1}, y_{n+1}) = y_n + \frac{h}{2} f(x_n, y_n)$$



$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \underbrace{\frac{1}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]}_{\phi(x,y,h)}$$ truncation error

$$|T_n| \le \frac{1}{12} h^2 M_3 \qquad M_3 = \max_{x \in [x_0, x_N]} |y'''(x)|$$

higher-order in h compared to Euler

theorem $\longrightarrow$ $$|e_n| \le \frac{M_3}{12 L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right) h^2 \qquad n = 0, 1, 2, \ldots$$

higher-order!

<u>Example:</u> $y' = y^2 - g(x)$, $y(0) = 2$

$$g(x) = \frac{x^4 - 6x^3 + 12x^2 - 14x + 9}{(1+x)^2}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[\underbrace{y_n^2 - g(x_n)}_{f(x_n, y_n)} + \underbrace{y_{n+1}^2 - g(x_{n+1})}_{f(x_{n+1}, y_{n+1})}\right]$$

quadratic equation in y_{n+1} — can be solved analytically or using Newton's method

$$y_{n+1}^{(1,2)} = \frac{1}{h} \pm \sqrt{\frac{1}{h^2} - \left(\frac{2}{h} y_n + y_n^2 - g(x_n) - g(x_{n+1})\right)}$$

Implicit / backwards Euler:

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$