§12 Initial value problems / ODEs

$$\begin{cases} y'' + 2y' = 3y \\ f''(x) + 2f'(x) = 3f(x) \\ \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 3y \end{cases}$$

$y = y(x)$ is a function of x

ODEs are relations between functions and their derivatives

solution is a function or a set of functions

identical but different notation

One solution is $y(x) = e^{-3x}$ since: $y'(x) = -3e^{-3x}$, $y''(x) = 9e^{-3x}$

$$y'' + 2y' = 9e^{-3x} + 2(-3e^{-3x}) = 3e^{-3x} = 3y$$

We consider <u>initial value problems</u>

$$(\text{IVP}) \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

← differential equation

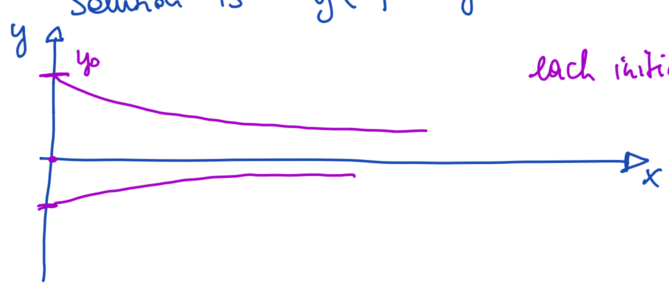
initial value

Solution is a curve / function $y : [x_0, x_M] \longrightarrow \mathbb{R}$ that starts at y_0

<u>Example:</u> $y' = -2xy$ on $x \in [0,1]$

$y(0) = y_0 \in \mathbb{R}$

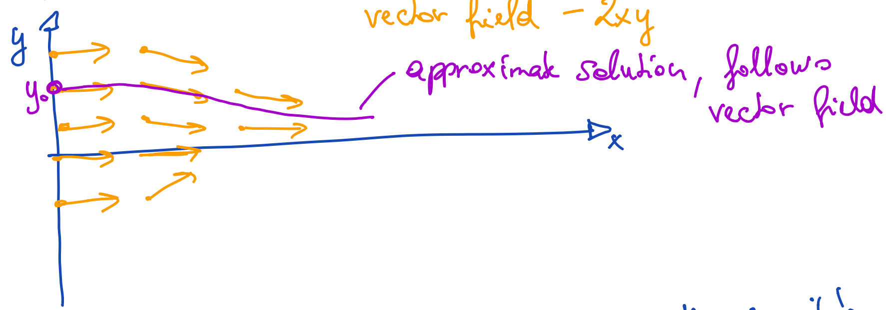
Solution is $y(x) = y_0 e^{-x^2}$



each initial condition gives a different solution.

How about if we cannot find solution analytically? Then we've to rely on numerical approximations

$y' = f(x,y) = -2xy$



We have to ensure that a unique solution exists – otherwise it's pointless to find numerical approximations.
In general, we cannot hope for a unique solution – a unique solution only exists if $f(\cdot,\cdot)$ satisfies certain properties.

<u>Theorem:</u> IVP $y' = f(x,y)$, $y(x_0) = y_0$

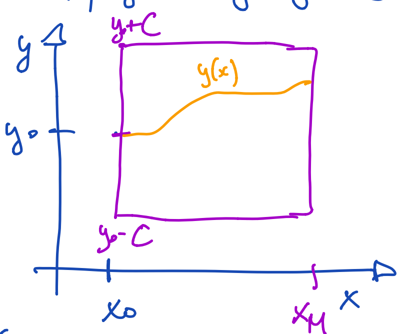
f continuous on $D = \{(x,y),\ x_0 \le x \le x_M,\ y_0 - C \le y \le y_0 + C\}$

$|f(x,y_0)| \le K$ for all x

f Lipschitz continuous in 2nd variable, i.e.:

$|f(x,u) - f(x,v)| \le L|u-v|$

$C \ge \frac{K}{L}\left(e^{L(x_M - x_0)} - 1\right)$



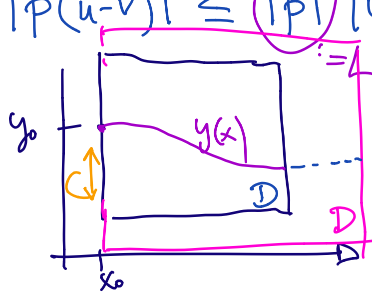
$\Longrightarrow$ there exists a unique solution $y \in C^1([x_0, x_M])$ inside D.

<u>Example:</u> $y' = py + q$ $\quad p, q \in \mathbb{R}$, $\quad y(x_0) = y_0$

$|f(x,y_0)| = |py_0 + q| \le |py_0| + |q| := K$

$|f(x,u) - f(x,v)| = |p(u-v)| \le |p|\,|u-v|$ $\quad := L$

$\longrightarrow$ unique solution for all $x \in [x_0, \infty)$



Example 2: $y' = y^2, \quad y(0) = 1$

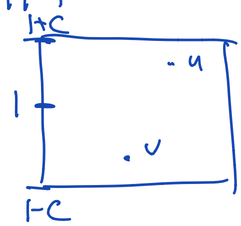
exact solution: $y(x) = \frac{1}{1-x}$, $\quad 0 \leq x < 1$

$$|f(x, y_0)| = |y_0^2| = 1 = K$$

$$|f(x,u) - f(x,v)| \leq |u^2 - v^2| = |v-u|\,|v+u| \leq L\,|v-u|$$

$L := 2(1+C)$ because

$|v+u| \leq |v| + |u|$



$$\left.\begin{array}{l} |u-1| \leq C \\ |v-1| \leq C \end{array}\right\} |u| + |v| \leq 2(1+C)$$

$$C \geq \frac{1}{2(1+C)}\left(e^{2(1+C)\,x_M} - 1\right)$$

$$\Longrightarrow x_M \leq \frac{1}{2(1+C)} \ln\left(1 + 2C + 2C^2\right)$$

$$\Longrightarrow x_M \leq 0.43$$

theory only guarantees solution for $x \in [0, 0.43]$.

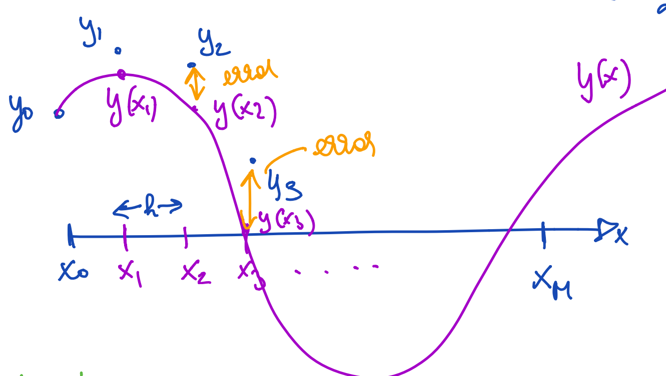
Approximation

points $x_0, x_1, x_2, \ldots$

$x_n = x_0 + nh \qquad h = \frac{x_M - x_0}{M}$

$y_n \approx y(x_n)$

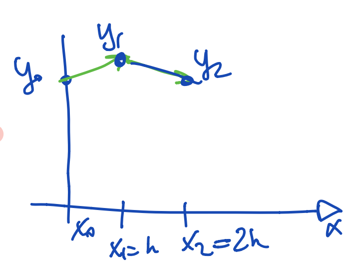
num. approx — real solution!



One step methods: y_{n+1} is computed just from y_n (different from k-step methods where y_{n+1} is computed from $y_n, y_{n-1}, \ldots, y_{n-k+1}$)

Simplest is **Euler's method**

$n = 0, 1, 2, \ldots \qquad y_{n+1} = y_n + h f(x_n, y_n)$



Where does this come from?

Taylor expansion of $y(x_{n+1}) = y(x_n + h)$

$$\underbrace{y(x_n+h)}_{\approx\, y_{n+1}} = \underbrace{y(x_n)}_{\approx\, y_n} + \underbrace{h\, \underbrace{y'(x_n)}_{f(x_n,\, y(x_n))}}_{h\, f(x_n,\, y(x_n))} + \dots O(h^2)$$

One step method, general form:

$$y_{n+1} = y_n + h\,\Phi$$